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Several Applications of Interval Mathematics to Electrical Network Analysis

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1. Introduction

This report offers several applications of interval mathematics to linear and nonlinear networks. The section 1 presents one of the applications of interval mathematics to the worst-case tolerance analysis of linear networks. Every parameter of a manufactured component or circuit has a tolerance associated with it. For example, the actual value of 10 percents of tolerance of 1kΩ resistor may be anywhere between 0.9kΩ and 1.1kΩ. Massively produced circuits with same specification, rigorously speaking, are practically different from each other. It is hence appropriate that the parameters can be considered as the interval numbers.

We usually use Monte Carlo method when we try to obtain solutions of equation of the linear circuits with interval parameters. The Monte Carlo method, as is known well, cosumes a lot of time to compute the solutions as the number of the trials increase to have an accurate range of the solutions. One of the methods to cope with this drawback is to use the interval arithmetics.

As the first step this report presents a method for obtaining solutions of interval linear resistive networks. The network equations suitable for interval operation are formulated. In the linear network analysis cutset or tieset equation is usually used. The elements of the coefficient matrix in these equations are given by the linear combination of parameters. This causes the elements of the coefficient matrix to have wide widths when parameters are given by interval numbers. The author proposes to formulate the network equation in a hybrid form. In addition, we use Hansen's preconditioning to carry out Gaussian elimination[1]. To have well-estimated solutions, we propose to formulate the network equations based on maximally distant trees in a graph theory[2,3].

In the section 2 the application of interval mathematics to nonlinear analysis of electric or electronic circuits is presented. The determination of the steady state of electrical or electronic circuits often is replaced to having the real solutions of a system of nonlinear equations with real coefficients. Some examples are the determination of the operating points of a circuit nonlinear elements such as transistors and diodes, and the determination of the amplitude and phase of the subharmonics or harmonics generated in a nonlinear network. The solution of the load flow equation of a electric power system is also one of such examples. Some of the earlier methods to solve these types of equations numerically are iterative method such as Newton method, piecewise linear methods and homotopy methods. Common questions in these methods are as follows;(a) How to set the initial values? (b) Whether or not all the solutions in the specified region can be determined? (c) When no solution exists in the given region, how can its nonexistence be decided?

Powerful method using interval mathematics to answer affirmatively these questions was proposed by R. Krawczyk, R. E. Moore and S. T. Jones (abbreviated as KMJ algorithm)[6,7]. The author proposed to apply KMJ algorithm to electronic circuit analysis and gave some modifications[9]. E. Hansen and S. Sengupta also proposed another operator[8]. We call it Hansen's operator. From the standpoint of application to electrical engineering problems, Krawczyk and Hansen's operator are compared with the aid of numerical experiments.

2. Application to Tolerance Analysis of Linear Network

2.1. Formulation of Interval Linear Network Equation

We consider a linear time-invariant resistive network. We define a branch of the network to be a single element: a resistor or a conductor. The typical branch is the resistor or the conductor which has connected across it an ideal current source and inserted into it an ideal voltage source. The resistances and conductances are given by the interval numbers. The values of the voltage and current sources are the point intervals. The network is connected and is assumed to contain neither loops of voltage sources nor cutset of current sourses. We suppose the network to have b such branches. The first step to formulate the network equation suitable for the interval analysis is to choose the tree. We select the tree so that it contains the branches of as many conductors as possible.

The KCL equation for the fundamental cutset associated with the network becomes

$$\mathbf{I}_t + \mathbf{Q}_l\mathbf{I}_l = \mathbf{0} \tag{1}$$

where $\mathbf{Q} = [\mathbf{1}_t, \mathbf{Q}_l]$ and $\mathbf{I} = [\mathbf{I}_t, \mathbf{I}_l]^T$. The symbol T means the transpose. The matrices $\mathbf{1}_t$ and $\mathbf{Q}_l$ are submatrices representing the cutsets for the conductive tree branches and the resistive links, respectively. The subvectors $\mathbf{I}_t$ and $\mathbf{I}_l$ represent the currents for the conductive tree branches and the resistive links, respectively. The KVL equation for the fundamental tieset becomes

$$\mathbf{B}_t\mathbf{V}_t + \mathbf{V}_l = \mathbf{0} \tag{2}$$

where $\mathbf{B} = [\mathbf{B}_t, \mathbf{1}_l]$ and $\mathbf{V} = [\mathbf{V}_t, \mathbf{V}_l]$. The submatrices $\mathbf{B}_t$ and $\mathbf{1}_l$ represent the tiesets for the conductive trees branches and the resistive links, respectively. The subvectors $\mathbf{V}_t$ and $\mathbf{V}_l$ represent the voltages for the conductive tree branches and resistive links, respectively.

The branch equations are given by

$$\mathbf{I}_t = \mathbf{G}\mathbf{V}_t + \mathbf{j}_t - \mathbf{G}\mathbf{e}_t \tag{3}$$

$$\mathbf{V}_l = \mathbf{R}\mathbf{I}_l + \mathbf{e}_l - \mathbf{R}\mathbf{j}_l \tag{4}$$

where

$$\mathbf{G} = diag(G_1, G_2, ..., G_\rho), \;\; \mathbf{R} = diag(R_{\rho+1}, R_2, ..., R_b).$$

The matrices $\mathbf{G}$ and $\mathbf{R}$ are the branch conductance and resistance matrices, respectively. The number ρ is the rank of the graph associated with the network neglecting the current and voltage sources. The vectors $\mathbf{j}_t$ and $\mathbf{j}_l$ are the current source vectors associated with the tree and the link, respectively. The vectors $\mathbf{e}_t$ and $\mathbf{e}_l$ are the voltage source vectors associated with the tree and the link, respectively.

Substituting Eqs.(3) and (4) into Eqs.(1) and (2) , we have

$$\mathbf{H}\mathbf{x} = \mathbf{a} \tag{5}$$

where

$$\mathbf{H} = \begin{bmatrix} \mathbf{G} & \mathbf{Q}_l \\ \mathbf{B}_t & \mathbf{R} \end{bmatrix}, \mathbf{x} = \begin{bmatrix} \mathbf{V}_t \\ \mathbf{I}_l \end{bmatrix}, \mathbf{a} = \begin{bmatrix} \mathbf{J}_s \\ \mathbf{E}_s \end{bmatrix}, \mathbf{J}_s = -\mathbf{j}_{st} + \mathbf{G}\mathbf{e}_{st}, \mathbf{E}_s = -\mathbf{e}_{sl} + \mathbf{R}\mathbf{j}_{sl}. \tag{6}$$

The vectors $\mathbf{j}_{st}$ and $\mathbf{j}_{sl}$ are the current sources vectors and $\mathbf{e}_{st}$ and $\mathbf{e} - sl$ the voltage source vectors, associated with the tree and the link, respectively. Eq.(5) is called hybrid equation of the linear resistive network.

2.2. Comparison with Cutset or Tieset equation

The linear circuit theory has been developed on the bases of the cutset analysis, tieset analysis or mixed analysis. One of the aspects of linear circuit theory is to reduce the number of the independent variables. Eventually the mixed analysis has been developed[4,5]. However, when the elements of the coefficient matrix are given by the interval numbers, these analyses are not necessarily suitable for the interval mathematics. We shall show this using the cutset equation.

Eliminating the link current vector $\mathbf{I}_l$ in Eq.(5), we have the cutset equation

$$\tilde{\mathbf{G}}\mathbf{V}_t = \tilde{\mathbf{J}}_s \tag{7}$$

where

$$\tilde{\mathbf{G}} = \mathbf{Q}(\mathbf{G}_\rho \oplus \mathbf{G}_\mu)\mathbf{Q}^T, \tilde{\mathbf{J}}_s = \mathbf{J}_s - \mathbf{Q}_l\mathbf{G}_\mu\mathbf{E}_s, \mathbf{G}_\rho = \mathbf{G}, \mathbf{G}_\mu = \mathbf{R}^{-1} \tag{8}$$

where $\oplus$ denotes the direct sum of the matrices.

In similar way, we have the tieset equation(the extention of loop equation)by eliminating $\mathbf{V}_t$ in Eq.(5). The elements of the matrix $\tilde{\mathbf{G}}$ are given by the linear combination of the interval resistances. Hence the sums of interval numbers give us the possibility to have the meaningless combinations of the parameters. For example, let G_1 and G_2 be $G_1 = [1,2]$ and $G_2 = [3,5]$. Then we have $G_1 + G_2 = [4,7]$. Let us choose $g_2 = 3.5 \in G_2$ and $g_1 + g_2 = 4.2 \in G_1 + G_2$. Then we have $g_1 = 0.7 \notin G_1$. Hence we can say that the conventional cutset equation is not pertinent if the parameters are given by the interval numbers.

2.3. Preconditioning by Hansen's Method

We try to obtain the interval solution of Eq.(5) using interval Gaussian algorithm. If an interval hybrid matrix $\mathbf{H}$ is a strictly diagonally dominant, then the Gaussian algorithm can be implemented for the interval matrix $\mathbf{H}$ without row or column interchanges[1]. However, if $\mathbf{H}$ does not hold this condition, Eq.(5) has possibility to be solved by a transformation given by Hansen, which tries to transform the matrix $\mathbf{H}$ into a strictly diagonally dominant interval matrix[1].

Let m be an operator which takes the midpoint of the interval. Then we have the point matrix $m(\mathbf{H})$ whose elements are the mean values of the interval elements. We assume that the inverse $m(\mathbf{H})^{-1}$ exists. Following Hansen, multiplying the real matrix $m(\mathbf{H})^{-1}$ on both sides of Eq.(5) we have

$$\tilde{\mathbf{H}}\mathbf{x} = \tilde{\mathbf{a}} \tag{9}$$

where

$$\tilde{\mathbf{H}} = m(\mathbf{H})^{-1}\mathbf{H},\ \tilde{\mathbf{a}} = \mathbf{m}(\mathbf{H})^{-1}\mathbf{a}. \tag{10}$$

This method is effective if the tolerance of parameters are not too large because the matrix $\tilde{\mathbf{H}}$ has possibility to be strongly diagonally dominant.

2.4. Introducing maximally distant tree

Let us consider again the processes to have the interval solutions of the branch voltages and currents. First we solve Eq.(9) to have the tree branch voltages $\mathbf{V}_t$ and the link currents $\mathbf{I}_l$. Second step is to have the voltages $\mathbf{V}_l$ by implementing the interval computation of Ohm's law of Eq.(4) or KVL of Eq.(2). The tree branch current $\mathbf{I}_t$ is also given by implementing the interval computation of Ohm's law of Eq.(3) or KVL of Eq.(1). In addition to the interval Gaussian algorithm the second step requires one more interval computation which makes the estimation of $\mathbf{V}_l$ and $\mathbf{I}_t$ worse. In order to obtain the well-estimated interval solutions only by using the interval Gaussian algorithm, we formulate and solve the several netwok equations deriving from the different trees. To do so, we introduce the maximally distant trees.

⟨The definition of distance between trees⟩

The distance between a pair of trees in a connected graph is defined as the number of branches contained in one tree but not in the other.

This definition leads us to define a pair of maximally distant tree of a graph as a pair of trees whose distance is maximal within the graph. Let T_1 and T_2 denote the maximally distant trees. Here, we assume that $T_1 \cup T_2$ cover all the branches of the graph. For each tree T_i(i=1,2) the cutset and tieset matrices are denoted as $\mathbf{Q}^{(i)}$ and $\mathbf{B}^{(i)}$. Then the hybrid equation for the maximally distant trees T_i is expressed by

$$\mathbf{H}_i \mathbf{x}_i = \mathbf{a}_i \;\; i = 1,2 \tag{11}$$

where

$$\mathbf{H}_i = \begin{bmatrix} \mathbf{G}^{(i)} & \mathbf{Q}_l^{(i)} \\ \mathbf{B}_t^{(i)} & \mathbf{R}^{(i)} \end{bmatrix}, \mathbf{x}_i = \begin{bmatrix} \mathbf{V}_t^{(i)} \\ \mathbf{I}_l^{(i)} \end{bmatrix}, \mathbf{a}_i = \begin{bmatrix} \mathbf{J}_s^{(i)} \\ \mathbf{E}_s^{(i)} \end{bmatrix} \tag{12}$$

The matrices $\mathbf{G}^{(i)}$ and $\mathbf{R}^{(i)}$ $(i = 1,2)$ are the diagonal matrices the element of which are given by the tree branch conductances and the link resistances corresponding to each tree T_i $(i = 1,2)$. The vectors $\mathbf{J}_s^{(i)}$ and $\mathbf{E}_s^{(i)}$ are the current and voltage source vectors for each tree. We solve Eq.(12) for each i and obtain

$$\mathbf{x}_i = [\mathbf{V}_t^{(i)}, \mathbf{I}_l^{(i)}]^T \;\; i = 1,2. \tag{13}$$

Ohm's law or KCL and KVL give us the branch voltage vectors

$$\mathbf{V}_i = [\mathbf{V}_t^{(i)}, \mathbf{V}_l^{(i)}]^T \;\; i = 1,2 \tag{14}$$

and current vectors

$$\mathbf{I}^{(i)} = [\mathbf{I}_t^{(i)}, \mathbf{I}_l^{(i)}]^T \;\; i = 1,2. \tag{15}$$

The true interval solutions of the network denoted by $\mathbf{V}_{true}$ and $\mathbf{I}_{true}$ hold the relation

$$\mathbf{V}_{true} \subseteq \mathbf{V}^{(1)}, \mathbf{V}^{(2)} \tag{16}$$

$$\mathbf{I}_{true} \subseteq \mathbf{I}^{(1)}, \mathbf{I}^{(2)} \tag{17}$$

where the relation $\subseteq$ denotes the inclusion of two interval vectors elementwise. Hence we have

$$\mathbf{V}_{true} \subseteq \mathbf{V}^{(1)} \cap \mathbf{V}^{(2)} \subseteq \mathbf{V}^{(1)}, \mathbf{V}^{(2)} \tag{18}$$

,

$$\mathbf{I}_{true} \subseteq \mathbf{I}^{(1)} \cap \mathbf{I}^{(2)} \subseteq \mathbf{I}^{(1)}, \mathbf{I}^{(2)} i = 1,2 \tag{19}$$

where the relation $\cap$ denotes the intersection of two interval vectors elementwise. Hence it is reasonable to consider the branch voltage vector $\mathbf{V} = \mathbf{V}^{(1)} \cap \mathbf{V}^{(2)}$ and the branch current vector $\mathbf{I} = \mathbf{I}^{(1)} \cap \mathbf{I}^{(2)}$ as the interval solution of the network equation.

2.5. Numerical Examples

This section demonstrates the effectiveness of a hybrid form in the numerical comparison with a cutset equation. The resistive network we deal with is illustrated in Fig.1. The graph of the network and the tree T_1 are shown in Fig. 2. One of the maximally distant trees from the tree T_1 is also shown in Fig.2. The interval branch conductances $G_i =< g_i, \varepsilon > (i = 1, 2, ..., 9)$ are given by $g_i = 1.0$, $0 \le \varepsilon \le 0.1\,(i = 1, 2, ..., 9)$,where the notation $G_i =< g_i, \varepsilon >$ means the interval with the center g_i and the tolerance ε. The values of the current sources are $J_1 = J_2 = 10$. In the hybrid form for the tree T_1, the resistances $R_i (i = 6, ..., 9)$ are computed by $R_i = G_i^{-1} (i = 6, ..., 9)$. In the similar way, we have the interval resistance for the hybrid form for the tree T_2.

Fig.3 shows the upper and lower bounds of the branch voltages for the torelance ε. The branch voltages obtained by Monte Carlo method, which are shown by the chain lines, are regarded as the nearest interval solution to the true interval solution. The branch voltage solutions of the cutset equation, the upper and lower bounds of which are shown by real lines, are in a big disagreement with those by Monte Carlo method.

Fig.4 illustrates the effectiveness of the hybrid equation scaled by Hansen's method. We can observe that the tree branch voltage solutions of the hybrid equation are in fairly good agreement with the tree branch voltages by Monte Carlo method. However, we can see the disagreement in the link voltages. The solutions by cutset equation are far away from the these solutions at $\varepsilon = 0.1$. Note that the scale of the V-axis is changed. Fig.5 shows the results obtained by taking the intersection of the solutions for maximally distant trees. The link voltages becomes faily well-estimated. Note again that the scale of V-axis is changed. As a whole, we can see the faily good agreement with the results by Monte Carlo method.

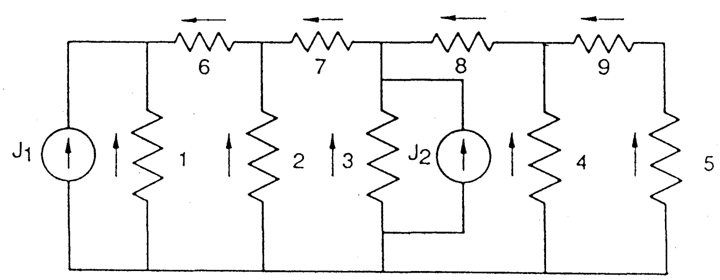

Fig. 1 Resistive ladder network. Arrow shows direction of branch voltage.

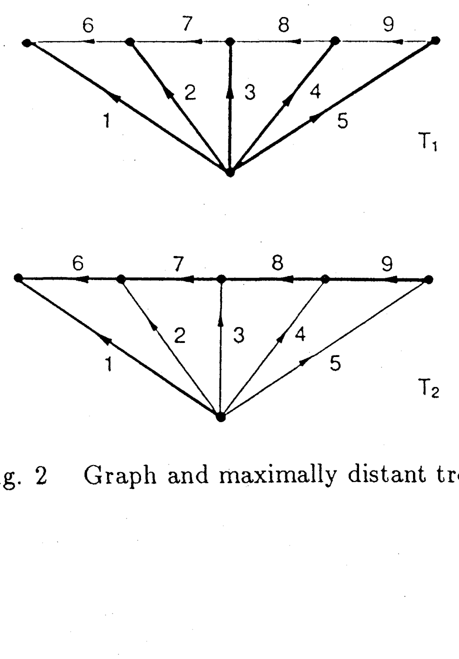

Fig. 2 Graph and maximally distant trees T_1, T_2.

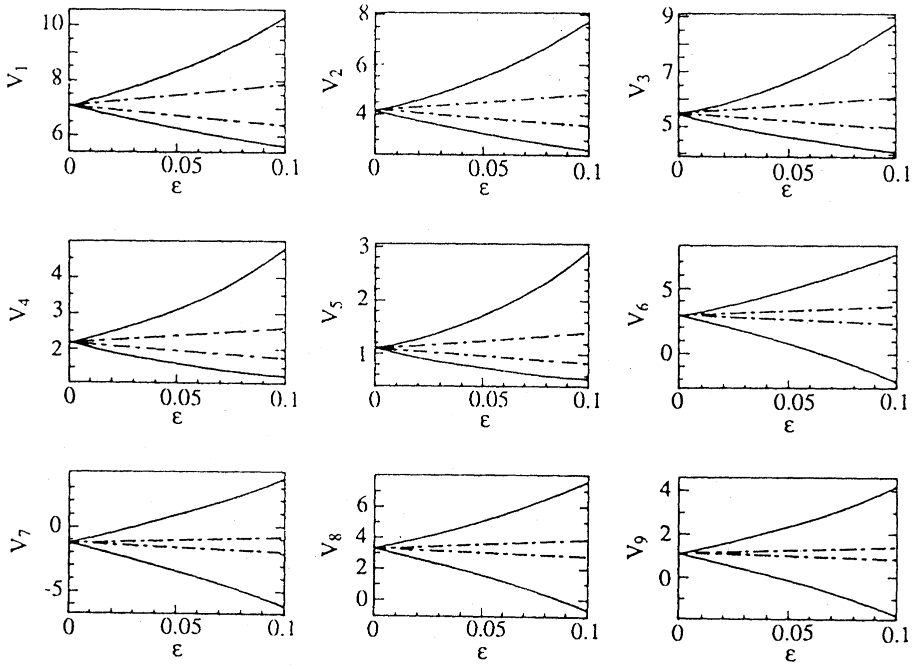

Fig. 3 Branch voltages obtained by Monte Carlo method and cutset equation implemented by interval Gaussian algorithm.

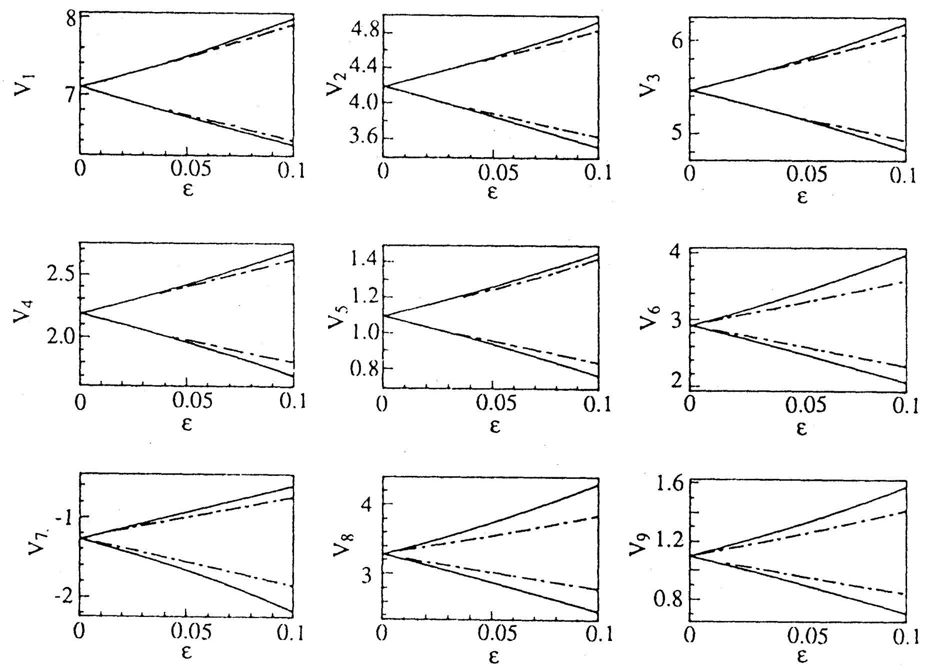

Fig. 4 Branch voltages obtained by hybrid equation modified by Hansen's method.

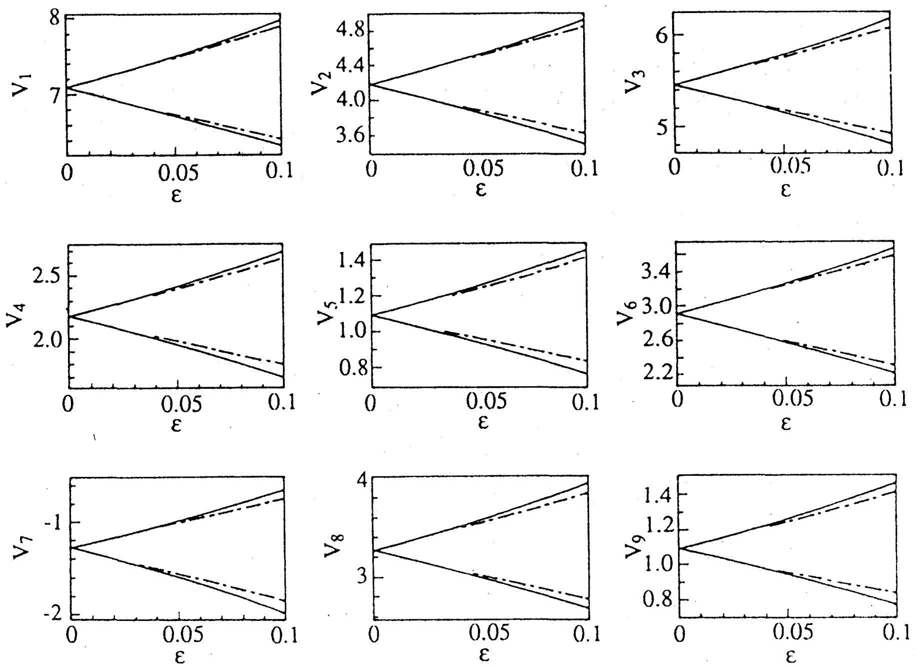

Fig. 5 Branch voltages given by intersection of solutions for maximally distant trees.

3. Application to Nonlinear Network Analysis

The Krawczyk's operator leads to the iteration of Jacobi type and Hansen's operator to Gauss-Seidel type. In this section we try to know which operator is a useful tool for finding the solutions of a nonlinear network equation. Numerical experiments will do this.

3.1. Krawczyk and Hansen's operators

Let the nonlinear network equation be

$$\mathbf{f}(\mathbf{x}) = \mathbf{0} \tag{20}$$

where $\mathbf{f}(\mathbf{x})$ is a vector valued function of dimension n and $\mathbf{x}$ is a real vector of dimension n. Usually the variable $\mathbf{x}$ represents the voltage and/or the current of the elements of the network. In a electric power system the variable $\mathbf{x}$ corresponds to the nodal voltage of the network.

Krawczyk's operator for Eq.(20) is defined by

$$K_i(\mathbf{X}) = y_i - g_i + \sum_{j=1}^{n} R_{ij}(X_j - y_j) \quad i = 1,2,...,n \tag{21}$$

where $\mathbf{g} = (g_i) = \mathbf{Y}\mathbf{f}(\mathbf{y})$, $\mathbf{R} = (R_{ij}) = \mathbf{I} - \mathbf{Y}\mathbf{F}'(\mathbf{X})$. The matrix $\mathbf{I}$ is the $n \times n$ unit matrix. The vector $\mathbf{y}$ and the matrix $\mathbf{Y}$ are defined by $\mathbf{y} = m(\mathbf{X})$ and $\mathbf{Y} = [m(\mathbf{F}'(\mathbf{X}))]^{-1}$.

On the other hand, Hansen's successive operator is defined by

$$H_i(\mathbf{X}) = y_i - g_i + \sum_{j=1}^{i-1} R_{ij}(H_j' - y_j) + \sum_{j=i}^{n} R_{ij}(X_j - y_j) \quad i = 1,2,...,n \tag{22}$$

where $H_j' = H_j \cap X_j$, $j = 1,2,...,n$. It is proved that Hansen's operator lies in Krawczyk's operator and $H(\mathbf{X})$ is usually smaller than $K(\mathbf{X})$. Hence $H(\mathbf{X})$ is more powerful for interval test. In practice, we will compare both operators by applying nonlinear equation derived from electric power networks.

3.2. Numerical Experiments

⟨*Example 1: Load Flow Equation of Power System*⟩

In recent years, the loads of a power system are getting heavier. Hence in the allowable region of the voltages of the buses we tend to have multiple load flow solutions close to nominal load flow solution. These multiple solutions seem to be deeply concerned with the voltage stability of the power system. Therefore, at first multiple load flow solutions are to be found out.

The load flow equation of $n+1$ node system is represented by

$$e_i \sum_{m=1}^{n+1} (G_{im} e_m - B_{2m} f_m) + f_i \sum_{m=1}^{n+1} (B_{im} e_m + G_{im} f_m) = P_i \tag{23a}$$

$$f_i \sum_{m=1}^{n+1} (G_{im} e_m - B_{2m} f_m) - e_i \sum_{m=1}^{n+1} (B_{im} e_m + G_{im} f_m) = Q_i \tag{23b}$$

$$i = 2, 3, ..., n+1.$$

This is a set of quadratic algebraic equations with $2n$ variables of e_i and f_i $(i = 2, 3, ..., n+1)$. The variables e_i and f_i are the real and imaginary part of complex voltage at node i. The parameters G_{im} and B_{im} are conductance and susceptance between node i and m, respectively. P_i and Q_i are effective and reactive power at node i, respectively.

We deal with 5 node system as shown in Fig.6. The dependence of the initial region is compared. The initial region of $f_k = [-0.25, -0.1]\, k = 2, ..., 5$ is fixed. The initial region of e_k is given by $e_k = [-s, 1.05]\, s = -0.1i$, $i = 0, 1, ..., 10$, $k = 2, ..., 5$. We inspect the number of partitionings of the initial regions and CPU time to be taken to compute all the solutions in the initial regions. The results in Fig.7 are shown by the ratio of Hansen's and Krawczyk's operator. For $-1.0 \leq s \leq -0.3$ there exists only one solution. In this initial region Hansen's operator is a little bit more effective than Krawczyk's operator. However, we can see that in the initial region for s=-0.2 Hansen'soperator takes more CPU time although the number of bisections of Hansen's operator is smaller than Krawczyk's operator. This is because another solution exists very close to the solution found first out, so that partitioning goes on until it is discriminated. For s=-0.1 we find two solutions and for s=0 three solutions. Except for s=-0.5, the number of partitionings of Hansen's operator is smaller than Krawczyk's operator. However, CPU time is not necessarily shorter because Hansen's operator takes a little more complicated process of computing.

⟨*Example 2: Equation with Many Variables*⟩

The number of bisections of the initial region is very important factor to compare both operators. We inspect dependence of the number of partitionings on the number of variables. We consider the algebraic equation

$$f_k(x) = (\sum_{i=1}^{n} x_i^3 + k)/(2n) - x_k = 0 \quad k = 1, 2, ..., n. \tag{24}$$

The initial region is fixed to $X_i = [-1, 1] (i = 1, 2, ..., n)$. The relation between number of variables and the number of partitionings are shown in Fig.8. As the number of variables increases, the number of partitionings in Hansen's operator decreases relatively compared with Krawczyk's operator. Hansen's operator seems to be more effective if the number of variables is large. However, as far as the CPU time is concerned, this is not necessarily hold as demonstrated in example 1.

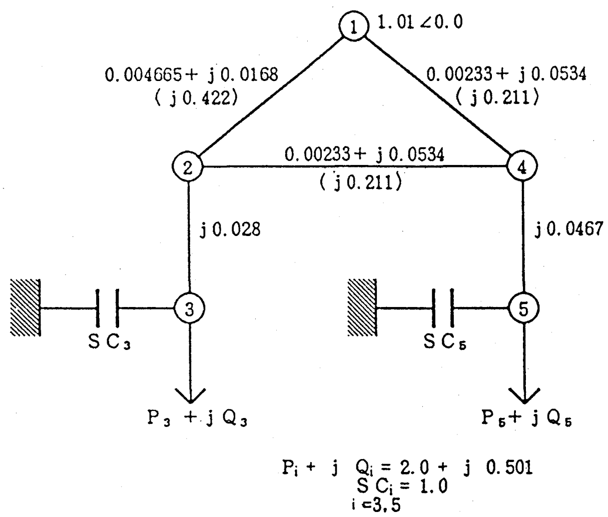

Fig. 6 Power system with 5 nodes.

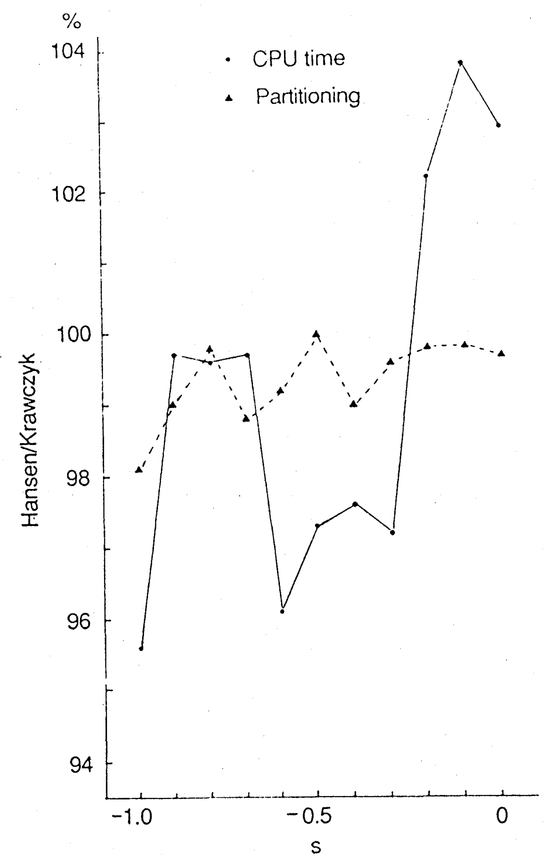

Fig. 7 Comparison of Krawczyk and Hansen's operator based on width of initial regions.

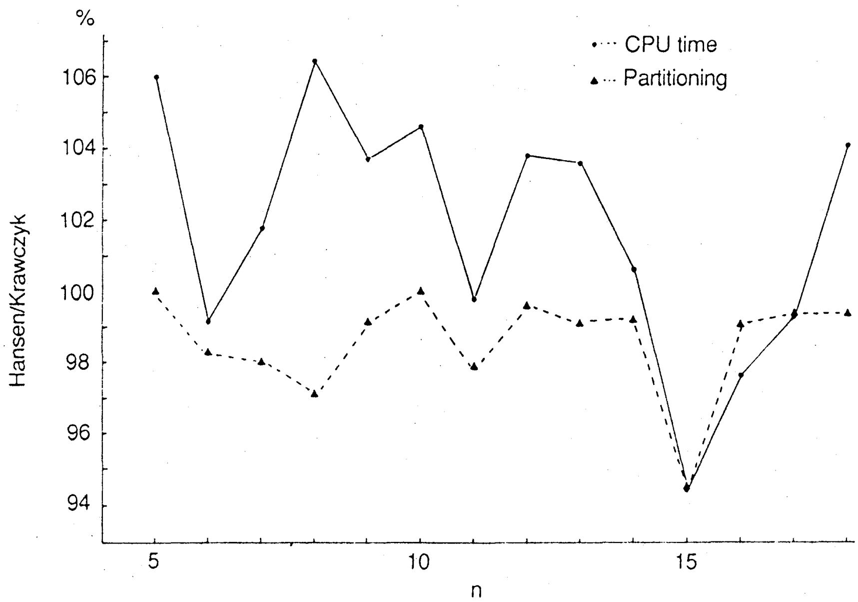

Fig. 8 Comparison of Krawczyk and Hansen's operator based on number of variables.

4. Conclusion

We have proposed to formulate the hybrid equation for the linear network with the interval resistive elements. The Monte Carlo method is time-consuming and the usual cutset or tieset equation(node or loop equation) is seen not to be suitable for such network. The implementation of the hybrid equation by interval Gaussian algorithm becomes easy by Hansen's method. However, the limitation of interval Gaussian algorithm alway exists. Further, we proposed to take the intersection of the two solutions of the hybrid equations based on two maximally distant trees if we require to have a good evaluation of the link voltages as well as the tree branch voltages.

Furthermore, in nonlinear equations Krawczyk and Hansen's operator are compared by some numerical experiments. The validity of Hansen's operator seems to be clarified if we apply it to nonlinear network equation with more number of variables than we have tried.

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