数理解析研究所講究録
第 829 巻 1993 年 77-87

"Indirect" Time Series Analysis for One-Dimensional Chaos Based on Perron-Frobenius Operator: "Generalized" Ulam-Li's Approximation to Invariant Density

Tohru KOHDA† and Kenji MURAO††

Kyushu University† and Miyazaki University††, Japan

Abstract A unified approach to time series analysis for one-dimensional discrete chaos is given which is based on the Galerkin approximation to the Perron-Frobenius operator. The proposed method gives approximations with high accuracy to statistics of various one-dimensional chaos. Numerical results for $1/f^{\delta}$ power spectrum of intermittent chaos also show that the observed exponent of the FFT power spectrum of long trajectories as $f \to 0$ is in good agreement not with the Procaccia-Schuster's estimate but with our estimate.

I. Introduction

There are two kinds of time series analysis for long-time chaotic trajectories $\{x_m\}_{m=0}^{\infty}$ generated by a recurrence formula $x_{m+1} = \tau(x_m)$ with an ergodic transformation $\tau : I = [0,1] \to I$. One of them is the "*time-average technique*", in which we evaluate certain statistics of a sample long-time trajectory $\{x_m\}_{m=0}^{n}$ with some initial value $x = x_0$; the other one is the "*ensemble-average technique*" under the assumption that τ is mixing with respect to an absolutely continuous invariant measure, denoted by $f^*(x)dx$. We give a unified approach to time series analysis for discrete chaos by such an ensemble-average technique.

The time-average technique which is a usual method[4] is referred to as the "*direct method*". On the contrary, the ensemble average technique is a kind of "*indirect methods*" because there is no need to directly calculate trajectories. Hence such an indirect method is expected to play an important role in theoretically understanding chaos. In fact, the Perron-Frobenius operator whose fixed point is $f^*(x)$ permits us to theoretically calculate the ensemble average of several statistics[5],[6]. This operator, denoted by P_τ, however, gives no practically calculating method because of its infinite dimensionality. Such a situation leads us to consider an efficient algorithm for systematically calculating statistics which is based on the Galerkin approximation to P_τ on a suitable function space[7]–[10]. This algorithm is referred to as a "generalized" Ulam-Li's method[7]. We used the word "Ulam-Li's method" because Li[2] gave an affirmative answer to the Ulam's conjecture[1] concerning a piecewise-constant approximation for $f^*(x)$.

Numerical experiments demonstrate that the proposed method can give approximations with high accuracy to statistics of various one-dimensional chaos.

II. Perron-Frobenius Operator and Statistics of Chaos

If $y = \tau(x)$ is mixing with respect to $f^*(x)dx$, then for almost initial value $x = x_0$ sequences $\{x_m\}_{m=0}^{\infty}$ can chaotically behave. From the Birchoff individual ergodic theorem, the time average of any L_1 function $F(x)$ along a trajectory $\{x_m\}_{m=0}^{\infty}$, which is defined by

$$\overline{F} = \lim_{T\to\infty} \frac{1}{T} \sum_{n=0}^{T-1} F(\tau^n(x)), \tag{1}$$

is equal almost everywhere to the ensemble average of $F(x)$ over I, defined by

$$< F >= \int_I F(x) f^*(x) dx. \tag{2}$$

The direct time series analysis is based on using $\overline{F}$. However, the *sensitive dependence on initial conditions*, one of chaotic properties[3], prevents us from precisely evaluating $\overline{F}$. On the other hand, the indirect time series analysis is based on using $< F >$. We begin with reviewing relations between typical statistics and P_τ. The operator P_τ is defined by

$$P_\tau f(x) = \int_I \delta(x - \tau(y)) h(y) dy. \tag{3}$$

For any L_1 functions of bounded variations $g(x)$ and $h(x)$, P_τ has the important property

$$(g(x), h(\tau(x))) = (P_\tau g(x), h(x)), \tag{4}$$

where

$$(g, h) = \int_I g(x)h(x)dx. \tag{5}$$

The invariant density $f^*(x)$ which plays a key role in our indirect method is the eigenfunction of P_τ belonging to the eigenvalue 1, that is,

$$P_\tau f^*(x) = f^*(x). \tag{6}$$

The autocorrelation function is defined by

$$\rho(k) = < x\tau^k(x) > - < x >^2 . \tag{7}$$

The first term of the rhs of this equation is rewritten as

$$< x\tau^k(x) > = (P_\tau^k(xf^*(x)), x), \tag{8}$$

where the above property of P_τ is repeatedly used. Let $h_i(x)$ be the eigenfunction of P_τ with the eigenvalue λ_i for the eigenvalue problem[4]

$$P_\tau h_i(x) = \lambda_i h_i(x). \tag{9}$$

If we can expand $xf^*(x)$ as

$$xf^*(x) = \sum_{i=1}^{\infty} \eta_i h_i(x), \tag{10}$$

then we have

$$\rho(k) = \sum_{i=2}^{\infty} u_i \lambda_i^k, \tag{11}$$

the Fourier Transform of which gives the power spectrum $S(\nu)$

$$S(\nu) = \sum_{i=2}^{\infty} u_i \frac{1-\lambda_i^2}{(1-\lambda_i z)(1-\lambda_i z^{-1})} \tag{12}$$

where

$$\lambda_1 = 1, u_i = \eta_i(x, h_i), \text{and} \quad z = exp(j2\pi\nu) \tag{13}$$

with $0 < \nu < 1$. Oono and Takahashi[5],[6] demonstrated that the Fredholm theory of P_τ plays an important role in discussions of the power spectrum. It is, however, difficult to find exact solutions of eigenvalues and eigenfunctions of P_τ, primalily because P_τ has the infinite dimensionality. Such a situation led us to consider an efficient algorithm of the indirect method.

III. Galerkin Approximations to Perron-Frobenius Operator

Let Δ be a function space which is spanned by a vector basis function $\vec{\ell}(x)$. The constructing method of Δ is as follows. We divide I into N subintervals $\{I_n\}$ with partition points $\{c_i\}_{i=0}^N$ satisfying $0 = c_0 < c_1 < c_2 < \cdots < c_N = 1$ such that

$$I = \bigcup_{n=1}^{N} I_n, \quad I_n = [c_{n-1}, c_n]. \tag{14}$$

Our Galerkin approximations depend on the appropriate selections of $\{c_i\}_{i=0}^N$ and of $\vec{\ell}(x)$[7]–[10]. A simple but efficient procedure, however, is omitted here for selecting $\{c_i\}_{i=0}^N$. Next, we take bases $\ell_{nk}(x)$ such as[10]

$$\ell_{nk}(x) = p_{nk}(x)s(x)\chi_n(x), \quad 0 \le k \le K, \quad 1 \le n \le N. \qquad (15)$$

In the above equation, $\chi_n(x)$ is the characteristic function of I_n and $p_{nk}(x)$ is the $k-$th order Legendre's polynomial which is orthogonal to each other on I_n. For most of practical usages, we use $K = 2$. When τ has a bounded invariant density, the function $s(x)$, referred to as a supplementary function, is taken to be 1. On the other hand, τ has an unbounded invariant density, $s(x)$ is chosen to be a singular function which approximates to singularities of the unbounded invariant density and the inner product (g, h) must be also replaced by the weighted inner product

$$(g, h)_w = \int_I g(x)h(x)w(x)dx \qquad (16)$$

with the weighting function

$$w(x) = s^{-2}(x). \qquad (17)$$

Each component $\ell_{nk}(x)$ is an appropriately chosen piecewise polynomial of at most K degree whose combination approximates to $xf^*(x)$ by the Galerkin method[7] such as

$$xf^*(x) \simeq \mathbf{f}^t\vec{\ell}(x), \qquad (18)$$

where the superscript t denotes the transpose of the vector $\mathbf{f}$. Using $\vec{\ell}(x)$, we get

$$< x\tau^k(x) > \simeq \mathbf{f}^t(P_\tau^k\vec{\ell}(x), x). \qquad (19)$$

Furthermore, using the Galerkin method with $\vec{\ell}(x)$ on Δ, we approximate to $P_\tau\vec{\ell}(x)$ such as

$$P_\tau \vec{\ell}(x) \simeq \widehat{P}_\tau^t \vec{\ell}(x) \tag{20}$$

which leads us to readily obtain

$$< x\tau^k(x) > \simeq \mathbf{f}^t (\widehat{P}_\tau^t)^k (\vec{\ell}(x), x), \tag{21}$$

where the $N(K+1) \times N(K+1)$ matrix $\widehat{P}_\tau$ is referred to as the *Galerkin-approximated matrix of the Perron-Frobenius operator* where N and K are integers to be given below. The explicit form of $\widehat{P}_\tau$ is given in[7]. Let $\mathbf{h}_i$ be the $i-$th right eigenvector of $\widehat{P}_\tau$ with the eigenvalue λ_i for the easily tractable eigenvalue problem

$$\widehat{P}_\tau \mathbf{h}_i = \widehat{\lambda}_i \mathbf{h}_i. \tag{22}$$

Let $\widehat{\lambda}_1$ be the maximum eigenvalue of $\widehat{P}$. It is easily shown that $\widehat{\lambda}_1$ when the supplementary function $s(x) = 1$, namely, when both the polynomial bases and the unweighted inner product are used. But $\widehat{\lambda}_1 \simeq 1$ when $s(x) \neq 1$, that is, when both the singular bases and the weighted inner product are used. For the latter case, numerical experiments show $\widehat{\lambda}_1$ is nearly equal to 1 with the eror less than 10^{-6} for $K = 2$ and $N = 32$. An approximate solution to the invariant density given by

$$\widehat{f}^*(x) = \mathbf{h}_1^t \vec{\ell}(x) \tag{23}$$

where $\mathbf{h}_1$ is normalized such that

$$\int_I \widehat{f}^*(x)dx = 1 \tag{24}$$

It is easily shown that Eq.(23) is an approximate solution to Eq.(6) by the Galerkin method and that $\widehat{f}^*(x)$ when $K = 0$ gives the results by the well-known Ulam-Li's method. Figure 1 shows convergence rates of approximate solution $\hat{f}^*(x)$ by our method[7].

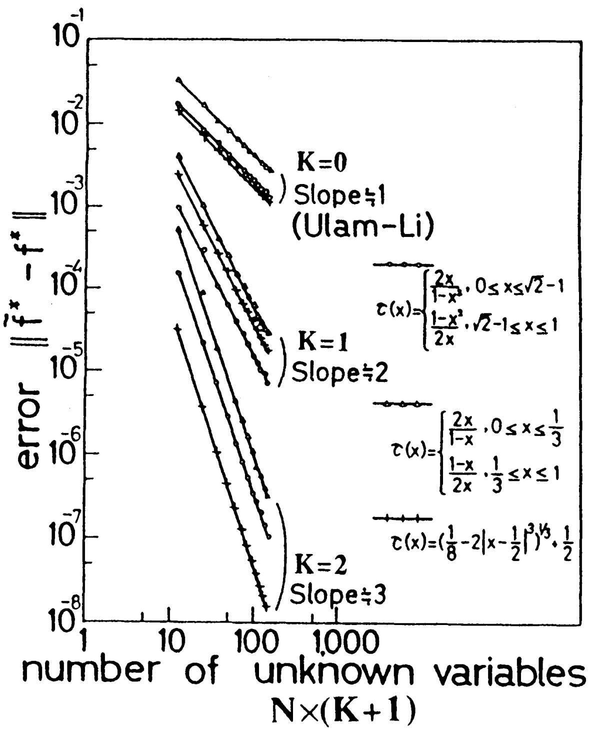

Fig. 1 Convergence rates of approximate solutions $\hat{f}^*(x)$ (the proposed method) to the bounded invariant densities $f^*(x)$ for mixing chaos in several examples.

IV. Numerical Examples

Example 1 Let

$$\tau(x) = \begin{cases} ax^z + (a+b-ab)/b & 0 \le x \le x_p = (1-1/b)^{1/z} \\ -b(x^z - 1) & x_p < x \le 1 \end{cases}$$

This map can generate periodic chaos for suitable parameters. Figures 2 and 3 show $f^*(x)$ and the power spectrum $\tilde{S}_T(\nu)$ for periodic chaos of period 6 which are calculated by our method[8],[9]. In this calculation, we take $\{\tau^n(0)\}_{n=1}^{30}$ as the partition points $\{c_i\}_{i=1}^{N-1}$ so that edges of the support of $f^*(x)$ will coincide with the partition points. In the calculation of $\tilde{S}_T(\nu)$, the finite discrete Fourier transform of $\{\rho(k)\}_{k=0}^{T-1}$ $(T = 1,024 \times 6)$ is used instead of using Eq.(12). On the other hand, $S_{T,m}(\nu)$ is obtained by averaging $m = 200$ discrete Fourier transforms of trajectories of length T. The spectrum $\tilde{S}_T(\nu)$ is in good agreement with $S_{T,m}(\nu)$ except for fluctuations in the latter.

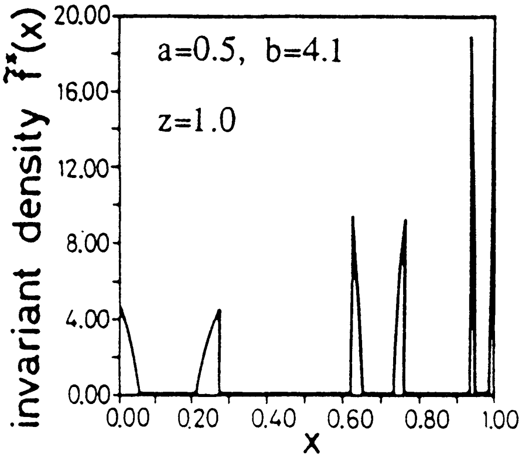

Fig. 2 Approximated invariant density $\hat{f}^*(x)$ (by our indirect method) for periodic chaos of period 6 in example 1.

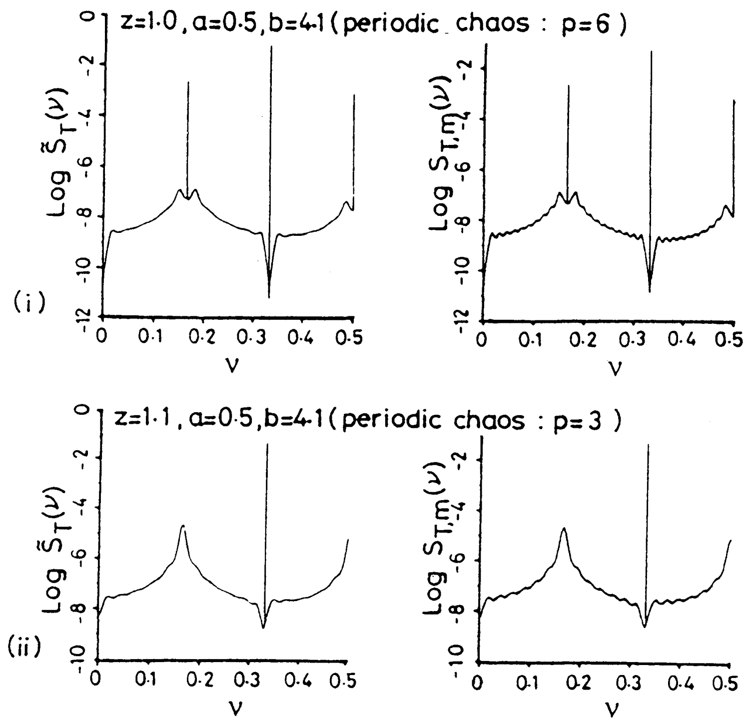

Fig. 3 Power spectra $\tilde{S}_T(\nu)$ (by our indirect method) and $\tilde{S}_{T,m}(\nu)$ (by the direct method) for periodic chaos of period 6 in example 1.

Example 2 Let

$$\tau(x) = \begin{cases} x + ux^z & 0 \le x \le x_p \\ (x - x_p)/(1 - x_p) & x_p < x \le 1 \end{cases}$$

where $\tau(x_p) = 1, u > 0, 1 < z < 2$. This map generates intermittent chaos with the power spectrum $1/f^\delta$. Figure 4 shows the power spectrum $S(\nu)$ by our method[10] (the smooth solid line) and $S_{T,m}(\nu)$ with $T = 2^{15}$ and $m = 100$ by the direct method (the fluctuated line), each of which is in good agreement each other in wide frequency range. In applying our method, we used $s(x) = x^{-(z-1)}$ because τ has the unbounded invariant density with a $(z-1)$-th order pole at $x = 0$. In this figure, the broken line shows the Procaccia and Schuster's estimate [11] of the spectrum when ν goes to 0 which does not coincide well with the former two.

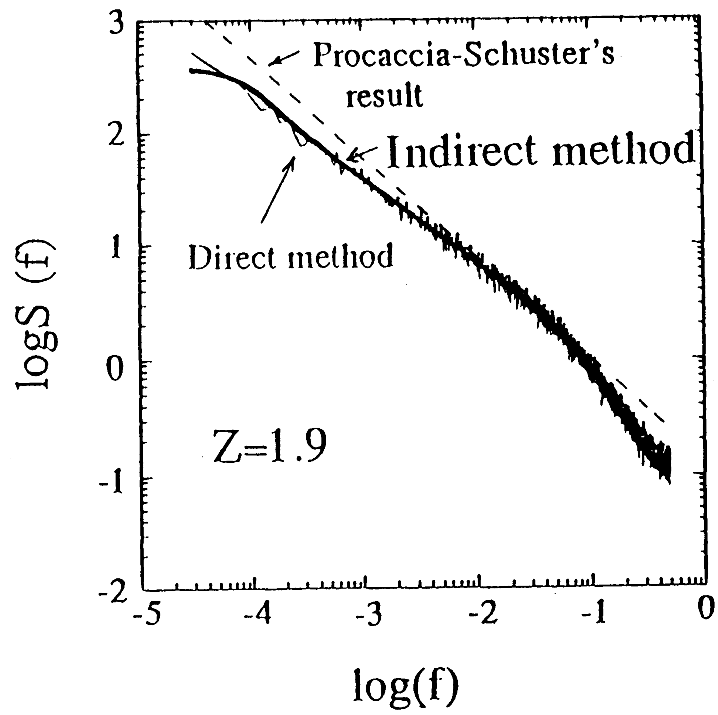

Fig. 4 Comparison of power spectra calculated by using three different methods for intermittent chaos in example 2.

References

[1] S.Ulam, *Problems in modern mathematics*, Intersience Pub. (1960).

[2] T.Y.Li, J.Approx. Theo., **17**, pp.177-186(1976).

[3] S.Grossmann and S.Thomae, Z. Naturforsch., **32a**, 1353-1363 (1977).

[4] S.J.Chang and J.Wright, Phys.Rev.A, **23**, 1419-1433 (1981).

[5] Y.Oono and Y.Takahashi, Progr.Theo.Phys., **63-5**, 1804-1807 (1980).

[6] Y.Takahashi and Y.Oono, Progr.Theo.Phys., **71-4**, 851-854 (1980).

[7] T.Kohda and K.Murao, Trans. IEICE, **J65-A**-6, 505–512 (1982).

[8] K.Murao and T.Kohda, Trans. IEICE, **J67-A**-5, 511-518 (1984).

[9] K.Murao and T.Kohda, Trans. IEICE, **J68-A**-7, 405-406 (1985).

[10] T.Kohda and K.Murao, Trans. IEICE, **E73**-6, 793-800 (1990).

[11] I.Procaccia and H.Schuster, Phys.Rev., A, **28**-2, 1210-1212 (1983).