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THREE TYPES OF MINIMAX THEOREMS FOR VECTOR-VALUED FUNCTIONS*

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Abstract. Given a real-valued function, it is well-known that the function possesses a saddle point if and only if the minimax value and the maximin value of the function are coincident.

If the function is a vector-valued function, how does the situation change? That is, if we give reasonable definitions for minimax and maximin of a vector-valued function, what minimax equation or inequation holds? Also, if we give a suitable definition for saddle points of the vector-valued function, under what conditions do there exist such saddle points? Moreover, what relationship holds among such minimax values and maximin values and saddle values?

In this paper, we will give interesting answers to such open questions and will show three types of minimax theorems for vector-valued functions.

Key Words. Minimax theorems, multicriteria games, multiobjective programming, multiple criteria decision making, vector optimization, Browder's coincidence theorem, ordered vector spaces, pointed convex cones.

1. Introduction and Preliminaries

In recent years, vector optimization theory has been widely developed, and the study of multicriteria games has accordingly come up again by many researchers; e.g., see [5] and [10]. The particular questions are minimax problems for vector-valued functions.

Recently, we can find some papers which give interesting answers to such open questions; see [3], [7], [8], [9], [21], [25], [26], [27], [29], and [30]. The research, in particular, of [21] formed an occasion for studies in this area to make development. However, the research is limited to a separated function of the type $f(x, y) = x+y$. A more general approach is done in [3], [8], and [9]. The research of [3] discusses minimax inequalities for a vector-valued function which maps in an ordered Banach space. On the other hand, the researches of [8] and [9] are given in more general setting, i.e., the image space of functions is a (ordered) locally convex Hausdorff topological vector space. Also, we have separately researched such minimax problems in general setting and proved minimax theorems, existence theorems for saddle points, and saddle point theorems in [25], [26], [27], [29], and [30]; all of such researches are contained in the author's doctoral thesis [32].

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The research of [25] generalizes the results given in [21] to more general cases, where a vector-valued function f is allowed to be of the type $f(x,y) = u(x) + v(y)$ and where existence theorems for saddle points of a vector-valued function are proved. The image space of functions in the research, however, is limited to a finite-dimensional space $\boldsymbol{R}^n$. The researches of [26], [27], and [29] are done either in real Hausdorff topological vector spaces or in real Hausdorff locally convex topological vector spaces, which are allowed to be infinite-dimensional. The research of [29] gives two minimax theorems for a vector-valued function and generalizes the results given in [26] and [27] to more general cases, where a vector-valued function is allowed to be one satisfying any of four convex-concave conditions. Also, in [26] and [27] we proved various existence theorems for saddle points of a vector-valued function. These minimax results come to the recent paper [30], which generalizes the results given in [27] and [29].

The aim of this paper is to present some of the most general versions of the author's results related to minimax problems for vector-valued functions. For this end we shall introduce new concepts of convexity and continuity of vector-valued functions, which are weaker than ordinary ones. One is the generalized quasiconvexity called "natural quasi C-convex"; the other is the continuity called "demicontinuous". Almost all the results in this paper are based on [31] and [32]. Only **Minimax Theorem II** is new and one of most general cases in the author's results.

Now, we give the preliminary terminology used throughout the paper. To begin with, the main spaces with mathematical structures on which our results work are a real topological vector space (t.v.s. for short) or a real locally convex space (l.c.s. for short) as a domain of functions and an ordered real topological vector space (ordered t.v.s. for short) as a range space of functions. We assume that the topologies are Hausdorff; one of the reasons why we work on a Hausdorff l.c.s. is the purpose of applying Browder's coincidence theorem; see [2] and [23]. (The coincidence theorem is a cyclical version of Fan-Glicksberg type's fixed-point theorems; see [6] and [11].)

If C is a convex cone of a real vector space S, the relation $\leq_C$ defined below is a (partial) vector ordering of S: for $x, y \in S$

$$x \leq_C y \Longleftrightarrow y - x \in C. \tag{1}$$

Conversely, let S be a real ordered vector space with a vector ordering $\leq$, and let $C := \{x \in S \mid 0 \leq x\}$. Then C is a convex cone of S, and its ordering $\leq_C$ is coincident with $\leq$; see page 2 in [17]. Thus, there is a one-to-one correspondence between vector orderings of a real ordered vector space S and convex cones in S, and hence we assume that such real ordered vector space [resp. ordered t.v.s.] has a convex cone C and that the ordering is defined by (1).

Throughout this paper, let Z be an ordered t.v.s. with a convex cone C. The convex cone C is assumed to be ***pointed***, i.e., $C \cap (-C) = \{0\}$, and hence the ordering is antisymmetric and $C \ni 0$. Also, the convex cone C is assumed to be ***solid***, i.e., its (topological) interior $\mathbf{int}C$ is nonempty, and hence $C^0 := (\mathbf{int}C) \cup \{0\}$ is a pointed convex cone and induces another (antisymmetric) vector ordering $\leq_{C^0}$ weaker than $\leq_C$ in Z. With respect to each of the orderings $\leq_C$ and $\leq_{C^0}$, we shall define minimal elements and maximal elements of a subset A of Z. As the concept, we will adopt "***cone extreme point***," the concept of which was proposed by P.L.Yu in [33].

An element z_0 of a subset A of Z is said to be a ***C-minimal point*** of A if $\{z \in A \,|\, z \leq_C z_0, z \neq z_0\} = \emptyset$, and a ***$C$-maximal point*** of A if $\{z \in A \,|\, z_0 \leq_C z, z \neq z_0\} = \emptyset$; which are equivalent to $A \cap (z_0 - C) = \{z_0\}$ and $A \cap (z_0 + C) = \{z_0\}$, respectively. We denote the set of such all C-minimal [resp. C-maximal] points of A by $\mathbf{Min}A$ [resp. $\mathbf{Max}A$]. Also, C^0-minimal and C^0-maximal points of A are defined similarly, and denoted by $\mathbf{Min_w}A$ and $\mathbf{Max_w}A$, respectively. These C^0-minimality and C^0-maximality are weaker concepts than C-minimality and C-maximality, respectively: it should be remarked that $\mathbf{Min}A \subset \mathbf{Min_w}A \subset A$ and $\mathbf{Max}A \subset \mathbf{Max_w}A \subset A$.

2. Minimax and Maximin of a Vector-Valued Function

Let A be a nonempty subset of an ordered t.v.s. Z with a (solid) pointed convex cone C. We say that the set A has the "***domination property***" (e.g., see page 697 in [19] and page 53 in [20]) if

$$\mathbf{Min}A \neq \emptyset \quad [\text{resp. } \mathbf{Max}A \neq \emptyset], \tag{2}$$

and

$$A \subset \mathbf{Min}A + C \quad [\text{resp. } A \subset \mathbf{Max}A - C]. \tag{3}$$

In particular, to produce conditions ensuring the condition (2) is one of the most important questions of vector optimization theory; e.g., see [12], [14], [18], [19], and [24]. In this paper, we need the following lemmas; see [30] and [32]:

Lemma 2.1. Let Z be an ordered t.v.s. with a (solid) pointed convex cone C, and A a subset of Z. If the convex cone C of Z satisfies the condition

$$\mathbf{cl}C + (C \setminus \{0\}) \subset C \tag{4}$$

and if A is nonempty and compact, then $\mathbf{Min}A \neq \emptyset$, $A \subset \mathbf{Min}A + C$ and $\mathbf{Max}A \neq \emptyset$, $A \subset \mathbf{Max}A - C$.

Lemma 2.2. Let Z be an ordered t.v.s. with a solid pointed convex cone C, and A a subset of Z. If A is nonempty and compact, then $\mathbf{Min_w}A \neq \emptyset$, $A \subset \mathbf{Min_w}A + C^0$ and $\mathbf{Max_w}A \neq \emptyset$, $A \subset \mathbf{Max_w}A - C^0$. Moreover, $\mathbf{Min_w}A$ and $\mathbf{Max_w}A$ are compact sets.

Next, we will give reasonable definitions for minimax values and maximin values of a vector-valued function. Let $f : X \times Y \to Z$ be a vector-valued function on a product $X \times Y$. We call the following subsets of Z

$$\mathbf{Min} \bigcup_{x \in X} \mathbf{Max_w} f(x, Y), \qquad \mathbf{Max} \bigcup_{y \in Y} \mathbf{Min_w} f(X, y) \tag{5}$$

the set of all ***minimax values*** of f and the set of all ***maximin values*** of f, respectively.

Let S_1 and S_2 be two topological spaces, respectively. A mapping F from S_1 into S_2 is said to be ***upper semicontinuous*** at $x \in S_1$, if for any open neighborhood V of $F(x)$, there exists a neighborhood U of x such that $F(y) \subset V$ for all $y \in U$. We say that F is upper semicontinuous (u.s.c. for short) if it is so at every $x \in S_1$; see Definition 1 in page 41 of [1]. If S is a compact set in S_1 and F is an u.s.c. compact-valued mapping from S into S_2, then the image $F(S)$ under F of S is compact; see Proposition 3 in page 42 of [1]. Based on this fact, we have the following theorem:

Theorem 2.1. Let X and Y be nonempty compact sets in two topological spaces, respectively, and Z an ordered t.v.s. with a solid pointed convex cone C. If a vector-valued function $f : X \times Y \to Z$ is continuous, and if C satisfies the condition (4), then

$$\left[\mathbf{Min} \bigcup_{x \in X} \mathbf{Max_w} f(x, Y)\right] + C \supset \mathbf{Max_w} f(x', Y) \neq \emptyset, \tag{6}$$

$$\left[\mathbf{Max} \bigcup_{y \in Y} \mathbf{Min_w} f(X, y)\right] - C \supset \mathbf{Min_w} f(X, y') \neq \emptyset \tag{7}$$

for each $x' \in X$ and $y' \in Y$.

Proof. From Lemma 2.2, it follows that $\mathbf{Max_w} f(x', Y) \neq \emptyset$ and $\mathbf{Min_w} f(X, y') \neq \emptyset$ for each $x' \in X$ and $y' \in Y$. Also, they are compact sets by Lemma 2.2, and hence the set-valued mappings $x \longmapsto \mathbf{Max_w} f(x, Y)$ and $y \longmapsto \mathbf{Min_w} f(X, y)$ are compact-valued.

Next, to prove the upper semicontinuity of the mappings, we consider their graphs. In the same way as in the proof of Theorem 2.1 in [21], we can verify that the graphs are closed in $X \times f(X, Y)$ and $Y \times f(X, Y)$, respectively. Since both the mappings map into the compact set $f(X, Y)$, they are u.s.c. by Corollary 1 in page 42 of [1]. Hence their images are compact. Thus, the results (6) and (7) follow from Lemma 2.1. ∎

This theorem yields a saddle point theorem located in Section 4 of a vector-valued function as its corollary.

3. Existence of Generalized Saddle Points

Under the previous notation given in Section 1, we will give reasonable definitions for saddle point and saddle value of a vector-valued function. The origin of such generalization for saddle point goes to [21] and [22]. Let Z be an ordered t.v.s. with a solid pointed convex cone C and $f : X \times Y \to Z$ a vector-valued function, respectively.

Definition 3.1. (i) A point (x_0, y_0) is said to be a C***-saddle point*** of f with respect to $X \times Y$, if $f(x_0, y_0) \in \mathbf{Max} f(x_0, Y) \cap \mathbf{Min} f(X, y_0)$;

(ii) A point (x_0, y_0) is said to be a ***weak*** C***-saddle point*** of f with respect to $X \times Y$, if $f(x_0, y_0) \in \mathbf{Max_w} f(x_0, Y) \cap \mathbf{Min_w} f(X, y_0)$.

For the convenience, we denote the set of all C-saddle points [resp. weak C-saddle points] of f by $\mathrm{SP}(f)$ [resp. $\mathrm{SP_w}(f)$] and the set of all C-saddle values [resp. weak C-saddle values] of f by $\mathrm{SV}(f)$ [resp. $\mathrm{SV_w}(f)$].

We note that any C-saddle point of f is a weak C-saddle point of f obviously. Also, in the case $C^0 = C$, the two concepts are coincident.

Now, we give the definition of C-semicontinuous; see Definition 2.4 in [4].

Definition 3.2. Let X be a topological space and Z an ordered t.v.s. with a pointed convex cone C. A vector-valued function $f : X \to Z$ is said to be C**-semicontinuous** if $f^{-1}(z - \mathbf{cl}C)$ is closed in X for each $z \in Z$.

First of all, we state the first existence theorem for generalized saddle points.

Theorem 3.1. (See Theorem 3.2.1 in [32].) Let X and Y be nonempty compact sets in two topological spaces, respectively, and Z an ordered t.v.s. with a solid pointed convex cone C. If a vector-valued function $f : X \times Y \to Z$ is of the type

$$f(x, y) = u(x) + v(y)$$

where u and v are C-semicontinuous, then f has at least one weak C-saddle point. Moreover, if C satisfies the condition (4), then f has at least one C-saddle point.

Second, to present the second existence theorem for generalized saddle points, we will introduce the notion of "semi-saddle point" for a pair of functionals, which is also known as "Nash equilibrium point" for a two-person nonzero-sum game in game theory.

Definition 3.3. Let two (real-valued) functionals g_1 and g_2 be defined on $X \times Y$.

(i) A point (x_0, y_0) is said to be a ***semi-saddle point*** of (g_1, g_2) with respect to $X \times Y$, if $g_1(x_0, y_0) \leq g_1(x, y_0)$ for any $x \in X$ and $g_2(x_0, y_0) \geq g_2(x_0, y)$ for any $y \in Y$;

(ii) A point (x_0, y_0) is said to be a ***strict semi-saddle point*** of (g_1, g_2) with respect to $X \times Y$, if $g_1(x_0, y_0) < g_1(x, y_0)$ for any $x \in X$, $x \neq x_0$ and $g_2(x_0, y_0) > g_2(x_0, y)$ for any $y \in Y$, $y \neq y_0$.

If $g_1 = g_2$, then a semi- [resp. strict semi-] saddle point (x_0, y_0) of (g_1, g_2) is an ordinary saddle [resp. strict saddle] point of g_1.

Now, we formulate the relationship between C-saddle points [resp. weak C-saddle points] and strict semi-saddle points [resp. semi-saddle points]. To this end we review Jahn's definition (cf. Definition 2.1 in [16]).

Definition 3.4. Let A be a nonempty subset of Z, and z_0 a vector of A.

(i) A functional $\varphi : A \to \boldsymbol{R}$ is called ***monotonically increasing*** with respect to the lower [resp. upper] section on A at z_0 if $\varphi(z) \leq \varphi(z_0)$ for any $z \in (\{z_0\} - C) \cap A$ [resp. $\varphi(z) \geq \varphi(z_0)$ for any $z \in (\{z_0\} + C) \cap A$];

(ii) A functional $\varphi : A \to \boldsymbol{R}$ is called ***strictly monotonically increasing*** with respect to the lower [resp. upper] section on A at z_0 if $\varphi(z) < \varphi(z_0)$ for any $z \in (\{z_0\} - \mathbf{int}C) \cap A$ [resp. $\varphi(z) > \varphi(z_0)$ for any $z \in (\{z_0\} + \mathbf{int}C) \cap A$].

Remark 3.1. Since we assume that $\mathbf{int}C \neq \emptyset$, there exists a nonzero continuous linear functional φ such that $\varphi(x) \geq 0$ for all $x \in C$; see page 18 of [15]. We denote the set of all such continuous linear functionals on Z by C^*. Also, each functional $\varphi \in C^*$ [resp. $\varphi \in C^* \setminus \{0\}$] is monotonically [resp. strictly monotonically] increasing with respect to both the lower section and the upper section on Z at any point of Z; see Corollary 2.3 in [16]. Hence, there exists at least one nonzero continuous linear functional which is monotonically and strictly monotonically increasing with respect to both the lower section and the upper section on Z at any point of Z.

Lemma 3.1. (See Theorem 2.4 in [27].) Let $f : X \times Y \to Z$ be a vector-valued function on $X \times Y$, φ_1 and φ_2 functionals from $f(X, Y)$ into $\boldsymbol{R}$, and a point $(x_0, y_0) \in X \times Y$ given.

(i) Suppose that the functionals φ_1 and φ_2 are monotonically increasing with respect to both the lower section and the upper section on $f(X, Y)$ at $f(x_0, y_0)$, respectively. If the point (x_0, y_0) is a strict semi-saddle point of $(\varphi_1 \circ f, \varphi_2 \circ f)$, then (x_0, y_0) is a C-saddle point of f;

(ii) Suppose that the functionals φ_1 and φ_2 are strictly monotonically increasing with respect to both the lower section and the upper section on $f(X,Y)$ at $f(x_0,y_0)$, respectively. If the point (x_0,y_0) is a semi-saddle point of $(\varphi_1 \circ f, \varphi_2 \circ f)$, then (x_0,y_0) is a weak C-saddle point of f.

In order to establish a general type of existence theorems for generalized saddle points, we introduce new concepts of convexity and continuity of vector-valued functions.

Definition 3.5. Let X be a convex set in a real vector space and Z an ordered t.v.s. with a (solid) pointed convex cone C. A vector-valued function $f : X \to Z$ is said to be ***natural quasi C-convex*** on X if

$$f(\lambda x_1 + (1-\lambda)x_2) \in \mathrm{co}\{f(x_1), f(x_2)\} - C \tag{8}$$

for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$, where co$A$ denotes the convex hull of the set A. The condition (8) is equivalent to the following condition: there exists $\mu \in [0,1]$ such that $f(\lambda x_1 + (1-\lambda)x_2) \leq_C \mu f(x_1) + (1-\mu)f(x_2)$.Also, it is said to be ***natural quasi C-concave*** on X if $-f$ is natural quasi C-convex on X.

Remark 3.2. In [28] and [30], we mentioned the relationship among various types of the convexity generalized to vector-valued functions: we note, in particular, that every C-convex function is natural quasi C-convex, and that every properly quasi C-convex function is natural quasi C-convex. (Let X be a convex set in a real vector space. A vector-valued function $f : X \to Z$ is said to be (i) C-convex on X if $f(\lambda x_1 + (1-\lambda)x_2) \leq_C \lambda f(x_1) + (1-\lambda)f(x_2)$ for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$; (ii) properly quasi C-convex on X if either $f(\lambda x_1 + (1-\lambda)x_2) \leq_C f(x_1)$ or $f(\lambda x_1 + (1-\lambda)x_2) \leq_C f(x_2)$ for every $x_1, x_2 \in X$ and $\lambda \in [0,1]$.)

Lemma 3.2. Let X be a convex set in a real vector space and Z an ordered t.v.s. with a (solid) pointed convex cone C, and we denote the set of all continuous linear functionals on Z by Z^*. If a mapping $f : X \to Z$ is natural quasi C-convex on X then for each $\varphi \in Z^*$, the composite mapping $\varphi \circ f$ is a (ordinary) quasi convex function.

Definition 3.6. Let X be a topological space and Z another topological space. A mapping $f : X \to Z$ is said to be ***demicontinuous*** on X if $f^{-1}(M) := \{x \in X \,|\, f(x) \in M\}$ is closed in X for each closed half-space $M \subset Z$.

Remark 3.3. Also, every continuous mapping is demicontinuous obviously.

Lemma 3.3. Let X be a topological space and Z a t.v.s. If a mapping $f : X \to Z$ is demicontinuous on X, then for each $\varphi \in Z^*$, the composite mapping $\varphi \circ f$ is continuous.

Then, we have the second existence theorem of weak C-saddle points, which generalizes Lemma 3.3 in [29] and Theorem 3.1 in [30], and the proof is based on ***Hartung's minimax theorem***; see [13].

Theorem 3.2. Let X and Y be nonempty compact convex sets in two t.v.s.'s, respectively, and Z an ordered t.v.s. with a solid pointed convex cone C. If a vector-valued function $f : X \times Y \to Z$ satisfies that

(i) $x \longmapsto f(x,y)$ is demicontinuous and natural quasi C-convex on X for every $y \in Y$;

(ii) $y \longmapsto f(x,y)$ is demicontinuous and natural quasi C-concave on Y for every $x \in X$,

then the vector-valued function f has at least one weak C-saddle point.

Proof. Since the pointed convex cone C is solid, it follows from Remark 3.1 that there exist nonzero functionals $\varphi_1, \varphi_2 \in C^* \setminus \{0\}$ (possibly $\varphi_1 = \varphi_2$). With these functionals we associate the following sets:

$$A_\alpha(x;\varphi) := \{y \in Y \mid \varphi(f(x,y)) \geq \alpha\}, \tag{9}$$

$$B_\beta(y;\varphi) := \{x \in X \mid \varphi(f(x,y)) \leq \beta\}, \tag{10}$$

for each $x \in X$, $y \in Y$, and $\alpha, \beta \in \boldsymbol{R}$. By Lemmas 3.2 and 3.3, the sets above are closed convex subsets in compact convex sets, respectively. Thus, the proof follows from Theorem 1 in [13] and (ii) of Lemma 3.1. ∎

Consequently, the following corollary is proved immediately by Remark 3.2 and the theorem above.

Corollary 3.1. (Lemma 3.3 in [29].) Let X and Y be nonempty compact convex sets in two t.v.s.'s, respectively, and Z an ordered t.v.s. with a solid pointed convex cone C. If a vector-valued function $f : X \times Y \to Z$ satisfies one of the following conditions:

(i) $x \longmapsto f(x,y)$ is continuous and properly quasi C-convex on X for every $y \in Y$,
$y \longmapsto f(x,y)$ is continuous and properly quasi C-concave on Y for every $x \in X$;

(ii) $x \longmapsto f(x,y)$ is continuous and properly quasi C-convex on X for every $y \in Y$,
$y \longmapsto f(x,y)$ is continuous and C-concave on Y for every $x \in X$;

(iii) $x \longmapsto f(x,y)$ is continuous and C-convex on X for every $y \in Y$,
$y \longmapsto f(x,y)$ is continuous and properly quasi C-concave on Y for every $x \in X$;

(iv) $x \longmapsto f(x,y)$ is continuous and C-convex on X for every $y \in Y$,
$y \longmapsto f(x,y)$ is continuous and C-concave on Y for every $x \in X$;

then the vector-valued function f has at least one weak C-saddle point.

At last, we shall give the third existence theorem for generalized saddle points.

Theorem 3.3. (See Theorem 4.1 in [25] and Theorem 3.1 in [26].) Let X and Y be nonempty compact convex sets in two l.c.s.'s, respectively, and Z an ordered t.v.s. with a solid pointed convex cone C. If a vector-valued function $f : X \times Y \to Z$ is continuous and if the following sets

$$T(y) := \{x \in X \,|\, f(x,y) \in \mathbf{Min_w} f(X,y)\}, \tag{11}$$

$$U(x) := \{y \in Y \,|\, f(x,y) \in \mathbf{Max_w} f(x,Y)\} \tag{12}$$

are convex for every $y \in Y$ and $x \in X$, respectively, then the vector-valued function f has at least one weak C-saddle point.

4. Minimax Theorems for Vector-Valued Functions

In a few of the author's papers, we have proposed some minimax theorems for vector-valued functions. In this section, we shall present some of most general versions of such minimax theorems. To this end, we need the following saddle point theorem of a vector-valued function, which is a corollary of Theorem 2.1.

Theorem 4.1. **(Saddle Point Theorem)** Let X and Y be nonempty compact sets in two topological spaces, respectively, and Z an ordered t.v.s. with a solid pointed convex cone C. If a vector-valued function $f : X \times Y \to Z$ is continuous and if C satisfies the condition (4), then

$$\left[\mathbf{Min} \bigcup_{x\in X} \mathbf{Max_w} f(x, Y)\right] + C \supset \mathrm{SV_w}(f),$$

$$\left[\mathbf{Max} \bigcup_{y\in Y} \mathbf{Min_w} f(X, y)\right] - C \supset \mathrm{SV_w}(f).$$

Hence, if f has a weak C-saddle point $(x_0, y_0) \in X \times Y$, then there exist

$$z_1 \in \mathbf{Min} \bigcup_{x\in X} \mathbf{Max_w} f(x, Y) \quad \text{and} \quad z_2 \in \mathbf{Max} \bigcup_{y\in Y} \mathbf{Min_w} f(X, y)$$

such that $z_1 \leq_C f(x_0, y_0)$ and $f(x_0, y_0) \leq_C z_2$.

Theorem 4.2. **(Minimax Theorem I)** Let X and Y be nonempty compact sets in two topological spaces, respectively, and Z an ordered t.v.s. with a solid pointed convex cone C. If a vector-valued function $f : X \times Y \to Z$ is of the type $f(x, y) = u(x) + v(y)$ where u and v are continuous and if C satisfies the condition (4), then

$$\mathbf{Max} \bigcup_{y\in Y} \mathbf{Min_w} f(X, y) \subset \left[\mathbf{Min} \bigcup_{x\in X} \mathbf{Max_w} f(x, Y)\right] + C$$

and

$$\mathbf{Min} \bigcup_{x\in X} \mathbf{Max_w} f(x, Y) \subset \left[\mathbf{Max} \bigcup_{y\in Y} \mathbf{Min_w} f(X, y)\right] - C,$$

and then there exist

$$z_1 \in \mathbf{Min} \bigcup_{x\in X} \mathbf{Max_w} f(x, Y) \quad \text{and} \quad z_2 \in \mathbf{Max} \bigcup_{y\in Y} \mathbf{Min_w} f(X, y)$$

such that $z_1 \leq_C z_2$.

Proof. In the same way as in the proof of Theorem 3.2 in [25], we can get the proof by using Theorem 4.1. ∎

Theorem 4.3. **(Minimax Theorem II)** Let X and Y be nonempty compact convex sets in two t.v.s.'s, respectively, and Z an ordered t.v.s. with a solid pointed convex cone C. If a vector-valued function $f : X \times Y \to Z$ is continuous and satisfies:

(i) $x \longmapsto f(x,y)$ is natural quasi C-convex on X for every $y \in Y$;

(ii) $y \longmapsto f(x,y)$ is natural quasi C-concave on Y for every $x \in X$,

and if C satisfies the condition (4), then there exist

$$z_1 \in \mathbf{Min} \bigcup_{x\in X} \mathbf{Max_w} f(x,Y) \quad \text{and} \quad z_2 \in \mathbf{Max} \bigcup_{y\in Y} \mathbf{Min_w} f(X,y)$$

such that $z_1 \leq_C z_2$.

Proof. The proof follows immediately from Theorems 3.2 and 4.1. ∎

Theorem 4.4. (**Minimax Theorem III**) Let X and Y be nonempty compact convex sets in two l.c.s.'s, respectively, and Z an ordered t.v.s. with a solid pointed convex cone C. Assume that a vector-valued function $f : X \times Y \to Z$ is continuous and that C satisfies the condition (4). If the sets of (11) and (12) are convex for every $y \in Y$ and $x \in X$, respectively, then there exist

$$z_1 \in \mathbf{Min} \bigcup_{x\in X} \mathbf{Max_w} f(x,Y) \quad \text{and} \quad z_2 \in \mathbf{Max} \bigcup_{y\in Y} \mathbf{Min_w} f(X,y)$$

such that $z_1 \leq_C z_2$.

Proof. The proof follows immediately from Theorem 3.3 and Theorem 4.1. ∎

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