数理解析研究所講究録
第 835 巻 1993 年 27-35

Gale's Theorem on an Infinite Network

椋　毅 (Takeshi MUKU)　山崎稀嗣 (Maretsugu YAMASAKI)
新潟大学自然科学研究科　島根大学理学部

1. Introduction with problem setting

Let X be a countable set of nodes, Y be a countable set of arcs and K be the node-arc incidence function. We assume that the graph $G = \{X, Y, K\}$ is connected, and has no self-loops as in [4]. Notice that G is not necessarily locally finite. Let R be the set of all real numbers and denote by $L(X; Z)$ the set of all functions from X to a set Z. In particular, we set $L(X) = L(X; R)$ and $L(Y) = L(Y; R)$.

For each $y \in Y$, the nodes $x^+(y)$ and $x^-(y)$ are determined uniquely by the relation:

$$K(x^+(y), y) = 1 \text{ and } K(x^-(y), y) = -1.$$

Intuitively, $x^-(y)$(resp. $x^+(y)$) is the initial (resp. terminal) node of y. For a nonempty subset A of X, we put for simplicity

$$Q_-(A) = \{y \in Y; x^-(y) \in A \text{ and } x^+(y) \in X - A\}$$
$$Q_+(A) = \{y \in Y; x^-(y) \in X - A \text{ and } x^+(y) \in A\}.$$

Notice that $Q_-(A) \cup Q_+(A)$ is a cut $A\Theta(X - A)$ in [4].

In this paper we always assume that the functions $V, W \in L(Y)$, $\lambda \in L(X; R \cup \{-\infty\})$ and $\mu \in L(X; R \cup \{\infty\})$ satisfy the following conditions:

$$V(y) \leq W(y) \text{ on } Y; \tag{1}$$

$$\sum_{y \in Y} |V(y)| < \infty, \sum_{y \in Y} |W(y)| < \infty; \tag{2}$$

$$\lambda(x) \leq \mu(x) \text{ on } X, \sum_{x \in \Lambda} \lambda(x) < \infty, -\infty < \sum_{x \in \Gamma} \mu(x), \tag{3}$$

where $\Lambda = \{x \in X; \lambda(x) > 0\}, \Gamma = \{x \in X; \mu(x) < 0\}$.

The feasibility problem of Gale is to find $w \in L(Y)$ which has the following properties:

$$(G.1) \qquad V(y) \leq w(y) \leq W(y) \text{ on } Y;$$
$$(G.2) \qquad \lambda(x) \leq \textstyle\sum_{y \in Y} K(x,y)w(y) \leq \mu(x) \text{ on } X.$$

The algebraic operations and order relation of R are extended to $R \cup \{-\infty\}$ or $R \cup \{\infty\}$ in the usual way, i.e.,

$$0 \cdot \infty = 0 \cdot (-\infty) = 0;$$
$$t + \infty = \infty, -\infty + t = -\infty \text{ for all } t \in R;$$
$$t \cdot \infty = \infty, t \cdot (-\infty) = -\infty \text{ for all } t > 0.$$

To state our main theorem, we introduce a notation. For a subset A of X and a function$f \in L(X; R \cup \{-\infty\}) \cup L(X; R \cup \{\infty\})$, we put

$$f(A) = \textstyle\sum_{x \in A} f(x)$$

if the sum is well-defined and $f(\emptyset) = 0$ for the empty set $\emptyset$. The quantity $w(Q)$ for a subset Q of Y and $w \in L(Y)$ is defined similarly.

Our aim of this paper is to prove the following theorem:

Theorem 1.1 *The feasibility problem of Gale has a solution if and only if the given functions V, W, λ and μ satisfy the relation:*

$$(H.1) \qquad \lambda(A), -\mu(X - A) \leq W(Q_+(A)) - V(Q_-(A))$$

for every nonempty subset A of X.

Gale [2] proved this theorem in the case where G is a finite graph without multiple arcs, i.e., for every two nodes, there exists at most one arc. An abstract Flow Theorem in B.Fuchssteiner and Lusky [1] and the theorem of Gale for infinite networks in M.M.Neumann [3] may be regarded as a generalization of the feasibility theorem of gale. In their problem settings, the set of nodes of the network is a nonempty set S endowed some algebra $\sum$ of subsets and a flow is a biadditive set functions from $\sum \times \sum$ to an ordered real vector space which is Dedekind complete. Note that even if $S = X$, their infinite network is assumed not to have multiple arcs. Notice that the feasible solution in [1] and [2] does not give an answer to our flow even if G has no multiple arcs.

2. Reduction of Theorem 1.1

First we prove the only if part of Theorem 1.1. Let w be a feasible solution of $(G.1)$ and $(G.2)$ and A be a nonempty subset of X. Then

$$\begin{aligned}\lambda(A) &\le \sum_{x\in A}\sum_{y\in Y} K(x,y)w(y) && \text{by } (G.2) \text{ and } (2)\\ &= \sum_{y\in Y} w(y)\sum_{x\in A} K(x,y)\\ &= \sum_{y\in Q_+(A)} w(y) - \sum_{y\in Q_-(A)} w(y)\\ &\le W(Q_+(A)) - V(Q_-(A)). && \text{by } (G.1)\end{aligned}$$

The inequality for $\mu(A)$ can be proved similarly.

To prove the "if" part, we may assume that $V = 0$. In fact, let $\tilde{V} = 0, \tilde{W} = W - V$,

$$\tilde{\lambda}(x) = \lambda(x) + \textstyle\sum_{y\in Y} K(x,y)V(y),$$
$$\tilde{\mu}(x) = \mu(x) + \textstyle\sum_{y\in Y} K(x,y)V(y).$$

If there exists $\tilde{w} \in L(Y)$ which satisfies the relation:

$$0 \le \tilde{w}(y) \le \tilde{W}(y) \text{ on } Y,$$
$$\tilde{\lambda}(x) \le \textstyle\sum_{y\in Y} K(x,y)\tilde{w}(y) \le \tilde{\mu}(x) \text{ on } X,$$

then $w(y) = \tilde{w}(y) + V(y)$ satisfies$(G.1)$ and $(G.2)$.

3. Preliminaries

A function $f \in L(X)$ is called simple if its range is a finite set. Denote by $L_S(X)$ the set of real valued simple functions on X. Hereafter we put

$$E = L_S(X) \text{ and } F = L_S(Y).$$

For a subset A of X and a subset Q of Y, denote by ϵ_A and φ_Q their characteristic functions respectively. Denote by $L_S(Y;E)$ the set of E-valued functions on Y, i.e., $\psi \in L_S(Y;E)$ can be written in the form $\psi = \sum_{i=1}^n f_i\varphi_{Q_i}$, where $f_1, \cdots, f_n \in E$ and $Q_1, \cdots, Q_n$ are mutually disjoint subsets of Y.

For each $f \in L(X)$, let us define $\theta(f) \in L(Y)$ by

$$\theta(f)(y)=\max\{0, \sum_{x\in X} K(x,y)f(x)\}$$

as in [1] and [2]. The following properties are easily seen:

$(\theta.1)$ $$\theta(\epsilon_A) = \varphi_{Q_+(A)};$$

$(\theta.2)$ $$\theta(-\epsilon_A) = \varphi_{Q_-(A)};$$

$(\theta.3)$ $$\theta(f-g) = \theta(f) + \theta(-g)$$

for $f, g \in L^+(X) \cap E$ such that $f(x)g(x) = 0$ on X;

$(\theta.4)$ $$\theta(\sum_{i=1}^n t_i \epsilon_{A_i}) = \sum_{i=1}^n t_i \theta(\epsilon_{A_i})$$

for all $t_1, \cdots, t_n \geq 0$ and all A_i such that $A_1 \supset \cdots \supset A_n$.

We prepare

Lemma 3.1 *For each $\psi \in L_S(Y; E)$, the function $\hat{\psi}$ defined by*

$$\hat{\psi}(y) := \theta(\psi(y))(y)$$

belongs to F.

Proof. By definition,

$$\psi(y) = \sum_{i=1}^n f_i \varphi_{Q_i}(y)$$

with $f_i \in L_S(X)$ and mutually disjoint subsets Q_i of Y. In case y does not belong any one of Q_i, $\psi(y) = 0 \in L(X)$ and $\theta(\psi(y))(y) = 0$. If $y \in Q_i$, then $\psi(y) = f_i$ and $\hat{\psi}(y) = \theta(f_i)(y)$. Thus

$$\hat{\psi}(y) = \sum_{i=1}^n [\theta(f_i)(y)] \varphi_{Q_i}(y)$$

and $\hat{\psi} \in F$. ∎

Lemma 3.2 *Let $f \in L_S^+(X) := L_S(X) \cap L^+(X)$ and assume that the number of elements in the range of f is equal to n. Then there exist non-negative numbers $t_1, \cdots, t_n$ and subsets $A_1, \cdots, A_n$ of X such that $A_1 \supset \cdots \supset A_n$ and*

$$f(x) = \sum_{i=1}^n t_i \epsilon_{A_i}(x).$$

Proof. There existe a class $\{B_i\}$ of mutually disjoint subsets of X such that $f(x) = \alpha_i$ on $B_i (i = 1, \cdots, n)$ and $\alpha_i \neq \alpha_j$ if $i \neq j$. Clearly we have

$$f(x) = \sum_{i=1}^n \alpha_i \epsilon_{B_i}(x).$$

Without any loss of generality, we may assume that $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n$. Define A_i and t_i as follows:

$$A_i = \cup_{j=i}^{n} B_j (1 \leq i \leq n);$$
$$t_1 = \alpha_1 \text{ and } t_i = \alpha_i - \alpha_{i-1} (2 \leq i \leq n).$$

Then our assertion is easily seen. ∎

4. Proof of Theorem 1.1

To prove the reduced "if" part of Theorem 1.1, we assume that condition $(H.1)$ holds and $V = 0$.

Let us put

$$G = L_S(Y; E).$$

Then G is a linear space. We shall identify each $f \in E$ with $\psi_f = f\varphi_Y \in G$. In this sense, $E \subset G$.

Let us introduce the convex cones K_λ and K_μ in E:

$$K_\lambda = \{f \in L_S^+(X); \sum_{x \in X} f(x)|\lambda(x)| < \infty\};$$
$$K_\mu = \{g \in L_S^+(X); \sum_{x \in X} g(x)|\mu(x)| < \infty\}.$$

Now assume that $W \in L^+(Y)$ and $W(Y) = \sum_{y \in Y} W(y) < \infty$. We shall consider a functional ρ on G as in [1] and [3]:

$$\rho(\psi) := \sum_{y \in Y} \hat{\psi}(y) W(y).$$

To verify $\rho(\psi)$ is finite, let $\psi = \sum_{i=1}^{n} f_i \varphi_{Q_i}$ as in Lemma 3.1. Let $m_i = \min\{f_i(x); x \in X\}$ and $M_i = \max\{f_i(x); x \in X\}$. Then

$$\theta(f_i)(y) \leq M_i - m_i,$$

$$0 \leq \hat{\psi}(y) \leq \max\{\theta(f_i)(y); i = 1, \cdots, n\} \leq c(\hat{\psi}) \text{ on } Y,$$

where $c(\hat{\psi}) = \max\{M_i - m_i; i = 1, \cdots, n\}$. Therefore

$$0 \leq \rho(\psi) \leq c(\hat{\psi}) W(Y) < \infty.$$

Notice that θ is sublinear, i.e., $\theta(\alpha f + \beta g)(y) \leq \alpha\theta(f)(y) + \beta\theta(g)(y)$ on Y for every $f, g \in L_S(X)$ and $\alpha, \beta \geq 0$. Therefore for $\psi = \psi_1 + \psi_2, (\psi_1, \psi_2 \in G)$, we have

$$\hat{\psi}(y) \leq \hat{\psi}_1(y) + \hat{\psi}_2(y) \text{ on } Y.$$

Namely the mapping $\psi \longrightarrow \hat{\psi}$ is sublinear.

Lemma 4.1 *Assume that condition* $(H.1)$ *holds with* $V = 0$. *Then*

$$\sum_{x\in X} \lambda(x)f(x) - \sum_{x\in X} \mu(x)g(x) \le \rho(f-g)$$

holds for every $f \in K_\lambda$ *and* $g \in K_\mu$.

Proof. For $f \in K_\lambda$ and $g \in K_\mu$, we put $\tilde{f} = (f-g)^+$ and $\tilde{g} = (f-g)^-$. Then

$$\tilde{f} \in K_\lambda, \tilde{g} \in K_\mu, \tilde{f}(x)\tilde{g}(x) = 0 \text{ on } X \text{ and}$$
$$f - \tilde{f} = g - \tilde{g} \in K_\lambda \cap K_\mu.$$

By Lemma 3.2, $\tilde{f}$ and $\tilde{g}$ can be expressed as follows:

$$\tilde{f} = \sum_{i=1}^{m} \alpha_i \epsilon_{A_i} \text{ and } \tilde{g} = \sum_{j=1}^{n} \beta_j \epsilon_{B_j},$$

where $\alpha_i, \beta_j \ge 0, A_1 \supset \cdots \supset A_m$ and $B_1 \supset \cdots \supset B_n$. By using the properties of θ, we have

$$\begin{aligned}
\rho(f-g) &= \rho(\tilde{f}-\tilde{g}) \\
&= \sum_{y\in Y} [\theta(\tilde{f}-\tilde{g})(y)\varphi_Y(y)]W(y) && \text{by } (\theta.3) \\
&= \sum_{y\in Y} [\theta(\tilde{f})(y) + \theta(-\tilde{g})(y)]W(y) \\
&= \sum_{y\in Y} [\theta(\sum_{i=1}^{m} \alpha_i \epsilon_{A_I})(y) + \theta(-\sum_{j=1}^{n} \beta_j \epsilon_{B_j})(y)]W(y) && \text{by } (\theta.4) \\
&= \sum_{y\in Y}\sum_{i=1}^{m} \alpha_i[\theta(\epsilon_{A_i})(y)]W(y) + \sum_{y\in Y}\sum_{j=1}^{n} \beta_j[\theta(-\epsilon_{B_j})(y)]W(y) && \text{by } (\theta.1) \text{ and } (\theta.2) \\
&= \sum_{i=1}^{m} \alpha_i W(Q_+(A_i)) + \sum_{j=1}^{n} \beta_j W(Q_-(B_j)) \\
&\ge \sum_{i=1}^{m} \alpha_i \lambda(A_i) - \sum_{j=1}^{n} \beta_j \mu(B_j) \\
&= \sum_{x\in X} \lambda(x)\tilde{f}(x) - \sum_{x\in X} \mu(x)\tilde{g}(x) && \text{by } (3) \\
&\ge \sum_{x\in X} \lambda(x)f(x) - \sum_{x\in X} \mu(x)g(x).
\end{aligned}$$

∎

For each $h \in K := K_\lambda - K_\mu$, define $\Phi(h)$ by

$$\Phi(h) = \sup\{\sum_{x \in X} \lambda(x)f(x) - \sum_{x \in X} \mu(x)g(x); h = f - g, f \in K_\lambda, g \in K_\mu\}.$$

Then it is easily seen that Φ is superlinear on K, i.e.,

$$\Phi(\alpha h_1 + \beta h_2) \geq \alpha\Phi(h_1) + \beta\Phi(h_2)$$

for every $h_1, h_2 \in K$ and $\alpha, \beta \geq 0$. Notice that by Lemma 4.1

$$\Phi(h) \leq \rho(h) \text{ for all } h \in K. \tag{4.1}$$

Clearly K is a convex subset of G. For a sublinear functional ρ on G and a superlinear functional Φ on K which satisfy (4.1), we can apply the Sandwich Theorem in [1]. Thus there exists a linear functional ξ on G such that

$$\Phi(h) \leq \xi(h) \text{ for every } h \in K, \tag{4.2}$$
$$\xi(\psi) \leq \rho(\psi) \text{ for every } \psi \in G. \tag{4.3}$$

For each $y \in Y$, let us put

$$\psi_y^+ := \epsilon_{\{x^+(y)\}}\varphi_{\{y\}} \text{ and } \psi_y^- := \epsilon_{\{x^-(y)\}}\varphi_{\{y\}}.$$

Then we have $\psi_y^+, \psi_y^- \in G$ and $\psi_y^+ + \psi_y^- = \epsilon_{e(y)}\varphi_{\{y\}}$ with $e(y) = \{x^+(y), x^-(y)\}$, so that

$$\rho(\psi_y^+) = W(y), \rho(-\psi_y^+) = 0 \text{ and}$$
$$\rho(\psi_y^+ + \psi_y^+) = \rho(-(\psi_y^+ + \psi_y^-)) = 0$$

Now we define $w \in L(Y)$ by

$$w(y) := \xi(\psi_y^+) = \xi(\epsilon_{\{x^+(y)\}}\varphi_{\{y\}}). \tag{4.4}$$

By (4.3) and the above observation, we obtain

$$0 \leq w(y) \leq W(y) \text{ on } Y \text{ and } \xi(\psi_y^-) = -w(y).$$

Our next goal is to prove that w satisfies $(G.2)$ with $V = 0$. Let $a \in X$ be any node such that $\lambda(a) \in R$ and put

$$Y' = \{y \in Y; a \notin e(y)\}.$$

Then, for every $y \in Y'$

$$\theta(\epsilon_{\{a\}}(y)) = \theta(-\epsilon_{\{a\}}(y)) = 0,$$

so that

$$\rho(\epsilon_{\{a\}}\varphi_{Y'}) = \rho(-\epsilon_{\{a\}}\varphi_{Y'}) = 0.$$

Therefore, by (4.4), $\xi(\epsilon_{\{a\}}\varphi_{Y'}) = 0$. For simplicity, put

$$\begin{aligned} Y_+(a) &= \{y \in Y; K(a,y) = 1\} \\ Y_-(a) &= \{y \in Y; K(a,y) = -1\}. \end{aligned}$$

By (4.2), we have

$$\begin{aligned} \lambda(a) &= \sum_{x\in X} \lambda(x)\epsilon_{\{a\}}(x) \\ &\leq \xi(\epsilon_{\{a\}}\varphi_Y) \\ &= \sum_{y\in Y_+(a)} \xi(\psi_y^+) + \sum_{y\in Y_-(a)} \xi(\psi_y^-) + \xi(\epsilon_{\{a\}}\varphi_{Y'}) \\ &= \sum_{y\in Y_+(a)} w(y) - \sum_{y\in Y_-(a)} w(y) \\ &= \sum_{y\in Y} K(a,y)w(y). \end{aligned}$$

Similarly we have

$$\sum_{y\in Y} K(a,y)w(y) \leq \mu(a)$$

for every $a \in X$ such that $\mu(a) \in R$. The resulting estimates are obvious if $\lambda(a) = -\infty(\mu(a) = \infty)$. This completes the proof.

Reference

[1] B.Fuchssteiner and W. Lusky, *Convex cones*, North-Holland Mathematics Studies **56**, North-Holland, 1981.

[2] D.Gale, A theorem on flows in networks, Pacific J. Math. **7**(1957), 1073-1082.

[3] M.M.Neumann, The theorem of Gale for infinite networks and applications In: Infinite Programming, E.J.Anderson and A.B.Philpott (ends.), Lecture Note in Economics and Mathematical Systems **259**, Springer-Verlag, 1984.

[4] M.Yamasaki, Extremum problems on an infinite netwrok, Hiroshima Math. J. **5**(1975), 223-250.