数理解析研究所講究録
第 831 巻 1993 年 129-140

Numerical Verification of Simple Bifurcation Points

Takuya Tsuchiya[†**]
(土屋 卓也)

Abstract. Nonlinear boundary value problems (NBVPs in abbreviation) with parameters are called parametrized nonlinear boundary value problems. This paper studies numerical verification of simple bifurcation points of parametrized NBVPs defined on one-dimensional bounded intervals. Around simple bifurcation points the original problem is extended so that the extented problem has an invertible Fréchet derivative. Then, the usual procedure of numerical verification of solutions can be applied to the extended problem. A numerical example is given.

Key words. parametrized nonlinear boundary value problems, numerical verification of solutions, simple bifurcation points

AMS(MOS) subject classifications. 65L10, 65L99

Abbreviated title. Numerical Verification

---

† Department of Mathematics, Ehime University, Matsuyama 790, Japan.
** Partially supported by Saneyoshi Scholarship Foundation.

1. Introduction.

For the past several years a theory for numerical verification of solutions of differential equations has been developed [N1-5], [TN], [WN]. By the theory the existence of *exact* solutions of differential equations are verified on computers by certain procedures in finite steps.

Let $\Lambda \subset \mathbb{R}$ be a bounded interval for a parameter. Here we deal with the following nonlinear two-point boundary value problem with a parameter $\lambda \in \Lambda$ on the bounded interval $J := (a, b)$:

$$(1.1)\qquad \begin{cases} -u'' = f(\lambda, x, u) & \text{in } J, \\ u(a) = u(b) = 0, \end{cases}$$

where $f : \Lambda \times J \times \mathbb{R} \to \mathbb{R}$ is a given smooth function. Since (1.1) has the parameter λ, the set of the solutions of (1.1) would form one dimensional curves. There, however, may exist singular points on the curves. For example, a solution curve might fold (the folding point is call a **turning point**), or several solution curves might intersect at one point (the intersecting point is called a **bifurcation point**).

Let (λ, u) be a solution of (1.1). The above singularities occur when the following eigenvalue problem has the eigenvalue $\mu = 0$:

$$(1.2)\qquad L\psi = \mu\psi,$$

where the differential operator L is defined by

$$L\psi := -\psi'' - f_y(\lambda, x, u)\psi,$$

and $f_y(\lambda, x, y)$ denotes the derivative of f with respect to y. More precisely, if $\mu = 0$ is *not* an eigenvalue of (1.2), by the implicit function theorem, there exists a unique solution curve around (λ, u), and it is parametrized by λ. Such a solution curve is called a **regular branch**. On regular branches the usual procedure of numerical verification of solutions of (1.1) can be applied.

Suppose that (1.2) has the eigenvalue 0. Then, we have some singularities there; we may have a turning point or, even worse, a bifurcatin point. In [TN] the case of turning points was considered.

In this paper we consider the case of bifurcation points. The difficulty of bifurcation points is as follows. A bifurcation point itself is not only very difficult to compute, but also very instable under perturbation: it may be disappeared by rounding errors or discretizations. Such a destroyed bifurcation is usually called a **numerical imperfect bifurcation.**

Our goal is to establish a new procedure for numerical verification of bifurcation points. The main idea is as follows: In [W] the original equation is extended around a simple bifurcation points so that the extended equation has an invertible Fréchet derivative. Then a straightforward modification of the usual numerical verification procedure works well at a bifurcation point.

In the last section a numerical examples is given.

2. Parametrized NBVP and Simple Bifurcation Points.

As is stated in Section 1, we consider the two-point boundary value problem

$$(2.1) \qquad \begin{cases} -u'' = f(\lambda, x, u) & \text{in } J, \\ u(a) = u(b) = 0, \end{cases}$$

where $J := (a, b) \subset \mathbb{R}$ is a bounded interval, and $\lambda \in \Lambda \subset \mathbb{R}$ is a parameter.

Let $H_0^1(J)$, $H^{-1}(J)$, etc. are the usual Sobolev spaces. In notation we omit '(J)' whenever there is no danger of confusion. The weak form of (2.1) is written as

$$(2.2) \qquad \textit{Find} \quad u \in H_0^1 \quad \textit{such that} \quad (u', v') = (f(\lambda, x, u), v), \quad \textit{for } \forall v \in H_0^1,$$

where $(\cdot, \cdot)$ is the inner product of L^2 defined by $(g, h) := \int_J ghdx$ for $g, h \in L^2$. Now, define the operators $L : \Lambda \times H_0^1 \to H^{-1}$ and $F : \Lambda \times H_0^1 \to L^2 \subset H^{-1}$ by, for $(\lambda, u) \in \Lambda \times H_0^1$,

$$(2.3) \qquad < L(\lambda, u), v > := \int_J u'v'dx, \qquad \forall v \in H_0^1,$$

$$(2.4) \qquad < F(\lambda, u), v > := \int_J f(\lambda, x, u)vdx, \qquad \forall v \in H_0^1,$$

where $< \cdot, \cdot >$ is the duality pair of H^{-1} and H_0^1. Since the inclusion $\iota : L^2 \hookrightarrow H^{-1}$ is compact, the operator $L - F : \Lambda \times H_0^1 \to H^{-1}$ is a Fredholm operator of index 1.

For F to be smooth, we suppose the following assumption holds:

A function $\psi : \Lambda \times J \times \mathbb{R} \to \mathbb{R}$ is called **Carathéodory continuous** if ψ satisfies the following conditions: for $(\lambda, x, y) \in \Lambda \times J \times \mathbb{R}$,

$$\begin{cases} \psi(\lambda, x, y) \text{ is continuous with respect to } \lambda \text{ and } y \text{ for almost all } x, \\ \psi(\lambda, x, y) \text{ is Lebesgue measurable with respect to } x \text{ for all } \lambda \text{ and } y. \end{cases}$$

If $\psi(\lambda, x, y)$ is Carathéodory continuous, $\psi(\lambda, x, u(x))$ is Lebesgue measurable with respect to x for any Lebesgue measurable function u.

Let $\alpha = (\alpha_1, \alpha_2)$ be usual multiple index with respect to λ and y. That is, for $\alpha = (\alpha_1, \alpha_2)$, $D^\alpha f(\lambda, x, y)$ means $\frac{\partial^{|\alpha|}}{\partial \lambda^{\alpha_1} \partial y^{\alpha_2}} f(\lambda, x, y)$.

Let $d \geq 1$ be an integer. For α, $|\alpha| \leq d$, we define the map $\mathbb{F}^\alpha(\lambda, u)$ for $(\lambda, u) \in \Lambda \times H_0^1$ by

$$\mathbb{F}^\alpha(\lambda, u)(x) := D^\alpha f(\lambda, x, u(x)). \tag{2.5}$$

We then assume that

Assumption 2.1. *Let $d \geq 2$. For all α, $|\alpha| \leq d$, we suppose that*

(1) *For almost all $x \in J$, $D^\alpha f(\lambda, x, y)$ exists at any $(\lambda, y) \in \Lambda \times \mathbb{R}$, and that is Carathéodory continuous.*

(2) *The mapping $\mathbb{F}^\alpha$ defined by (2.5) is a continuous operator from $\Lambda \times H_0^1$ to L^2, and the image $\mathbb{F}^\alpha(U)$ of any bounded subset $U \subset \Lambda \times H_0^1$ is bounded.* $\lhd$

Assumption 2.1 is satisfied if $f : \Lambda \times J \times \mathbb{R} \to \mathbb{R}$ is, for instance, C^d function.

Lemma 2.2. *Suppose that Assumption 2.1 holds. Then, the operator $F : \Lambda \times H_0^1 \to H^{-1}$ is of C^d class, and its partial derivatives are written as*

$$\begin{aligned} < D_u F(\lambda, u)\psi, v > &= \int_J f_y(\lambda, x, u(x))\psi v dx, \\ < D_\lambda F(\lambda, u)\eta, v > &= \eta \int_J f_\lambda(\lambda, x, u(x)) v dx, \end{aligned}$$

for $\psi, v \in H_0^1$, and $\eta \in \mathbb{R}$. $\lhd$

Now, suppose that $(\lambda_0, u_0) \in \Lambda \times H_0^1$ satisfies the following:

$$\begin{cases} (L - F)(\lambda_0, u_0) = 0 \in H^{-1}, \\ \dim\mathrm{Ker} D_u(L - F)(\lambda_0, u_0) = 1, \\ D_\lambda(L - F)(\lambda_0, u_0) \in \mathrm{Im} D_u(L - F)(\lambda_0, u_0). \end{cases} \tag{2.6}$$

We denote $D_u(L - F)(\lambda_0, u_0)$, $D_\lambda(L - F)(\lambda_0, u_0)$ by $D_u(L - F)^0$, $D_\lambda(L - F)^0$, respectively. Since $L - F$ is a nonlinear Fredholm operator of index 1 and (2.6), we know that that $\dim\mathrm{Ker} D(L - F)^0 = 2$ and $\dim\mathrm{Im} D(L - F)^0 = 1$. We also find that H_0^1 can be decomposed into subspaces

$$H_0^1 = V_0 \oplus \mathrm{Ker} D_u(L - F)^0,$$

and $D_u(L-F)^0|_{V_0}$ is an isomorphism from V_0 to $\mathrm{Im}D_u(L-F)^0$. Let P be the inverse operator of $D_u(L-F)^0|_{V_0}$. It follows from (2.6) that there exists $\phi_0 \in H_0^1$ and $\psi_0 \in H_0^1$ such that

$$
\text{(2.7)} \qquad \begin{cases} D_u(L-F)^0\phi_0 = 0, \qquad \|\phi_0\|_{H_0^1} = 1, \\ \mathrm{Ker}D_u(L-F)^0 = \mathrm{span}\{\phi_0\}, \\ \mathrm{Im}D_u(L-F)^0 = \{v \in H^{-1} | < v, \psi_0 >= 0\}. \end{cases}
$$

Then we assume that

$$
\text{(2.8)} \qquad B_0^2 - A_0C_0 > 0,
$$

where A_0, B_0, and C_0 are defined by

$$
\begin{aligned}
A_0 &:=< D_{uu}^2(L-F)^0\phi_0^2, \psi_0 >, \\
B_0 &:=< D_{\lambda u}^2(L-F)^0\phi_0 + D_{uu}^2(L-F)^0(\phi_0, -PD_\lambda^0), \psi_0 >, \\
C_0 &:=< D_{\lambda\lambda}^2(L-F)^0 + 2D_{\lambda u}^2(L-F)^0(-PD_\lambda^0) + D_{uu}^2(L-F)^0(-PD_\lambda^0)^2, \psi_0 >,
\end{aligned}
$$

and $PD_\lambda^0 := P(D_\lambda(L-F)^0)$.

By the Morse Theorem we have the following theorem:

Theorem 2.3. *Suppose that Assumption 2.1 holds for $d \geq 2$. Assume also that* (2.6) *and* (2.8) *are satisfied. Then, there exists an open set of (λ_0, u_0) such that the solutions of*

$$
(L-F)(\lambda, u) = 0
$$

consist in two C^{d-2} branches which intersect transversally at (λ_0, u_0).

A point at which (2.6) and (2.8) are satisfied is called a **simple bifurcation** point.

To resolve the singularity at a simple bifurcation point, Weber presented in [W] an extended operator which has an invertible Fréchet derivative at the simple bifurcation point. Let X and Y be Banach spaces defined by

$$
\text{(2.9)} \qquad X := H_0^1 \times H_0^1 \times H_0^1 \times \mathbb{R} \times \mathbb{R}, \qquad Y := H^{-1} \times H^{-1} \times H^{-1} \times \mathbb{R} \times \mathbb{R},
$$

with the norms

$$
\begin{aligned}
\|(x,y,z,a,b)\|_X &:= \max\Big\{\|x\|_{H_0^1}, \|y\|_{H_0^1}, \|z\|_{H_0^1}, |a|, |b|\Big\}, \\
\|(p,q,r,c,d)\|_Y &:= \max\Big\{\|p\|_{H^{-1}}, \|q\|_{H^{-1}}, \|r\|_{H^{-1}}, |c|, |d|\Big\},
\end{aligned}
$$

for $(x, y, z, a, b) \in X$ and $(p, q, r, c, d) \in Y$. Let $X_0 := (H_0^1)^3 \times \Lambda \times \mathbb{R} \subset X$. Suppose that $(\lambda_0, u_0) \in \Lambda \times H_0^1$ is a simple bifurcation point of the equation $L - F = 0$. Then we define an extended operator $G : X_0 \to Y$ by

$$(2.10) \qquad H(u, r, s, \lambda, \mu) := \begin{pmatrix} (L-F)(\lambda, u) + \mu\phi \\ D_u(L-F)(\lambda, u)r \\ D_u(L-F)(\lambda, u)s + D_\lambda(L-F)(\lambda, u) \\ (r, r) - 1 \\ (r, s) \end{pmatrix},$$

where $(\cdot, \cdot)$ is the L^2 inner product on the interval J, and $\phi \in H^{-1}$ is taken in a certain way so that $\phi \notin \mathrm{Im} D_u(L-F)(\lambda_0, u_0)$. Weber proved that the equation $H = 0$ is an isolated solution $w_0 := (u_0, r_0, s_0, \lambda_0, 0) \in X_0$ if $(\lambda_0, u_0) \in \Lambda \times H_0^1$ is a simple bifurcation point of the equation $L - F = 0$.

Now, we rewrite the equation $H = 0$ as a fixed point problem. Since $L|_{H_0^1} : H_0^1 \to H^{-1}$ is an isomorphism, we define $\Phi \in \mathcal{L}(H^{-1}, H_0^1)$ by $\Phi := (L|_{H_0^1})^{-1}$. It is easy to show that there exists a positive constant C_1 such that

$$(2.11) \qquad \|\Phi f\|_{H^2} \leq C_1 \|f\|_{L^2}$$

for any $f \in L^2$. Note that in this case the constant C_1 is easily determined, that is, C_1 is available in numerical verification procudure.

We define the operator $G : X_0 \to X$ by

$$(2.12) \qquad G(u, r, s, \lambda, \mu) := \begin{pmatrix} \Phi(F(\lambda, u) - \mu\phi) \\ \Phi(D_u F(\lambda, u)r) \\ \Phi(D_u F(\lambda, u)s + D_\lambda F(\lambda, u)) \\ \lambda + (r, r) - 1 \\ \mu + (r, s) \end{pmatrix}.$$

Then, the equation $H = 0$ is now equivalent to the fixed point problem

$$(2.13) \qquad (u, r, s, \lambda, \mu) = G(u, r, s, \lambda, \mu).$$

We try to modify the usual procedure of numerical verification to the fixed point problem (2.13).

3. Formulation of Numerical Verification.

Let $S_h \subset H_0^1$ be a finite element space. The projection $P_{h0} : H_0^1 \to S_h$ is defined by

$$((u - P_{h0}u)', v_h') = 0, \qquad \forall v_h \in S_h.$$

For S_h, we suppose the following assumption:

Assumption 3.1. *There exists a computable constant C_2 which is independent of h and u, and satisfies the following estimate:*

$$\|u - P_{h0}u\|_{H_0^1} \leq C_2 h |u|_{H^2}, \qquad \forall u \in H_0^1 \cap H^2. \quad \triangleleft \tag{3.1}$$

It is well known that the finite element space of piecewise linear functions satisfies Assumption 3.1.

Let $X_h = Y_h := (S_h)^3 \times \mathbb{R}^2$. The projection $P_h : X \to X_h$ is defined by

$$P_h(u, r, s, \alpha, \beta) := (P_{h0}u, P_{h0}r, P_{h0}s, \alpha, \beta), \quad \text{for} \quad (u, r, s, \alpha, \beta) \in X. \tag{3.2}$$

Suppose now that we are dealing with a simple bifurcation point. A finite element solution of the equation $H = 0$ could be defined by

$$\begin{aligned}
&(u_h', v_{1h}') - (f(\lambda_h, x, u_h) + \mu_h \phi, v_{1h}) = 0, \\
&(r_h', v_{2h}') - (f_y(\lambda_h, x, u_h) r_h, v_{2h}) = 0, \\
&(s_h', v_{3h}') - (f_y(\lambda_h, x, u_h) s_h - f_\lambda(\lambda_h, x, u_h, v_{3h}) = 0, \\
&(r_h, r_h) - 1 = 0, \\
&(r_h, s_h) = 0,
\end{aligned} \tag{3.3}$$

for any $(v_{1h}, v_{2h}, v_{3h}) \in (S_h)^3$. As is stated in Section 1, however, bifurcation may be destroied by discretization. Therefore, in general, a finite element solution of a simple bifurcation point does not exist.

Hence we just take $w_h := (u_h, r_h, s_h, \lambda_h, \mu_h) \in X_h$ which is "close enough" to the solution $w_0 \in X_0$ of the equation $H = 0$. In practical computation we try to make the absolute values of the left-hand side of (3.3) as small as possible. We then assume the following.

Assumption 3.2. *Let $w_h := (u_h, r_h, s_h, \lambda_h, \mu_h) \in X_h$ be "finite element solution" computed by the above manner. We assume that the restricted operator $P_h(I - DG(w_h))|_{X_h}$ has the inverse*

$$[I - DG^h]_h^{-1} : X_h \to X_h. \quad \triangleleft$$

If our "finite element solution" is close enough to the simple bifurcation point, Assumption 3.2 is satisfied.

In the sequel, we denote $DG(w_h)$ and $DF(w_h)$ by DG^h and DF^h, respectively.

Next, we introduce notions of *rounding* and *rounding error*. Let ϵ, $(0 < \epsilon < 1)$ be a parameter. We first define the operator $T_\epsilon : X_0 \to X$ by

$$T_\epsilon := I - ([I - DG^h]_h^{-1} P_h + \epsilon I)(I - G). \tag{3.4}$$

Note that if $[I - DG^h]_h^{-1} P_h + \epsilon I$ has an inverse operator, the two fixed point equations $w = G(w)$ and $w = T_\epsilon(w)$ are equivalent. Our main tool of numerical verification has been the following fixed point theorem (for instance, see [Z]):

Theorem 3.3 (Sadovskii's Fixed Point Theorem). *Let X be a Banach space and $U \subset X$ a nonempty, bounded, convex, closed subset. Suppose that the nonlinear operator $T : U \to U$ is a condensing map. Then, there exists a fixed point $u \in U$ of T:*

$$\exists u \in U \quad \textit{such that} \quad u = Tu. \quad \lhd$$

Since T_ϵ can be rewritten as

$$T_\epsilon = (1 - \epsilon) I + [I - DG^h]_h^{-1} P_h (I - G) + \epsilon G,$$

T_ϵ is a condensing map from X_0 to X. Hence, if we have a nonempty, bounded, convex, closed subset $U \subset X_0$ such that $T_\epsilon U \subseteq U$, we can conclude that there exists a fixed point of T_ϵ. Moreover, if $[I - DG^h]_h^{-1} + \epsilon I$ is invertible, the fixed point of T_ϵ is a solutuon of the equation $H = 0$. Hence, our verification is reduced to the construction of such U on the memory of computer.

The approximations of an element $u \in H_0^1$, a sebset $U \subset H_0^1$, and operators defined on H_0^1 in a certain finite element space S_h are called their **rounding**. The error of the rounding is called **rounding error**. These notions are defined by projection.

The rounding $\tilde{T}_\epsilon$ of T_ϵ is defined by $\tilde{T}_\epsilon := P_h \circ T_\epsilon$, where P_h is the projection defined by (3.2). Then, we see that

$$\tilde{T}_\epsilon = \tilde{I} - ([I - DG^h]_h^{-1} + \epsilon \tilde{I})(\tilde{I} - \tilde{G}), \tag{3.5}$$

where $\tilde{I} := P_h \circ I$ and $\tilde{G} := P_h \circ G$. Let $U \subset X$. The rounding $R(T_\epsilon U)$ is defined as the image of $\tilde{T}_\epsilon$:

$$R(T_\epsilon U) := \left\{ w_h \in X_n \;\middle|\; w_h = \tilde{T}_\epsilon(w),\ w \in U \right\}. \tag{3.6}$$

We define the rounding error $RE(T_\epsilon U)$ by

$$\alpha := \sup_{w\in U} \|T_\epsilon(w) - \tilde{T}_\epsilon(w)\|_X,$$

$$C := C_1C_2, \qquad (C_1, C_2 \text{ are defined by (2.11), (3.1), respectively.}),$$

$$RE(T_\epsilon U) := \left\{(\psi_1.\psi_2,\psi_3) \in (S_h^\perp)^3 \;\middle|\; \max_i \|\psi_i\|_{H_0^1} \le \alpha,\ \max_i \|\psi_i\|_{L^2} \le Ch\alpha\right\} \times \{(0,0)\}.$$

Note that $RE(T_\epsilon U)$ is a subset of $(H_0^1)^3 \times \{(0,0)\}$. Then, as in [TN] and [WN], we have

Theorem 3.4. *Let $U \subset X_0$ be a nonempty, bounded, convex, closed subset. If*

$$R(T_0U) \oplus RE(T_0U) \overset{\circ}{\subset} U, \tag{3.7}$$

then, there exists a solution $w \in U$ of the fixed point problem $w = G(w)$. Here, $A \overset{\circ}{\subset} B$ means $closure(A) \subset interior(B)$.

4. Numerical Verification.

By Theorem 3.4, in the set $U \subseteq X_0$ which satisfies (3.7), there exists at least one solution $w \in X_0$ of the fixed point problem $w = G(w)$. Therefore, if we construct such U on the memory of computer, the solution of the fixed point problem is said to verified numerically. This is what we shall do in this section.

Let $\{\phi_j\}_{j=1}^M$ be the basis of $(S_h)^3$. Let Θ_h be the set of linear combinations of intervals and ϕ_j:

$$\Theta_h := \left\{\left(\sum_{j=1}^M A_j\phi_j, A_{M+1}, A_{M+2}\right) \;\middle|\; A_j \subset \mathbb{R} \text{ are intervals}\right\}. \tag{4.1}$$

That is, an element $\omega \in \Theta_h$ is the set

$$\omega = \left(\sum_{j=1}^M A_j\phi_j, A_{M+1}, A_{M+2}\right) := \left\{\left(\sum_{j=1}^M a_j\phi_j, a_{M+1}, a_{M+2}\right) \;\middle|\; a_j \in A_j\right\}.$$

Let $\mathbb{R}^+$ be the set of nonnegative reals. For $\alpha \in \mathbb{R}^+$, we define the set $[\alpha] \subset (S_h^\perp)^3 \times \{(0,0)\} \subset (H_0^1)^3 \times \{(0,0)\}$ by

$$[\alpha] := \left\{(\psi_1.\psi_2,\psi_3) \in (S_h^\perp)^3 \;\middle|\; \max_i \|\psi_i\|_{H_0^1} \le \alpha,\ \max_i \|\psi_i\|_{L^2} \le Ch\alpha\right\} \times \{(0,0)\}. \tag{4.2}$$

We define the following iteration:

Definition 4.1. *Let $w_h \in X_h$ be the "finite element solution."*

(1) *We set $\Delta w_h^0 := \{w_h\}$ and $\alpha_0 := 0$ as the initial values.*

(2) *For $n \geq 1$, we define $U^{n-1} \subset X_0$, $\Delta w^n \subset X_h$, and $\alpha_n \in \mathbf{R}^+$ inductively by*

$$\left\{ \begin{array}{l} U^{n-1} := \Delta w_h^{n-1} + [\alpha_{n-1}], \\ \Delta w_h^0 := \tilde{T}_0 U^{n-1}, \\ \alpha_n := Ch \sup_{y \in U^{n-1}} \|\tilde{G}(y)\|, \end{array} \right. \tag{4.3}$$

where $\|\tilde{G}(y)\|$ is defined, for $y := (y_1, y_2, y_3, a, b) \in X$, by

$$\|\tilde{G}(y)\| := \max\left\{\|f(a,x,y_1)-b\phi\|_{L^2}, \|f_y(a,x,y_1)y_2\|_{L^2}, \|f_y(a,x,y_1)y_3+f_\lambda(a,x,y_1)\|_{L^2}\right\}. \quad \triangleleft$$

Note that it is very difficult or impossible to estimate Δw_h^n and α_n in (4.3) exactly. It is, however, possible and easy to enclose each coefficient interval by a slightly bigger interval, that is, overestimate them (cf. [WN]).

Now, let $\delta > 0$ be a small real. We define

$$\left\{ \begin{array}{l} \Delta \tilde{w}_h^n := \Delta w_h^n + \left(\sum_{j=1}^{M} [-1,1]\delta\phi_j, [-1,1]\delta, [-1,1]\delta \right), \\ \tilde{\alpha}_n := \alpha_n + \delta. \end{array} \right. \tag{4.4}$$

The definition of (4.4) is called **δ-extension**. Let $\tilde{U} := \Delta \tilde{w}_h^n + [\tilde{\alpha}_n]$. Let $\Delta \bar{w}_h \subset X_h$ and $\bar{\alpha}_n \in \mathbf{R}^+$ be obtained by the iteration (4.3) from $\tilde{U}$:

$$\left\{ \begin{array}{l} \Delta \bar{w}_h := \tilde{T}_0 \tilde{U}, \\ \bar{\alpha}_n := Ch \sup_{y \in \tilde{U}} \|\tilde{G}(y)\|. \end{array} \right. \tag{4.5}$$

For these sets, the inclusion $\Delta \bar{w}_h \mathring{\subset} \Delta \tilde{w}_h^n$ is defined by $B_j \mathring{\subset} A_j$ $(j = 1, \dots, M+2)$, where $\Delta \tilde{w}_h^n = \left(\sum_{j=1}^{M} A_j \phi_j, A_{M+1}, A_{M+2} \right)$ and $\Delta \bar{w}_h = \left(\sum_{j=1}^{M} B_j \phi_j, B_{M+1}, B_{M+2} \right)$.

To judge whether or not $\tilde{U}$ is what we want, we have the following theorem:

Theorem 4.2. *If we find*

$$\left\{ \begin{array}{l} \Delta \bar{w}_h \mathring{\subset} \Delta \tilde{w}_h^n, \\ \bar{\alpha}_n < \tilde{\alpha}_n, \end{array} \right. \tag{4.6}$$

we conclude that there exists a solution $w \in \tilde{U}$ of the fixed point problem $w = G(w)$.

5. A Numerical Example.

In this section we present an example of numerical verification for the following equation: $J := (0,1)$ and

$$(5.1)\qquad \begin{cases} -u'' = \lambda u + \sin(2\pi x), & \text{in } J, \\ u(0) = u(1) = 0. \end{cases}$$

It is easy to check that $(\pi^2, \sin(2\pi x/(3\pi^2))$ is a simple bifurcation point of (5.1). We set $\phi(x) := \sin(\pi x)$. We consider the extended equation

$$(5.2)\qquad \begin{cases} -u'' = \lambda u + \sin(2\pi x) + \mu\phi, \\ -r'' = \lambda r, \\ -s'' = \lambda s + u, \\ (r,r) - 1 = 0, \qquad (r,s) = 0 \\ u(0) = u(1) = r(0) = r(1) = s(0) = s(1) = 0. \end{cases}$$

By a simple calculation we find that the equation (5.2) has an isolated solution $w_0 := (u_0, r_0, s_0, \lambda_0, 0) \in X_0$, where $u_0 := \sin(2\pi x)/(3\pi^2)$, $r_0 := \sqrt{2}\sin(\pi x)$, $s_0 := \sin(2\pi x)/(9\pi^4)$, and $\lambda_0 := \pi^2$.

Let $N := 100$. We divide J equally into N small intervals. Let S_h the finite element space of piecewise linear funtions.

We try to verify the solution w_0. The following are the result of verification. We show $\tilde{\alpha}_n$ and the constructed set $\tilde{U} = (\sum_{j=1}^{297} A_j\phi_j, A_{298}, A_{299})$, where $A_j := [a_j, b_j]$.

The iteration number $= 5$,

$\tilde{\alpha}_n = 3.147062\text{D–}2$,

$\lambda_h = 9.87042 \in A_{298} = (9.86224, 9.87860)$ and $|A_{298}| = 1.45585\text{D–}2$,

$\mu = 4.80\text{D–}15 \in A_{299} = (-1.03775\text{D–}2, 1.03775\text{D–}2)$ and $|A_{299}| = 2.07550\text{D–}2$,

$\max_{1\le i\le 99} |A_i| = 2.29400\text{D–}2$,

$\max_{100\le i\le 198} |A_i| = 9.01306\text{D–}4$,

$\max_{199\le i\le 297} |A_i| = 9.95609\text{D–}4$,

References

[BRR1] **F. Brezzi, J. Rappaz, and P.A. Raviart**, Finite Dimensional Approximation of Nonlinear Problems, Part I: Branches of Nonsingular Solutions, Numer. Math., 36 (1980), pp.1-25.

[BRR2] **F. Brezzi, J. Rappaz, and P.A. Raviart**, Finite Dimensional Approximation of Nonlinear Problems, Part II: Limit Points, Numer. Math., 37 (1981), pp.1-28.

[BRR3] **F. Brezzi, J. Rappaz, and P.A. Raviart**, Finite Dimensional Approximation of Nonlinear Problems, Part III: Simple Bifurcation Points, Numer. Math., 38 (1981), pp.1-30.

[N1] **M.T. Nakao**, A numerical approach to the proof of existence of solutions for elliptic problems, Japan J. Appl. Math., 5, (1988), 313-332.

[N2] **M.T. Nakao**, A computational verification method of existence of solutions for nonlinear elliptic equations, Lecture Notes in Num. Appl. Anal., 10, (1989), 101-120.

[N3] **M.T. Nakao**, A numerical approach to the proof of existence of solutions for elliptic problems II, Japan J. Appl. Math., 7, (1990), 477-488.

[N4] **M.T. Nakao**, Solving nonlinear parabolic problems with result verificaion, *to appear in J. Comp. Appl. Math., 38 (1991).*

[N5] **M.T. Nakao**, A numrical verification method for the existence of weak solutions for nonlinear BVP, *to appear in J. Math. Anal. Appl.*

[R] **W.C. Rheinboldt**, Numerical Analysis of Parametrized Nonlinear Equations, Wiley, 1986.

[TB] **T. Tsuchiya and I. Babuška**, A prioir error estimates of finite element solutions of parametrized nonlinear equations, *submitted.*

[TN] **T. Tsuchiya and M.T. Nakao**, Numerical verification of solutions of parametrized nonlinear boundary value problems with turning points, *submitted.*

[WN] **Y. Watanabe and M.T. Nakao**, Numerical verifications of solutions for nonlinear elliptic equations, Research Report of Mathematics of Computation, Kyushu University, RMC 66-09, (1991), 15 pages.

[W] **H. Weber** On the numerical approximation of secondary bifurcation problems, in *Lecture Note in Mathematics 878*, Springer, (1981).

[Z] **E. Zeidler**, Nonlinear Functional Analysis and Its Application I, Springer, (1986).