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On the Confluence of Weakly Normalizing TRSs *
(Note)

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abstract

This note investigates the confluence(CONF) and unique normal form property w.r.t. reduction($UN^{\rightarrow}$) of weakly normalizing(WN) TRSs. The main devices are the transformation of WN TRSs into a kind of membership conditional TRSs called normalized MCTRSs, and the observation of critical pairs in them. By these two, it becomes possible to determine $UN^{\rightarrow}$ of WN TRSs, and it seems promising to extend them for detections of CONF.

1. Introduction.

TRSs are ubique scheme as formal models of functional programming languages, automated theorem proving, and program synthesis/transformation/verification. The research on TRSs focuses to show their two important properties, namely, confluence(CONF) and normalizability. As for the latter property, there exist two subproperties, i.e., strongly and weakly normalizing(SN and WN). SN means every reduction sequence is not infinite and WN does every term has at least one finite sequence from it. Oftenly, it is difficult to show TRSs to be SN and it happens that TRSs are only WN. Thus, we aim to propose a method to detect CONF for WN TRSs.

The following TRS defines factorial f on the set of natural numbers $\mathcal{N} = \{0, s0, s^2 0, \cdots\}$.

$$R : \begin{cases} f0 \rhd 1 \ (\equiv s0) & \text{0 is a constant, } s \text{ the } \textit{successor}, \\ fx \rhd x * fpx & p \text{ the } \textit{predecessor}, \text{ and} \\ psx \rhd x & * \text{ the } \textit{multiplication} \text{ denoted} \\ spx \rhd x & \text{by infix notation with right associativity.} \end{cases}$$

This R is WN but non-SN, as is illustrated by the next two reduction sequences:

$$fs^2x \rightarrow s^2x * fps^2x \rightarrow s^2x * ps^2x * fp^2s^2x \rightarrow ps^2x * p^2s^2x * fp^3s^2x \rightarrow \cdots \quad (\infty),$$

$$fs^2x \rightarrow s^2x * fps^2x \rightarrow_{in} s^2x * fsx \rightarrow s^2x * sx * fpsx \rightarrow_{in} s^2x * sx * fx \quad (\textit{finite})$$

where $\rightarrow_{in}$'s are *innermost* reductions.

The author came to an idea to approximate WN TRSs by a kind of MCTRSs[6]. But soon [1] appeared and executed research in this direction to a full extent based the one for *typed λ-calculi*[3]. Though [6] objected Knuth-Bendix like completion of first order TRSs, and this note is in its direction. Another motivation of this note is to make *non-overlapping* in [4] more precise. Because it is mentioned only as "... *similar to unconditional case* ..."

* This note is based on the presentation at RIMS, Kyoto University on 1st February 1993.

2. Preliminaries.

First of all, some necessary notions are defined. For general notions and facts, please refer to some literature, e.g., [2].

Definition. A TRS R is

confluent(CONF), iff $\forall t, s[(t \overset{*}{\longleftrightarrow} s \wedge t, s \in \mathrm{NF}) \Longrightarrow t \equiv s]$,

uniquely normalizing(UN), iff $\forall t, s[(t \overset{*}{\longleftrightarrow} s \wedge t, s \in \mathrm{NF}) \Longrightarrow t \equiv s]$, and,

uniquely normalizing w.r.t. reduction($\mathrm{UN}^{\rightarrow}$), iff $\forall t, s_1, s_2[(t \to^* s_j \wedge s_1, s_2 \in \mathrm{NF}) \Longrightarrow t \equiv s]$.

where $\overset{*}{\longleftrightarrow}$ is the transitive, reflexive and symmetric closure of $\to$. Now, the relationship between these three properties is noticed:

$$\mathrm{CONF} \subset \mathrm{UN} \subset \mathrm{UN}^{\rightarrow}.$$

For the simplicity, the next additional assumption will be added:

Assumption. $\mathrm{WN} \wedge \forall t, s \in T[(t \longrightarrow^* s \wedge s \in \mathrm{NF}) \Longrightarrow t \longrightarrow^*_{in} s]$. $(*)$

The membership conditional TRS (MCTRS) is a kind of TRS, whose rules are attached with the membership conditions on their variables. In a MCTRS, rules are applied only when membership conditions on their variables hold. The membership conditions are assumed to be decidable apart from the reduction relations to prevent from their harmful interferences. More details can be found in [4] and [7].

The notion of normalized MCTRS was originally introduced in [4] to prove that non-*left-linear*(LL) and non-*overlapping*(OVLP) MCTRSs are CONF. The normalized MCTRS $R_{nf} = \langle T, \to_{nf}\rangle$ for a TRS $R = \langle T, \to\rangle$ can be obtained by posing the membership condition to the set of normal forms(NF) on the every variable in its rules, as is exemplified below:

$$R : \begin{cases} f0 \rhd 1 \\ fx \rhd x * fpx \\ psx \rhd x \\ spx \rhd x \end{cases} \Longrightarrow \quad R_{nf} : \begin{cases} f0 \rhd 1 & \\ fx \rhd x * fpx & : \quad x \in \mathrm{NF} \\ psx \rhd x & : \quad x \in \mathrm{NF} \\ spx \rhd x & : \quad x \in \mathrm{NF} \end{cases}$$

A fact on CONF of MCTRSs is quoted, and it will be utilized in the next section.

Fact. (Toyama[5]) For MCTRSs, quasi-LL $\wedge$ non-OVLP $\Longrightarrow$ CONF.

A MCTRS is called *quasi-LL* iff every non-linear variable in LHS of every rule is restricted by some non-trivial membership condition. Thus every normalized MCTRS is quasi-LL.

3. CONF and $\mathrm{UN}^{\rightarrow}$ of Weakly Normalizing TRSs.

Preliminarily, the correspondence between TRSs and their normalized MCTRSs is stated. The former part of the next lemma is clear from the definition of $\to_{nf}$, and immediately the latter follows.

Lemma. $\to \supset \to_{nf}$ and $\mathrm{NF}(\to) \subset \mathrm{NF}(\to_{nf})$.

From the assumption $(*)$ and this lemma:

Lemma. Let R be a WN TRS satisfying $(*)$. Then

$$t_1 \to t_2 \to \cdots \to t_k \to \tilde{t} \in \mathrm{NF}(\to) \quad \Longrightarrow \quad t_1 \to_{nf} t'_2 \to_{nf} \cdots \to_{nf} t'_l \to_{nf} \tilde{t} \in \mathrm{NF}(\to_{nf}).$$

After these preparations, central device of this note and main result are presented.

Definition. Let $l_j \triangleright r_j$ for $j = 1, 2$ be rules overlapping at $u \in \mathcal{O}(l_1)$, $\langle P, Q\rangle$ the CP at u with the m.g.u. $\theta = \{x_j/t_j\}$. $\langle P\sigma, Q\sigma\rangle : x'_1 \in \mathrm{NF}, \cdots, x'_n \in \mathrm{NF}$ is a *normalized CP* (at u) where σ is a most general substitution such that $x'_1 \in \mathrm{NF} \wedge \cdots \wedge x'_n \in \mathrm{NF}$ implies $\bigwedge_i x_i\sigma \in \mathrm{NF} \wedge \bigwedge_j t_j\sigma \in \mathrm{NF}$, where $\{x'_1, \cdots, x'_n\} = Var(Im(\sigma))$.

It can be easily understood that a classical CP may have multiple σ's and normalized CPs. Therefore, all the normalized CPs of R_{nf} is all the ones derived from any CP of original TRS R. As the condition $\bigwedge_i x_i \in \mathrm{NF} \wedge \bigwedge_j t_j \in \mathrm{NF}$ cannot appear in normalized MCTRSs, CPs with such conditions must be interpreted as equivalent sets of normalized CPs defined here.

An example of generation of normalized CPs is demonstrated for the normalized MCTRS below:

$$R_{nf} : \begin{cases} fx \triangleright fgx & : \ x \in \mathrm{NF} \\ fgx \triangleright gx & : \ x \in \mathrm{NF} \\ gfx \triangleright gx & : \ x \in \mathrm{NF} \\ g^2x \triangleright gx & : \ x \in \mathrm{NF} \end{cases}$$

on the set of terms $T = (\{f^1, g^1, h^1\}, V)$ where the superfixes of function symbols denote their arities and V does the set of variables.

The first two rules of R_{nf} overlap at root and a conventional CP $\langle gx, fg^2x\rangle$ is generated from fgx. To have *normalized CP*, σ's must be found under $x \in \mathrm{NF} \wedge gx \in \mathrm{NF}$ and the LHSs of rules in R_{nf} are obeserved by a method similar to covering set induction. For this case, $\{x/hz\}$ with $x \in \mathrm{NF}$ is only possible:

$$\langle ghz,\ fg^2hz\rangle \ : \ z \in \mathrm{NF}$$

Theorem. Let R be a WN TRS satisfying the condition $(*)$, and R_{nf} the normalized MCTRS of R.

$$R_{nf} \text{ has no normalized CP } \Longrightarrow R_{nf} \text{ is CONF and } R \text{ is } \mathrm{UN}^{\rightarrow}.$$

Proof. R_{nf} is non-OVLP from the absence of normalized CP, and obviously R_{nf} is always quasi-LL. Then by the fact by Toyama, R_{nf} is CONF. Let $t \rightarrow^* t_j \in \mathrm{NF}(\rightarrow)$ for $j = 1, 2$. These t_j's exist by WN of R. By lemmas and $(*)$, there are reduction sequences $t \rightarrow^*_{nf} t_j \in \mathrm{NF}(\rightarrow_{nf})$ for $j = 1, 2$ and $t_1 \equiv t_2$ by $\mathrm{UN}^{\rightarrow_{nf}}$. Thus R is $\mathrm{UN}^{\rightarrow}$.

If the TRS under discussion fulfills some stronger demands on it as $(**)$ in the remark below, then we have:

Corollary. If every normalized CP converges in R_{nf}, then R is CONF.

Once this corollary is obtained, Knuth-Bendix like completion becomes possible by adding normalized CPs to original systems as [7]:

$$\begin{array}{ccc} R : \mathrm{WN} + \text{etc.} & ==\Rightarrow & R' : \text{obtained by dropping } \in \mathrm{NF}\text{'s} \\ \Downarrow & & \Uparrow \\ R_{nf} : \text{the normalized MCTRS of } R & \Longrightarrow & R'_{nf} : \text{completed} \end{array}$$

but it is still open which of properties of original R are preserved for R'.

Remark.

Even if R_{nf} is known to be CONF, much stronger assumption seems to be necessitated for R to be CONF, for example:

$$\forall t, s \in T[(t \rightarrow t_1 \rightarrow t_2 \rightarrow \cdots \rightarrow t_k \rightarrow s \wedge s \in \mathrm{NF}) \Longrightarrow t \rightarrow_{in} t'_1 \rightarrow_{in} t'_2 \rightarrow_{in} \cdots \rightarrow_{in} t'_l \rightarrow_{in} s] \qquad (**)$$

where $\{t_1, t_2, \cdots, t_k\} \subset \{t'_1, t'_2, \cdots, t'_l\}$.

Furthermore R_{nf} is only WN and not necessarily SN, because not all the innermost reduction sequences in R are not guaranteed to terminate.

Discussion.

Here, the result above is compared with the preceding results in [1] and [5]. In this note, the following (1)-(4) hold for the relationship between $\rightarrow$ and $\rightarrow_{nf}$:

(1) $\rightarrow$ is WN and $(**)$ for CONF of $\rightarrow$ ($(*)$ for UN$^{\rightarrow}$),

(2) $\rightarrow_{nf}$ is CONF,

(3) $\stackrel{*}{\longleftrightarrow} \supset \rightarrow_{nf}$, and

(4) NF$(\rightarrow) \subset$ NF$(\rightarrow_{nf})$.

On the other hand, (3′) $\stackrel{*}{\longleftrightarrow} \subset \stackrel{*}{\longleftrightarrow}_{nf}$ in both [1] and [5], and (4′) NF$(\rightarrow) =$ NF$(\rightarrow_{nf})$ in [5]** for CONF of $\rightarrow$.

Thus, our result is not included in either of [1] and [5], but it assumes a much stronger condition $(**)$. It will be desirable to relax $(**)$ and to find some result which is applicable for more wider cases.

References.

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[5] Y. Toyama : How to Prove Equivalence of Term Rewriting Systems without Induction, Theoretical Computer Science 90 (1991), pp.369-390

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[7] J. Yamada : Confluence of Terminating Membership Conditional TRSs, Springer LNCS 656 (1993), pp.378-392.

** [5] has proven more stronger results on equivalence relations, and corollary 3.2 deduces CONF in a similar setup.