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可算状態マルコフ連鎖における定常絶対確率の連続性について

SUFFICIENT CONDITIONS FOR CONTINUITY OF STATIONARY ABSOLUTE PROBABILITIES IN COUNTABLE STATE MARKOV CHAINS

和歌山大教育 門田良信 (Yoshinobu Kadota)

1. Introduction

We consider a space of discrete time stationary Markov chains, which are abbreviated as "MC's" henthforth, with a countable state space. In the earlier paper[8], Kadota proposed recurrent conditions which establish a Laurent series expansions of the MC. The present paper has two purposes, defining the similar type conditions on a space of MC's. The first purpose is to examine the conditions by comparing with each other or by finding counter examples. The second is to show that those conditions assure continuity of cofficients of the Laurent series on a parameter space.

Our problems have been considered in the theory of Markov decision processes with average and sensitive discount criteria. The Laurent series expansion were first obtained in a complete sense by Miller and Veinott[10] and Veinott[12] in a finite state space. When the state space is countable, the series does not exist in general. It requires some recurrence conditions. Wijngaard[13] and Dietz and Nallaw[3] have obtained the series, using a quasi-compact condition for transition probabilities. Kadota[7] has obtained it, using a Doèblin condition which is equivalent to the quasi-compactness. Zijm[14] shows the existence and continuity of the first two terms of the serieses, in tha case that all recurrent classes of the MC's are aperiodic. Our conditions are so loose that those results are contained as special cases and they admit that each MC have countably many periodic recurrent classes. They supply a space of MC's to the conditons of Dekker and Hordijk[2].

We now state our model. Let a state spase S be a non-empty countable set. A MC is denoted by it's transition probabilities $\{p_{ij}; i,j \in S\}$, where p_{ij} is the conditional probability that a system moves from $i \in S$ to $j \in S$ in a unit time. We assume throughout the paper that $0 \leqq p_{ij} \leqq 1$ for any $i,j \in S$ and that $\sum_{j \in S} p_{ij} = 1$ for any $i \in S$. Let $p_{ij}^1 = p_{ij}$ and $p_{ij}^n = \sum_{k \in S} p_{ik} p_{kj}^{n-1}$ for $i,j \in S$ and $n = 1, 2, \ldots$. We denote by P^n the Markov matrix whose (i,j)-component is given by p_{ij}^n. If $n = 0$, let $P^0 = I$ the unit matrix. Associated with P^n, a transition probability function is given by $p^n(i, E) = \sum_{j \in E} p_{ij}^n$ for any $i \in S$ and $E \subset S$.

From the mean ergodic theorem of the MC, there exists $p_{ij}^* = \lim_{n \to \infty}(\sum_{k=1}^n p_{ij}^k)/n$ for any i, $j \in S$. Denote by P^* the matrix whose (i, j)-component is given by p_{ij}^*. Then we have $PP^* = P^*P = P^* = (P^*)^2$. Associated with P^*, let $p^*(i, E) = \liminf_{n \to \infty}(\sum_{k=1}^n p^k(i, E))/n$ for any $i \in S$ and $E \subset S$.

We need notation related to the ergodic theory of the MC's. A MC separates S into the subsets R of *recurrent* states and T of *transient* states. Then R consists of recurrent classes $\{E_a; a \in \mathcal{A}\}$. Each recurrent class E_a is *positive* or *null*. A recurrent class E_a is divided into *periodic* classes ${}_aC_1, {}_aC_2, \ldots, {}_aC_{d_a}$. We call d_a a period of E_a. If $d_a = 1$, we call E_a *aperiodic*. A subset $E \subset S$ is called *closed* if $p(i, E) = 1$ for any $i \in E$. If E is closed, we can consider a sub-MC $\{p_{ij}; i,j \in E\}$. Then we have d_a-step sub-MC's $\{p_{ij}^{d_a}; i,j \in {}_aC_r\}$ for $r = 1,2,\ldots, d_a$. Since $\{p_{ij}^{d_a}; i,j \in {}_aC_r\}$ is aperiodic, we have $\lim_{n\to\infty} p_{ij}^{nd_a} = p_{ij}^{d_a *} = p_j^{d_a *}$ for $i \in {}_aC_r$. We define $p^{d_a *}(i,E)$ from $p_{ij}^{d_a *}$ by lim inf similar to $p^*(i,E)$. If $d = \text{l.c.m.}\{d_a\ ;\ a \in \mathcal{A}\}$ exists, let $\{p_{ij}^d; i,j \in S\}$ be a d-step MC. Note that $p^{d*}(i,E)$ is also defined for $i \in S$ and $E \subset S$ and that $p^{d*}(i,E) = p^{d_a *}(i,E)$ for any $i \in {}_aC_\alpha$ and $E \subset S$.

Let $A = (a_{ij}; i,j \in S)$ be a countable-dimensional matrix. Define the norm of A by $\| A \| = \sup\{\sum_{j\in S} \mid a_{ij} \mid; i \in S\}$. If $\| A \| < \infty$, A is called bounded. We denote by $\sigma(S)$ the set of bounded matrices. Note that if A, $B \in \sigma(S)$, the matrix product $AB \in \sigma(S)$ and that $\| AB \| \leqq \| A \| \| B \|$.

If $p^*(i,E) = \sum_{j\in E} p_{ij}^*$ for any $E \subset S$, then $P^n - P^* \in \sigma(S)$. Denoting $S(i)_n^+ = \{j \in S;\ p(f)_{ij}^n \geqq p_{ij}^*\}$, we have

$$\| P^n - P^* \| = 2 \sup_{i\in S}\{p^n(i, S(i)_n^+) - p^*(i, S(i)_n^+)\}.$$

For the sub-MC's $\{p_{ij}; i,j \in R\}$, $\{p_{ij}; i,j \in E_a\}$ and $\{p_{ij}^{d_a}; i,j \in {}_aC_\alpha\}$, we denote their norms by $\| \cdot \|_R$, $\| \cdot \|_a$ and $\| \cdot \|_{a,\alpha}$, respectively. For instance, $\| P^{nd_a} - P^* \|_{a,\alpha} = \sup\{\sum_{j\in {}_aC_\alpha} \mid p_{ij}^{nd_a} - p_{ij}^* \mid; i \in {}_aC_\alpha\}$. Note that $\| A \|_{a,\alpha} \leqq \| A \|_a \leqq \| A \|_R \leqq \| A \|$ for any $A \in \sigma(S)$.

We sometimes denote $p^n(i,E) - p^*(i,E)$ by $(p^n - p^*)(i,E)$ for simplicity. For $A = (a_{ij})$, $B = (b_{ij}) \in \sigma(S)$, We often denote $\sum_{j\in S} a_{ij} b(j,E)$ by $ab(i,E)$.

We are mainly concerned with a space of MC's, denoting $\{P(f); f \in F\}$ with a parameter f in a non-empty set F. The definitions and notation above are applied to each $P(f)$. Then, we should denote P^n, p_{ij}^*, P^*, R, T, E_a, $\mathcal{A}$, ${}_aC_r$, d_a, $p^{d_a *}$, p^{d*}, etc, for any $f \in F$ by $P(f)^n$, $p(f)_{ij}^*$, $P(f)^*$, $R(f)$, $T(f)$, $E_a(f)$, $\mathcal{A}(f)$, ${}_aC(f)_r$, $d(f)_a$, $p(f)^{d_a *}$, $p(f)^{d*}$, etc, respectively. But those notation is bothersome, then we often omit the notation f. For instance, we denote ${}_{R(f)}p(f)^{nd(f)_a}(i, E \cap E(f)_a)$ by ${}_R p(f)^{nd_a}(i, E \cap E_a)$.

2. Conditions and results.

In this section we set up our conditions and state our results. The proofs of them are explained in the next section.

Condition (A). (A-1) For a family of MC's, there exist a positive integer N, a positive number δ and a family of subsets $K(f) \subset S$ satisfying the following conditions.

(A-1-1) $p(f)^N(i, K(f)) \geqq \delta > 0$ for any $i \in S$ and $f \in F$, where N and δ are constants independent of $i \in S$ and $f \in F$.

(A-1-2) $K(f) \cap T(f) = \emptyset$ for any $f \in F$.

(A-1-3) Each ${}_aC(f)_\alpha \cap K(f)$ consists of only one state for $a \in \mathcal{A}$, α=0, 1,..., $d_a - 1$ and $f \in F$.

(A-2) $\sup\{d(f)_a;\ a \in \mathcal{A}(f), f \in F\} < \infty$.

From (A-2) there is a least common multiplier of all $d(f)_a$. We denote it by d. According to Hordijk[6], we say a MC satisfies the *Doèblin condition,* if there are an integer $N \geqq 1$, a number $\delta > 0$ and a finite set K such that $p^N(i, K) \geqq \delta$ for any $i \in S$. A family of MC's satisfies *simultaneous Doèblin conditions (sim D)* if evry MC satisfies the Doèblin condition and if N, δ, K are taken independent of $f \in F$. Condition (A) does not assume $P(f)$ satisfies Doèblin condotion, since $K(f)$ is not necessarily finite.

Condition (B). (B-1) The same condition with (A-2).

(B-2) There exist a positive integer N and $\varepsilon_0 > 0$ such that

$$| p(f)^{Nd}(i, E) - p(f)^{d*}(i, E) | \leqq \frac{1}{2}(1 - \varepsilon_0)$$

for all $i \in S$, $E \subset S$ and $f \in F$.

Condition (C). There exists a constant $B > 0$ such that

$$| \sum_{k=1}^{n} \{p(f)^k(i, E) - p(f)^*(i, E)\} | \leqq B$$

for all $i \in S$, $E \subset S$, $f \in F$ and $n = 1, 2, \ldots$.

Condition (D). There exists $\lim_{n\to\infty}(\sum_{k=1}^{n} p(f)^k(i, E))/n = p(f)^*(i, E)$ uniformly in $i \in S$, $E \subset S$, and in $f \in F$.

It is shown that Condition (D) implies (A-2) and $\sum_{j\in S} p_{ij}^* = 1$ for any $i \in S$ and $f \in F$. Next Theorem 1 gives the relations of Conditions (A) ~ (D). Corollary 2 gives equivalent conditions to (B). If d=1, the (a-3) of Corollary 2 is reduced to to the condition used by [14].

Theorem 1. *Condition (A) implies (B), (B) implies (C), and (C) implies (D).*

Corollary 2. *Supposes Condition (B-1) holds. Then,*

(a) following conditions are equivalent.

(a-1) There are positive integers M, N and a positive number ε_0 satisfying that $\sup_{i\in T} p(f)^M(i, T(f)) \leqq 1 - \varepsilon_0$ for all $f \in F$ and that $\| P(f)^{Nd} - P(f)^{d*} \|_\alpha \leqq 1 - \varepsilon_0$ for all $f \in F$, $a \in \mathcal{A}(f)$ and $\alpha = 0, 1, \ldots, d_a - 1$.

(a-2) Condition (B-2).

(a-3) There exists $\lim_{n\to\infty} \| P(f)^{nd} - P(f)^{d*} \| = 0$ uniformly in $f \in F$.

(b) If (a) holds, there exists a positive number B such that $\sum_{n=0}^{\infty} \| \sum_{r=0}^{d-1} p(f)^r$

$(P(f)^{nd} - P(f)^*) \|\leqq C$, *converging uniformly in* $f \in F$.

We give a Laurent series expansion which is seen in [8]. Let any $f \in F$ be fixed. Then, we omit the notation f. For $0 \leqq \beta < 1$, let

$$H_\beta(i, E) = \sum_{n=0}^{\infty}(p^n(i, E) - p^*(i, E))\beta^n.$$

If Condition (C) holds, there exists $H(i, E) = \lim_{\beta\to 1+} H_\beta(i, E)$ for any $i \in S$ and $E \subset S$. We denote $H_{ij} = H(i, \{j\})$ for $j \in S$. Then it holds that $H(i, E) = \sum_{j\in E} H_{ij}$ for any $i \in S$ and $E \subset S$ and that $\| H \|\leqq 2B$. Letting $\beta = 1/(1+\rho)$ for $0 < \beta < 1$, denote βH_β by H_ρ and H by $H_0 = \lim_{\rho\to 0+} H_\rho$.

Theorem 3. *Suppose Condition (C) holds.*

(a) Let $\rho_0 > 0$ *be* $\rho_0 \| H \|< 1$. *Then,*

$$\sum_{n=0}^{\infty} \beta^{n+1} p^n(i, E) = \frac{1}{\rho} p^*(i, E) + \sum_{n=0}^{\infty}(-\rho)^n H^{n+1}(i, E) \tag{1}$$

for any $i \in S$ *and* $E \subset S$ *and* $0 < \rho \leqq \rho_0$.

(b) For $0 \leqq \rho < \infty$, H_ρ *satisfies uniqly in* $\sigma(S)$ *that* $(I - \beta P)H_\rho = H_\rho(I - \beta P)$ $= \beta(I - \beta P^*)$ *and that* $P^* H_\rho = H_\rho P^* = O$, *where* O *is the zero matrix whose components are all zero.*

In order to show continuity of the coefficients on the right of (1), we assume F is a subset of a metric space (X, dist), where dist(f, g) is a metric from $(f, g) \in X \times X$ to the set of real numbers. We study two kinds of continuity conditions for the transition probabilities.

Condition(I). For any i, $j \in S$, $p(f)_{ij}$ is continious in $F \in F$.

Condition(II). The Markov matrix $P(f)$ is continuous on F with respect to the norm $\|\cdot\|$. Namely, $\lim_{g \to f} \| P(g) - P(f) \| = 0$ for any $f \in F$.

It is clear that Condition (II) implies Condition (I). Really, Condition (II) asserts uniform continuity of $p(f)(i, E)$ in $i \in S$ and $E \subset S$.

Lemma 4. *(a) Suppose Condition (I) holds. Then $p(f)^n(i, E)$ is continuous uniformly in $E \subset S$ for any $i \in S$ and $n = 1, 2, \ldots$.*

(b) Suppose Condition (II) holds. Then $P(f)^n$ is continuous on F with respect to the norm.

Theorem 5. *Suppose Condition (D) holds.*

(a) If $\{P(f); f \in F\}$ satisfies Condition (I), then $p(f)^*(i, E)$ is continuous on F uniformly in $E \subset S$ for any $i \in S$.

(b) If $\{P(f); f \in F\}$ satisfies Condition (II), then $p(f)^*$ is continuous on F with respect to the norm.

Theorem 6. *Suppose Condition (C) holds.*

(a) If $\{P(f); f \in F\}$ satisfies Condition (I), then $H(f)^n(i, E)$ is continuous on F uniformly in $E \subset S$ for any $i \in S$ and $n = 1, 2, \ldots$.

(b) If $\{P(f); f \in F\}$ satisfies Condition (II), then $H(f)^n$ is continuous on F with respect tothe norm for $n = 1, 2, \ldots$.

Note that if Condition (C) and Condition (I) (or (II)) hold, then, any MC $P(f)$ has the Laurnt series of which all the coefficients are continuous in the sense of (a) (or (b)) in Theorem 5 and 6.

3. Lemmas.

This section gives Lemmas and short comments for obtaining the results in section

2.

Lemma 1 of [8], which shows equivalent conditions to $\sum_{j\in S} p^*_{ij} = 1$ for $i \in S$, play a fundamental role in our arguments. Next Lemma 7 gives an ergodic theorem. Lemma 8 considers properties of the norm.

Lemma 7. *Suppose (A-2) and $\sum_{j\in S} p^*_{ij} = 1$ for any $i \in S$. Then,*

(a) $p^*(i,E) = \frac{1}{d}\sum_{r=0}^{d-1} p^r p^{d*}(i,E)$ *and*

(b) $\sum_{r=0}^{d-1} p^r (p^{nd} - p^*)(i,E) = \sum_{r=0}^{d-1} p^r (p^{nd} - p^{d*})(i,E)$ *for any $i \in S$ and $E \subset S$,*

Lemma 8. *Suppose the same conditions with Lemma 7. Then, we have following equalities and inequalities.*

(a) $\| \sum_{r=0}^{d-1} P^r (P^{nd} - P^*) \| \leq d \| P^{nd} - P^{d*} \|$ *for $n = 0, 1, 2, \ldots$.*

(b) $\| P^{nd} - P^{d*} \|_a = \max\{\| P^{nd} - P^{d*} \|_{a,\alpha}\,; \alpha = 0, 1, \ldots, d_a - 1\}$ *and*

$\| P^{nd} - P^{d*} \|_R = \sup\{\| P^{nd} - P^{d*} \|_a\,; a \in \mathcal{A}\}$.

(c) *For any $n > m \geq 1$, it holds that*

$$\| P^{nd} - P^{d*} \| \leq 4 \sup_{i\in T} p^{md}(i,T) + \{\sup_{i\in T}(1 - p^{md}(i,T))\} \| P^{(n-m)d} - P^{d*} \|_R .$$

Next Lemma 9 is an application of Case (b) in Doob[4, p197] to the space of MC's. It can be proved in the same manner as that of [4]. (See also the proof of Theorem 2.4 in A. Federgruen, A. Hordijk and H. C. Tijms[5] and Lemma 2 of [8].)

Lemma 9. *Suppose Condition (A) holds. Then there exist a positive integer N and $\varepsilon_0 > 0$ such that $\| P(f)^{Nd} - P(f)^{d*} \|_R \leq 1 - \varepsilon_0$ for any $f \in F$. denotes the norm $\| \cdot \|$ taken on the restricted d-step MC $\{p^d_{ij}\,; i,j \in R\}$.*

Lemmas 7, 8, 9 are used in the proofs of Theorem 1, 3 and Corollary 2. We omit the proof of Theorem 3, since it is lengthy. Theorem 5(a) is obtained using the analytic theorem that if a sequence of continuous functions uniformly converges, the limit function is continuous. To prove the continuity of $H(f)$ in Theorem 6(a), we need next Lemma 10 which follows from Theorem 5.

Lemma 10. *Suppose Condition (C) holds. Then,*

$$H(i,E)=\lim_{N\to\infty}\frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=0}^{n}\{p^k(i,E)-p^*(i,E)\}.$$

The convergence is uniform both in $i\in S$ and in $E\subset S$.

The statements (b) in Theorem 5, 6 can be proved similar to (a) in them.

4. Examples.

Even when S is finite, there exists a space of MC's where $P(f)^*$ is discontinuous. Among them the following Example 1, 2 have simple structure. Example 1 is seen in Schweitzer[11] and Example 2 is seen in Wijngaard[13] and Zijm[14].

Example 1. Let $S=\{1,2\}$ and $F=[0,1]\times[0,1]$, where $[0,1]$ is the closed interval of real numbers. For any $f=(p,q)\in F$, let $p(f)_{11}=1-p$, $p(f)_{12}=p$, $p(f)_{21}=q$, $p(f)_{22}=1-q$.

For $0=(0,0)\in F$, $P(0)=P(0)^*=I$ and $H(0)=O$ the zero-matrix. If $p+q>0$, the MC makes one recurrent class for each $f\in F$. Note that any subset $K(f)\subset S$ does not satisfy both Condition (A-1-2) and (A-1-1). If $p+q>0$, we have $p(f)^*_{i1}=q/(p+q)$ and $p(f)^*_{i2}=p/(p+q)$ for $i=1,2$. It is clear that $\lim_{n\to\infty}P(f_n)^*\neq P(0)^*$ for any sequence $\{f_n\}$ converging to $0\in F$. On the other-hand, it always holds that $\|\sum_{k=0}^{n}(P(f)^k-P(f)^*)\|\leqq B(f)$ for any $f\in F$, when S is finite. We have $H(f)_{11}=-H(f)_{12}=p/(p+q)^2$ and $H(f)_{21}=-H(f)_{22}=-q/(p+q)^2$. Let $f_n=(p_n,q_n)$

be a sequence converging 0 satisfying $p_n \geqq q_n$. Then we have $\lim_{n\to\infty} H(f_n)_{11} = -\lim_{n\to\infty} H(f_n)_{12} = \infty$, which contradicts to Condition (C).

Example 2. Let $S = \{1,2,3\}$ and $F = [0,\ (-1+\sqrt{5})/2)]$, the closed interval. For any $f = p \in F$, let $p(f)_{11} = 1 - p - p^2$, $p(f)_{12} = p$, $p(f)_{13} = p^2$, $p(f)_{22} = p(f)_{33} = 1$ and $p(f)_{21} = p(f)_{23} = p(f)_{31} = p(f)_{32} = 0$.

If $f = p = 0$, $P(0) = P(0)^* = I$ and $H(0) = 0$. If $f = p \neq 0$, the states 2, 3 make recurrent classes respectively and the state 1 is a transient. Note that any subset $K(f) \subset S$ does not satisfies both Condition (A-1-3) and (A-1-1). When $f = p \neq 0$, we have $p(f)^*_{12} = 1/(1+p)$, $p(f)^*_{13} = p/(1+p)$ and $p(f)^*_{22} = p(f)^*_{33} = 1$ and other $p(f)_{ij}$ are all zero. We have $H(f)_{11} = 1/p(1+p)$, $H(f)_{12} = -1/p(1+p)^2$, $H(f)_{13} = -1/(1+p)^2$ and other components are all zero. Then $\lim_{p\to 0} H(f)_{11} = -\lim_{p\to 0} H(f)_{12} = \infty$, which contradicts to Condition (C).

REFERENCES

[1] K. A. Chung, *Markov Chains with Stationary Transition Probabilities*, Springer-verlag Berlin, 1960.

[2] R. Dekker and A. Hordijk, *Average, sensitive and Blackwell optimal policies in denumerable Markov decision chains with unbounded rewards*, Mathematics of Operations Reseach **13** (1988), 395-420.

[3] H. M. Dietz and V. Nollau, *Markov Decision Problems with Countable State Spaces*, Academie-Verlag, Berlin, 1983.

[4] J. Doob, *Stochastic Processes*, Wiley New York, 1953.

[5] A. Federgruen, A. Hordijk and H. C. Tijms, *Recurrence conditions in denumerable state Markov decision processes* (1978), Dynamic Programming and Its Applications,ed. by M. L. Puterman, 3–22.

[6] A. Hordijk, *Dynamic Programming and Potential Theory* (1974), Math. Centre Tracts No. 51, Mathematisch Centrum, Amsterdam.

[7] Y. Kadota, *Countable state Markovian decision processes under the Doeblin-Conditions*, Bull. Math. Statist. **19** (1979), 85–94.

[8] 門田 良信 (Kadota), *Kasan jyootai marukohu rensa no rooran tenkai ni tuite*, Suuri kaiseki kenkyuu syo koukyuu roku **798** (1992), 180–190. (in Japanese).

[9] J. G. Kemeny, J. L. Snell and A. W. Knapp, *Denumerable Markov Chains*, Springer-verlag New York Inc, 1976.

[10] B. L. Miller and A. F. Jr. Veinott, *Discrete dynamic programming with a small interest rate*, Ann. Math. Statist. **40** (1969), 366–370.

[11] P. J. Schweitzer, *Perturbation theory and finite Markov chains*, J. Appl. Prob. **5** (1968), 401–413.

[12] A. F. Jr. Veinott, *On discrete dynamic programming with sensitive discount optimality criteria*, Ann. Math. Statist. **40** (1969), 1635–1660.

[13] J. Wijngaard, *Stationary Markovian decision problems and perturbation theory of quasi-compact linear operators*, Math. Oper. Res. **2** (1977), 91–102.

[14] H. Zijm, *The optimality equations in multichain denumerable state Markov decision poocesses with the average cost criterion: the bounded cost cace*, Stat. & Decisions. **3** (1985), 143–165.