数理解析研究所講究録
第 835 巻 1993 年 89-98

Martingale limit theorem and its application to an ergodic controlled Markov chain

Akio TANIKAWA (谷川明夫)
Faculty of Engineering, Kanazawa University (金沢大・工)

1. Introduction.

Let $\{X_i, \mathcal{F}_i,\ i\geq 1\}$ be an adapted sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, namely, $\{\mathcal{F}_i,\ i\geq 1\}$ is an increasing sequence of σ-fields with $\mathcal{F}_i \subset \mathcal{F}$ for each $i \geq 1$ and X_i is $\mathcal{F}_i$-measurable for each $i \geq 1$. We assume that X_i is integrable: $E|X_i| < \infty$, $i \geq 1$. In this paper we study the martingale limit theorem (Hall and Heyde [6], p. 36, or Loève [7], p. 53)

$$(1.1)\qquad n^{-1}\sum_{i=1}^{n}[X_i - E(X_i|\mathcal{F}_{i-1})] \longrightarrow 0 \quad \text{a.s.}$$

under the condition

$$(1.2)\qquad \sup_i E[\psi(X_i)] < \infty .$$

Here, $\psi(x)$ is a nonnegative, even and continuous function which satisfies convexity near ∞ and some additional condition (see Remark 2). Later on (in Section 4) this result will be applied to show existence of an optimal control for an ergodic controlled Markov chain which was considered by Borkar([2], [3], [4] and [5]).

In order to investigate the martingale limit theorem (1.1) we define a random variable Y_i by

$$(1.3)\qquad Y_i = X_i - E(X_i|\mathcal{F}_{i-1}) .$$

Since $\{Y_i, \mathcal{F}_i,\ i\geq 1\}$ is a martingale difference sequence (i.e., $\{S_i \equiv$

$\Sigma_{j=1}^{i} Y_j, \mathcal{F}_i, i \geq 1\}$ is a martingale), we can obtain the strong law of large numbers for martingales (see Stout [9]):

$$(1.4) \qquad n^{-1} \sum_{i=1}^{n} Y_i \longrightarrow 0 \quad \text{a.s.}$$

under the condition

$$(1.5) \qquad \sup_i E[\psi(Y_i)] < \infty .$$

In [9], p. 156 , as an example of $\psi(x)$, Stout gave

$$(1.6) \qquad \psi(x) = |x|(\log^+|x|)^{\alpha} , \qquad \alpha > 1 ,$$

where $\log^+|x| = (\log|x|) \vee 0$. In this paper, we consider a more general form

$$(1.7) \qquad \psi(x) = |x|L(|x|) ,$$

where $L(x)$ is a nonnegative, even and continuous function which has the following representation: There exists a positive number B such that for all $x \geq B$ we have

$$(1.8) \qquad L(x) = c \exp\{\int_B^x \varepsilon(y)y^{-1}dy\} ,$$

where c is a constant $(0 < c < \infty)$ and $\varepsilon(x)$ is a continuous function on $[B, \infty)$ such that $\lim_{x\to\infty} \varepsilon(x) = 0$. Namely, $L(x)$ is a slowly varying function at infinity such that its c-*function* $c(x)$ is a constant c (see Seneta [8], p. 2). In Bingham, Goldie and Teugels [1], p. 15 , it is called a normalized slowly varying function. Since we are concerned with the measuring of scales of growth at ∞ , slowly varying functions are of interest only to within asymptotic equivalence. In this sense we lose nothing by restricting attention to the case of constant c-function.

In Section 2, we study sufficient conditions on $\psi(x)$ or $\varepsilon(x)$ for (1.5) to imply (1.4). In Section 3, we show that the strong law of large numbers for martingales (1.4)-(1.5) implies the martingale limit theorem (1.1)-(1.2) provided that $\psi(x)$ is convex on $[r, \infty)$ for some $r > 0$. In Section 4, as an application of the martingale limit theorem, we discuss on existence of an optimal control for an ergodic controlled Markov chain with long-run average cost criterion under a weaker condition than Borkar's.

2. The strong law of large numbers.

Let $\{Y_i, \mathcal{F}_i, i \geq 1\}$ be a martingale difference sequence. It is easy to show the following strong law of large numbers for martingales by modifying Theorem 3.3.9 in Stout [9].

THEOREM 1. Let $\psi(x)$ be a positive even function satisfying the following conditions. There exists a positive number r such that on $[r, \infty)$ $\psi(x)/x$ is nondecreasing, $\psi(x)/x^2$ is nonincreasng and that

$$(2.1) \qquad \int_r^\infty (1/\psi(x))dx < \infty .$$

If a martingale difference sequence $\{Y_i, \mathcal{F}_i, i \geq 1\}$ satisfies

$$(2.2) \qquad \sup_i E[\psi(Y_i)] < \infty ,$$

then the strong law of large numbers (1.4) holds.

We now investigate sufficient conditions on $\varepsilon(x)$ in (1.8) for $\psi(x)$ to satisfy all the conditions in Theorem 1. For this purpose we make the following assumption.

ASSUMPTION 1. There exists a positive number r_1 ($\geq B$) such that $\varepsilon(x) \geq 0$ for $x \geq r_1$.

PROPOSITION 1. Under Assumption 1, there exists a positive number r ($\geq r_1$) such that on $[r, \infty)$ $\psi(x)/x$ is nondecreasing and

$\psi(x)/x^2$ is nonincreasing.

We can give an example which satisfies all the requirements in Theorem 1 by checking Assumption 1.

EXAMPLE 1. For $\alpha \geq 1$, we define the function $\varphi_{k,\alpha}(x)$ by

(2.3) $$\varphi_{k,\alpha}(x) = |x|(\log^+|x|)(\log_2^+|x|)\cdots(\log_{k-1}^+|x|)(\log_k^+|x|)^\alpha ,$$

where $\log^+|x| = (\log|x|) \vee 0$, $\log_2^+|x| = (\log \log|x|) \vee 0$, and so on. It is easy to see that there exists a positive number r such that $\int_r^\infty \varphi_{k,\alpha}(x)^{-1}dx < \infty$ ($= \infty$, resp.) if $\alpha > 1$ ($\alpha = 1$, resp.). Hence we obtain the following corollary.

COROLLARY 1. Let $\{Y_i, \mathcal{F}_i, i\geq 1\}$ be a martingale difference sequence. If it satisfies

(2.4) $$\sup_i E[\varphi_{k,\alpha}(Y_i)] < \infty$$

for some $\alpha > 1$, then the strong law of large numbers (1.4) holds.

3. The martingale limit theorem.

Let $\{X_i, \mathcal{F}_i, i\geq 1\}$ be an adapted sequence of random variables. In this section, we show that the strong law of large numbers for martingales implies the martingale limit theorem if $\psi(x)$ satisfies some additional conditions as well as conditions in Section 2.

We prepare preliminary inequalities, (3.1) and (3.2), which connect (1.2) with (1.5) for (1.3).

LEMMA 1. Let $h(x)$ be an even, continuous and convex functon. Let X be a random variable and $\mathcal{F}$ be a σ-field. Then we have the inequality

(3.1) $$E\, h(X - E(X|\mathcal{F})) \leq E\, h(2X) .$$

DEFINITION. A function $h(x)$ is said to belong to $\mathcal{G}$ if it is a nonnegative even function which satisfies the following growth condition: For some $a > 0$ and $b > 0$

(3.2) $h(2x) \le a + b\, h(x)$

holds for any x .

REMARK 1. Since in our case $\psi(x)$ is continuous (and hence locally bounded) and has the form $\psi(x) = xL(x)$ (see (1.7)), it is easy to observe that $\psi(x)$ belongs to $\mathcal{G}$.

THEOREM 2. Let $\psi(x)$ be a convex function which belongs to $\mathcal{G}$ and satisfy all the conditions in Theorem 1. Let $\{X_i, \mathcal{F}_i, i \ge 1\}$ be an adapted sequences of random variables. If

(3.3) $\sup_i E[\psi(X_i)] < \infty$,

then the martingale limit theorem (1.1) holds.

REMARK 2. In view of proof of Theorem 3, we need not require the convexity of $\psi(x)$ on the whole space $(-\infty, +\infty)$, namely, the convexity can be replaced by the following weaker condition: There exists a positive number $r > 0$ such that $\psi(x)$ is convex on $[r, \infty)$ and $\partial^+\psi(r) \ge 0$, where $\partial^+\psi\ (r)$ is the right differential coefficient at r .

In the next proposition, we study the condition which assures that the function $\psi(x)$ with the representation (1.7)-(1.8) is convex on $[r, \infty)$ for some $r \ge B$. For simplicity we make the following assumption.

ASSUMPTION 2. There exists a positive number r_2 ($\ge B$) such that $\varepsilon(x)$ is absolutely continuous on $[r_2, \infty)$.

PROPOSITION 2. For some r ($\ge r_2$) , $\psi(x)$ is convex on $[r, \infty)$ if and only if $\varepsilon(x)$ satisfies

(3.4) $\qquad \varepsilon(x) + \varepsilon(x)^2 + x\varepsilon'(x) \geq 0$ a.e. on $[r,\infty)$.

It is easy to show the following main theorem by virtue of Theorem 2.

THEOREM 3. Let $\psi(x)$ be a nonnegative, even and continuous function with the representation (1.7)-(1.8) where its ε-function $\varepsilon(x)$ satisfies Assumptions 1 and 2. We also assume that there exists a positive number r ($\geq B \vee r_1 \vee r_2$) such that $\psi(x)^{-1}$ is integrable on $[r, \infty)$ (i.e., (2.1) holds) and that the inequality (3.4) holds on $[r, \infty)$. If an adapted sequence of random variables $\{X_i, \mathcal{F}_i,\ i\geq 1\}$ satisfies

(3.5) $\qquad \sup_i E[\psi(X_i)] < \infty$,

then the martingale limit theorem (1.1) holds.

We can easily check the inequality (3.4) with some $r > 0$ for $\varphi_{k,\alpha}$ defined by (2.3). Hence we obtain the following corollary.

COROLLARY 2. Let $\{X_i, \mathcal{F}_i,\ i\geq 1\}$ be an adapted sequence of random variables. If it satisfies

(3.6) $\qquad \sup_i E[\varphi_{k,\alpha}(X_i)] < \infty$

with $\alpha > 1$, then the martingale limit theorem holds:

(3.7) $\qquad n^{-1} \sum_{i=1}^{n} [X_i - E(X_i | \mathcal{F}_{i-1})] \longrightarrow 0$ a.s.

REMARK 3. When $\alpha = 1$, we can give a simple counterexample to the above corollary by modifying Example 15.3 (p.141) of Stoyanov [10].

4. An application to a controlled Markov chain.

In this section, let X_n , $n = 1, 2, \cdots$ be a controlled Markov chain on a state space $S = \{1, 2, \cdots\}$ with a transition matrix

$P_u = [p(i,j,u_i)]$, $i, j \in S$, where $u = [u_1, u_2, \cdots]$ is the control vector satisfying $u_i \in D$ for some prescribed compact metric space D . The functions $p(i,j,\cdot)$ are assumed to be continuous. Define L as the countable product of copies of D with the product topology. A control strategy (CS) is a sequence of L-valued random variables $\{\xi_n\}$, $\xi_n = [\xi_n(1), \xi_n(2), \cdots]$, such that for all $i \in S$, $n \geq 1$,

(4.1) $$P(X_{n+1} = i \mid X_m, \xi_m, m \leq n) = p(X_n, i, \xi_n(X_n)) .$$

The controlled Markov chain $\{X_n\}$ is said to be governed by the CS $\{\xi_n\}$ whenever (4.1) holds. If $\{\xi_n\}$ are identically distributed with a common law Φ and ξ_n is independent of X_m , $m \leq n$, ξ_m , $m < n$, for each n , we call $\{\xi_n\}$ a stationary randomized strategy (SRS) , denoted by $\gamma[\Phi]$. If, in addition, Φ is a Dirac measure at some $\xi \in L$, we call it a stationary strategy (SS) denoted by $\gamma\{\xi\}$. If the controlled Markov chain $\{X_n\}$ is govered by a SRS $\{\xi_n\}$, then $\{X_n\}$ is a stationary Markov chain (see Borkar [5]). We shall assume throughout that the chain has a single communicating class under all SRS (see Borkar [5]). Thus, under all SRS , the Markov chain is irreducible. If, in addition, it is positive recurrent under some $\gamma[\Phi]$ or $\gamma\{\xi\}$, we call it a stable SRS (SSRS) or a stable SS (SSS) respectively, and denote the corresponding unique invariant probability measure by $\pi[\Phi]$ or $\pi\{\xi\}$ respectively.

Let $k: S \times D \to \mathfrak{R}$ be a nonnegative continuous function. We consider the following optimal control problem:

Minimize a.s. over all CS

$$\text{(4.2)} \qquad \limsup_{n\to\infty} n^{-1} \sum_{m=1}^{n-1} k(X_m, \xi_m(X_m)) .$$

This is an 'ergodic' or 'long-run average cost' control problem with 'running cost' k .

We review some of Borkar's consequences ([2], [3], [4] and [5]). He studied existence of an optimal SSS for two distinct set-ups, i.e., the near-monotone case and the stable case. We are concerned only with the stable case here. Let $\tau = \min\{n > 1 | X_n = 1\}$ ($= \infty$ if $X_n = 1$ for all $n > 0$) . He made the following assumption.

<u>CONDITION A</u>. The family $\{\tau(\xi)\}$ of random variables is uniformly integrable, where $\tau(\xi) = \tau$ corresponds to the chain governed by $\gamma\{\xi\}$ with initial condition $X_0 = 1$.

He showed that Condition A is equivalent to the tightness of $\{\pi\{\xi\} | \xi \in L\}$(Theorem 3.1 in [4], Lemma V.2.1 in [5]).

In order to prove existence of an optimal SSS among all CS , he needed the following additional assumption.

<u>CONDITION B</u>. $\sup_{CS} E[\tau^2 | X_0 = 1] < \infty$.

Under Condition B, he obtained the following theorem.

<u>THEOREM 4</u>. (Theorem 7.2 of [4], Theorem V.2.1 of [5]) Under Condition B, an optimal SSS exists.

He proved this theorem as follows. First, he considered the following $\mathfrak{P}(S\times D)$-valued empirical process $\{\nu_n, n\geq 1\}$ defined by

$$\text{(4.3)} \qquad \nu_n(A\times B) = n^{-1} \sum_{i=0}^{n-1} I(X_i \in A , \ \xi_i(X_i) \in B)$$

for A , B Borel in S , D respectively. Then he showed that if the empirical process $\{\nu_n, n\geq 1\}$ for a chain $\{X_n\}$ governed by an arbitrary CS $\{\xi_n\}$ forms a tight sequence with probability one then there exists an optimal SSS exists. In order to prove the tightness of the empirical process, he needed Condition B because he made use of the martingale limit theorem in Loève [7]. Since in Section 3 we obtained a more convenient result (Corollary 2) for our purpose, we can prove Theorem 4 under a weaker assumption by the same arguments as those of Theorem V.2.1 in [5].

CONDITION C. $\sup_{CS} E[\varphi_{k,\alpha}(\tau) | X_0 = 1] < \infty$ for some $\alpha > 1$, where $\varphi_{k,\alpha}$ was defined by (2.4) in Example 1.

THEOREM 4'. Under Condition C, an optimal SSS exists.

REMARK 4. In [4] and [5], Borkar conjectured that Condition A will be sufficient and so Condition B can be dropped in the proof. His conjecture cannot be proved here. However, we showed Theorem 4' by making use of Condition C instead of Condition B. Hence the gap between Condition A and Condition B was considerably reduced in this paper.

Acknowledgements.

I would like to thank Professor M. Tsuchiya of Kanazawa University for his helpful comments and support. I also thank Professor D.A. Dawson, Professor M. Csörgő and Dr. V.V. Vinogradov for their kindness while I was visiting Carleton University, Ottawa.

References

[1] Bingham, N.H., Goldie, C.M., and Teugels, J.L.: Regular Variation, Encyclopedia of Math. Appl., Cambridge University Press, Cambridge, 1987.

[2] Borkar, V.S.: Controlled Markov chains and stochastic networks,

SIAM J. Control and Opt. 21 (1983), 652-666.

[3] Borkar, V.S.: On minimum cost per unit time control of Markov chains, SIAM J. Control and Opt. 22 (1984), 965-978.

[4] Borkar, V.S.: Control of Markov chains with long-run average cost criterion, in Proc. of the I.M.A. Workshop on Stochastic Differential Systems, Stochastic Control Theory and Appl., Fleming, W., and Lions P.L. (eds.), Springer, 1987, 57-77.

[5] Borkar, V.S.: Topics in Controlled Markov Chains, Pitman Research Notes in Math., No. 240, Longman Scientific and Technical, 1991.

[6] Hall, P., and Heyde, C.C.: Martingale Limit Theory and Its Application, Academic Press, 1980.

[7] Loève, M.: Probability Theory II, 4th. ed., Springer, 1978.

[8] Seneta, E.: Regularly Varying Functions, Lecture Notes in Math. 508, Springer, 1976.

[9] Stout, W.F.: Almost Sure Convergence, Academic Press, 1974.

[10] Stoyanov, J.M.: Counterexamples in Probability, John Wiley and Sons, 1987.