

A POTENTIAL OF FUZZY RELATIONS WITH A LINEAR STRUCTURE : THE CONTRACTIVE CASE

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Abstract: In this paper, by introducing a linear structure for fuzzy relations, we develop potential theory for fuzzy relations on the positive orthant R_+^n of an n -dimensional Euclidean space. Under the assumption of contraction and compactness, we derive the existence of a potential and its characterization by fuzzy relational equation. In one-dimensional unimodal case, a potential is given explicitly. Also, a numerical example is shown to illustrate our approaches.

Keyword: Fuzzy potential; linear structure; fuzzy relation; fuzzy relational equation; contraction property.

1. Introduction and notation.

The convergence theorems for a sequence of fuzzy sets defined successively by fuzzy relations are firstly found in Bellman and Zadeh[1]. They considered a sequence of fuzzy numbers in a finite space by solving a fuzzy linear equation written in the matrix form. Kurano, et al.[3], by introducing a contractive property, has studied a limit of a sequence of fuzzy sets defined by the dynamic fuzzy system with a compact state space. These works would present the basic tool for the limiting behaviour of fuzzy sets and the prerequisite for fuzzy potential theory.

Our objective is to develop potential theory for fuzzy relations on the positive orthant R_+^n of an n -dimensional Euclidean space. We introduce a linear structure for fuzzy relations. In this paper, under the assumption of contraction and compactness, we prove the existence theorem of a potential, which is characterized by fuzzy relational equation. Moreover, we deal with one-dimensional unimodal case, where a potential is given explicitly. A numerical example is shown to illustrate our approaches.

Let n be a positive integer. R^n denotes an n -dimensional Euclidean space with a basis $\{e_1, e_2, \dots, e_n\}$. For $x, y \in R^n$, the sum of x and y and the product of a scalar λ and x are written by $x + y$ and λx respectively. Let w_i be an orthogonal projection from R^n to the subspace $\{\lambda e_i \mid \lambda \in R^1\}$:

$$x = \sum_{i=1}^n w_i(x)e_i \quad \text{for } x \in R^n.$$

We put a norm $\|\cdot\|$ and a metric d by $\|x\| = (\sum_{i=1}^n (w_i(x))^2)^{\frac{1}{2}}$ and $d(x, y) = \|x - y\|$ for $x, y \in R^n$. A positive orthant of R^n , $R_+^n := \{x \in R^n \mid w_i(x) \geq 0 \text{ for all } i = 1, 2, \dots, n\}$, is a closed convex cone and (R_+^n, d) is a complete separable metric space.

Throughout this paper, we denote a fuzzy set on R_+^n by its membership function $\tilde{s} : R_+^n \rightarrow [0, 1]$. For the details, refer to Novák[7] and Zadeh[8]. For any fuzzy set \tilde{s} on R_+^n and $\alpha \in [0, 1]$, its α -cut is defined by

$$\tilde{s}_\alpha := \{x \in R_+^n \mid \tilde{s}(x) \geq \alpha\} \quad (\alpha > 0) \quad \text{and} \quad \tilde{s}_0 := \text{cl}\{x \in R_+^n \mid \tilde{s}(x) > 0\},$$

where cl means the closure of a set. We call \tilde{s}_0 a support of \tilde{s} .

The linear structure of fuzzy sets are defined as follows: For fuzzy sets \tilde{s}, \tilde{r} and a scalar λ ,

$$(\tilde{s} + \tilde{r})(x) := \sup_{y+z=x; y,z \in R_+^n} \{\tilde{s}(y) \wedge \tilde{r}(z)\} \quad \text{and} \quad (\lambda\tilde{s})(x) := \begin{cases} \tilde{s}(x/\lambda) & \text{if } \lambda > 0, \\ I_{\{0\}}(x) & \text{if } \lambda = 0, \end{cases} \quad x \in R_+^n,$$

where $\lambda \wedge \mu = \min\{\lambda, \mu\}$ for $\lambda, \mu \in R_+^1$, and $I_A(\cdot)$ is the classical characteristic function for any ordinary subset A of R_+^n . Then the corresponding α -cut representations are as follows (see Madan, et al.[5]) :

$$(\tilde{s} + \tilde{r})_\alpha = \tilde{s}_\alpha + \tilde{r}_\alpha \quad \text{for fuzzy sets } \tilde{s}, \tilde{r} \text{ and } \alpha \in [0, 1], \quad (1.1)$$

$$(\lambda\tilde{s})_\alpha = \lambda\tilde{s}_\alpha \quad \text{for a fuzzy set } \tilde{s}, \lambda \in R_+^1 \text{ and } \alpha \in [0, 1], \quad (1.2)$$

where $A + B = \{x + y \mid x \in A \text{ and } y \in B\}$ ($A, B \subset R_+^n$) and $\lambda A = \{\lambda x \mid x \in A\}$ ($A \subset R_+^n, \lambda \in R_+^1$). We also represent the finite sum of fuzzy sets $\{\tilde{s}_i\}_{i=0}^k$ and the 'formal' infinite sum of fuzzy sets $\{\tilde{s}_i\}_{i=0}^\infty$ respectively by

$$\sum_{i=0}^k \tilde{s}_i := \tilde{s}_1 + \tilde{s}_2 + \cdots + \tilde{s}_k \quad \text{and} \quad \sum_{i=0}^\infty \tilde{s}_i := \tilde{s}_1 + \tilde{s}_2 + \cdots.$$

The finite sum is well-defined, however the infinite sum is precisely defined later.

Let $\mathcal{F}(R_+^n)$ be the set of all the fuzzy sets \tilde{s} on R_+^n , being upper semi-continuous, which have a compact support and satisfy $\sup_{x \in R_+^n} \tilde{s}(x) = 1$. Let $\tilde{q} : R_+^n \times R_+^n \rightarrow [0, 1]$ be a fuzzy relation on R_+^n . We assume that

(i) \tilde{q} is continuous on $R_+^n \times R_+^n \setminus \{(0, 0)\}$ and

(ii) $\tilde{q}(\cdot, y) \in \mathcal{F}(R_+^n)$ for all $y \in R_+^n$.

Note that \tilde{q} has the discontinuity at $(0, 0)$. The reason will be seen in the following section(see Fig. 2). We introduce the following transition. For any $\tilde{p} \in \mathcal{F}(R_+^n)$,

$$\tilde{q}(\tilde{p})(x) = \sup_{y \in R_+^n} \{\tilde{q}(x, y) \wedge \tilde{p}(y)\}, \quad x \in R_+^n. \quad (1.3)$$

Using (1.3), we can inductively define the sequence of fuzzy sets $\{\tilde{q}^k(\tilde{p})\}_{k=0}^\infty$ by

$$\tilde{q}^0(\tilde{p}) = \tilde{p} \quad \text{and} \quad \tilde{q}^k(\tilde{p}) = \tilde{q}(\tilde{q}^{k-1}(\tilde{p})), \quad k = 1, 2, \dots. \quad (1.4)$$

Then if the 'formal' infinite sum

$$Q(\tilde{p}) := \sum_{k=0}^\infty \tilde{q}^k(\tilde{p}) \quad (1.5)$$

converges (in the sense of the definition of Section 3), we call it a fuzzy potential or simply a potential.

In Section 2 fundamental assumptions are discussed in order to develop potential theory. The existence theorem of a potential and its characterization by a fuzzy relational equation are obtained in Section 3. In Section 4 a potential is given explicitly in one-dimensional unimodal case and numerical example is also given.

2. Assumptions and preliminary lemmas.

We call a fuzzy set $\tilde{s} \in \mathcal{F}(R_+^n)$ to be convex if its α -cut \tilde{s}_α is a convex set for all $\alpha \in [0, 1]$. From now on, we assume the convexity and linearity concerning the fuzzy relation \tilde{q} in the following Assumption A.

Assumption A (Convexity and linearity). The fuzzy relation \tilde{q} satisfies the conditions (A1) and (A2) :

(A1) $\tilde{q}(\cdot, y)$ is convex for all $y \in R_+^n$,

(A2) $\tilde{q}(\cdot, \lambda y + \mu z) = \lambda \tilde{q}(\cdot, y) + \mu \tilde{q}(\cdot, z)$ for all $y, z \in R_+^n$ and $\lambda, \mu \in R_+^1$.

We note that Assumption A2 is equivalent to the following representation of α -cuts ;

$$\tilde{q}_\alpha(\lambda y + \mu z) = \lambda \tilde{q}_\alpha(y) + \mu \tilde{q}_\alpha(z) \quad \text{for all } y, z \in R_+^n, \lambda, \mu \in R_+^1, \text{ and } \alpha \in [0, 1], \quad (2.1)$$

where

$$\tilde{q}_\alpha(y) := \{x \in R_+^n \mid \tilde{q}(x, y) \geq \alpha\}, \quad \alpha \in [0, 1].$$

Lemma 2.1. For a convex subset A of R_+^n , it holds that

$$\lambda A + \mu A = (\lambda + \mu)A \quad \text{for all } \lambda, \mu \in R_+^1. \quad (2.2)$$

Theorem 2.1. Suppose that a convex fuzzy set $\tilde{q}(\cdot, e_i) \in \mathcal{F}(R_+^n)$ is given for each basis $e_i (i = 1, 2, \dots)$. Let the fuzzy relation \tilde{q} on R_+^n by

$$\tilde{q}(\cdot, y) := \sum_{i=1}^n w_i(y) \tilde{q}(\cdot, e_i), \quad y \in R_+^n. \quad (2.3)$$

Then this satisfies Assumption A.

We note by (2.3) that

$$\tilde{q}(\cdot, 0) = I_{\{0\}}, \quad (2.4)$$

which is a natural consequence of the linearity in the Assumption A2.

In this paper, we deal with the contraction case in fuzzy sets on R_+^n with a compact support, so that we need the following assumption, which is assumed from now on.

Assumption B (Contraction) . The fuzzy relation \tilde{q} satisfies the condition:

$$\tilde{q}_\alpha(e_i) \subset \{x \in R_+^n \mid \|x\| < \frac{1}{n}\} \quad \text{for all } i = 1, 2, \dots \text{ and } \alpha \in [0, 1].$$

Later it is seen that Assumption *B* for \tilde{q} corresponds to the contraction property introduced by Kurano, et al.[3].

Let $\mathcal{C}(R_+^n)$ be the collection of all the non-empty closed subsets of R_+^n . For $\alpha \in [0, 1]$, we define the map $\tilde{q}_\alpha : \mathcal{C}(R_+^n) \rightarrow \mathcal{C}(R_+^n)$ by

$$\tilde{q}_\alpha(D) := \begin{cases} \{x \in R_+^n \mid \tilde{q}(x, y) \geq \alpha \text{ for some } y \in D\} & \text{for } \alpha > 0, D \in \mathcal{C}(R_+^n) \\ \text{cl}\{x \in R_+^n \mid \tilde{q}(x, y) > 0 \text{ for some } y \in D\} & \text{for } \alpha = 0, D \in \mathcal{C}(R_+^n). \end{cases} \quad (2.5)$$

From the continuity of \tilde{q} , \tilde{q}_α maps any closed subset of R_+^n to a closed subset of R_+^n . So, the definition of \tilde{q}_α is consistent. From (2.2) and (2.5) we note that $\tilde{q}_\alpha(y) = \tilde{q}_\alpha(\{y\})$ for $y \in R_+^n$ and that

$$\tilde{q}_\alpha(D) = \bigcup_{y \in D} \tilde{q}_\alpha(y) \quad \text{for all } D \in \mathcal{C}(R_+^n). \quad (2.6)$$

Here, using the map \tilde{q}_α , for each $k = 0, 1, 2, \dots$, we define the map $\tilde{q}_\alpha^k : \mathcal{C}(R_+^n) \rightarrow \mathcal{C}(R_+^n)$ by

$$\tilde{q}_\alpha^0 \text{ is an identity map and } \tilde{q}_\alpha^k = \tilde{q}_\alpha(\tilde{q}_\alpha^{k-1}) \quad (k = 1, 2, \dots).$$

Then we obtain the following lemma (see Kurano, et al.[3, Lemma 1]).

Lemma 2.2. For $\tilde{p} \in \mathcal{F}(R_+^n)$, it holds that

$$(\tilde{q}^k(\tilde{p}))_\alpha = \tilde{q}_\alpha^k(\tilde{p}_\alpha) \quad \text{for all } k = 0, 1, 2, \dots \text{ and } \alpha \in [0, 1],$$

where $\tilde{q}^k(\tilde{p}), k = 0, 1, 2, \dots$, are defined by (1.4).

Next we shall show the contraction property for the fuzzy relation \tilde{q} . For any positive number a , we define a rectangle $J(a)$ of R_+^n :

$$J(a) := \{x = \sum_{i=1}^n w_i(x)e_i \in R_+^n \mid 0 \leq w_i(x) \leq a\}.$$

Then $(J(a), d)$ is a compact metric space. Further let $\mathcal{C}(J(a))$ be the collection of all the closed subsets of $J(a)$. Then $(\mathcal{C}(J(a)), \rho)$ becomes a compact metric space with a Hausdorff metric ρ (see [2,4]). The following lemma holds for the map \tilde{q}_α .

Lemma 2.3. For $\alpha \in [0, 1]$ and real $a > 0$, the map \tilde{q}_α satisfies the following (i) – (iii) and there exists a real number $\beta (0 < \beta < 1)$ satisfying (ii) and (iii), which is independent of α and a :

- (i) $\tilde{q}_\alpha(D) \in \mathcal{C}(J(a))$ for all $D \in \mathcal{C}(J(a))$,
- (ii) $\tilde{q}_\alpha(J(a)) \subset J(\beta a)$,
- (iii) $\rho(\tilde{q}_\alpha(A), \tilde{q}_\alpha(B)) \leq \beta \rho(A, B)$ for all $A, B \in \mathcal{C}(J(a))$.

3. Main results.

In this section we shall show the existence of potentials defined by (1.5). Further we shall give its characterization by a fuzzy relational equation.

First we shall define the convergence in $\mathcal{F}(R_+^n)$.

Definition (see [3,6]). For $\{\tilde{s}_k\}_{k=0}^\infty \subset \mathcal{F}(R_+^n)$ and $\tilde{s} \in \mathcal{F}(R_+^n)$,

$$\lim_{k \rightarrow \infty} \tilde{s}_k = \tilde{s}$$

means that there exists $a > 0$ satisfying $\tilde{s}_{k,0} \subset J(a) (k = 0, 1, 2, \dots)$ and $\tilde{s}_0 \subset J(a)$ and that

$$\sup_{\alpha \in [0,1]} \rho(\tilde{s}_{k,\alpha}, \tilde{s}_\alpha) \rightarrow 0 \quad (k \rightarrow \infty).$$

The following lemma, which can be easily check(c.f. [3,7]), is needed to get results.

Lemma 3.1. Let \tilde{s} be a fuzzy set on R_+^n . Then $\tilde{s} \in \mathcal{F}(R_+^n)$ if and only if \tilde{s} satisfies the following three conditions (i) – (iii) :

- (i) There exists a positive number a satisfying $\tilde{s}_\alpha \in \mathcal{C}(J(a))$ for all $\alpha \in [0, 1]$.
- (ii) $\lim_{\alpha' \uparrow \alpha} \tilde{s}_{\alpha'} = \tilde{s}_\alpha$.
- (iii) $\tilde{s}_1 \neq \phi$.

The following theorem holds for the sequence of fuzzy sets $\{\tilde{q}^k(\tilde{p})\}_{k=0}^\infty$, which is defined by (1.4).

Theorem 3.1. Let $\tilde{p} \in \mathcal{F}(R_+^n)$. Then

- (i) $\tilde{q}^k(\tilde{p}) \in \mathcal{F}(R_+^n)$ for $k = 0, 1, 2, \dots$,
- (ii) $\lim_{k \rightarrow \infty} \tilde{q}^k(\tilde{p}) = I_{\{0\}}$.

Lemma 3.2. For $a > 0$ and $A, B, C, D \in \mathcal{C}(J(a))$, the following (i) and (ii) hold :

- (i) $\rho(A, A + B) \leq \max_{x \in B} d(0, x)$.
- (ii) $\rho(A + B, C + D) \leq \rho(A, C) + \rho(B, D)$.

Lemma 3.3. For any $a > 0$ with $J(a) \supset \tilde{p}_0$, it holds that

$$J(\beta^k a) \supset (\tilde{q}^k(\tilde{p}))_\alpha, \quad k = 0, 1, 2, \dots \text{ and } \alpha \in [0, 1],$$

where $\beta (0 < \beta < 1)$ is the number given by Lemma 2.3.

Lemma 3.4. For $\tilde{s}, \tilde{r} \in \mathcal{F}(R_+^n)$ and $\lambda, \mu \geq 0$, it holds that

$$\tilde{q}^k(\lambda \tilde{s} + \mu \tilde{r}) = \lambda \tilde{q}^k(\tilde{s}) + \mu \tilde{q}^k(\tilde{r}), \quad k = 0, 1, 2, \dots$$

Lemma 3.5. Let $\{\tilde{p}_k\}_{k=0}^\infty \subset \mathcal{F}(R_+^n)$ and $\tilde{p} \in \mathcal{F}(R_+^n)$. If $\lim_{k \rightarrow \infty} \tilde{p}_k = \tilde{p}$, then $\lim_{k \rightarrow \infty} \tilde{q}(\tilde{p}_k) = \tilde{q}(\tilde{p})$.

Lemma 3.6 (c.f.[6]). For a real number $a > 0$ we suppose that a family of subsets $\{A_\alpha \mid \alpha \in [0, 1]\} \subset \mathcal{C}(J(a))$ satisfies the following conditions (i) – (iii) :

- (i) $A_\alpha \subset A_{\alpha'}$ for $0 \leq \alpha' < \alpha \leq 1$.
- (ii) $\lim_{\alpha' \uparrow \alpha} A_{\alpha'} = A_\alpha$ for $\alpha \in (0, 1]$.
- (iii) $A_1 \neq \phi$.

Then

$$\tilde{s}(x) := \sup_{\alpha \in [0,1]} \{\alpha \wedge I_{A_\alpha}(x)\}, \quad x \in R_+^n$$

satisfies $\tilde{s} \in \mathcal{F}(R_+^n)$ and $\tilde{s}_\alpha = A_\alpha$ for all $\alpha \in [0, 1]$.

Now, we shall show the main theorem.

Theorem 3.2. For any $\tilde{p} \in \mathcal{F}(R_+^n)$, the potential

$$Q(\tilde{p}) := \sum_{k=0}^{\infty} \tilde{q}^k(\tilde{p})$$

converges in $\mathcal{F}(R_+^n)$ and has the next linearity :

$$Q(\lambda \tilde{s} + \mu \tilde{r}) = \lambda Q(\tilde{s}) + \mu Q(\tilde{r}) \quad \text{for all } \tilde{s}, \tilde{r} \in \mathcal{F}(R_+^n) \text{ and } \lambda, \mu \geq 0. \quad (3.3)$$

Nextly we give a fundamental characterization for potentials.

Theorem 3.3. For any $\tilde{p} \in \mathcal{F}(R_+^n)$, the fuzzy relational equation

$$\tilde{u} = \tilde{p} + \tilde{q}(\tilde{u}) \quad (3.10)$$

has a unique solution $\tilde{u} = Q(\tilde{p}) \in (\mathcal{F}(R_+^n))$, which is the potential of \tilde{p} .

4. One-dimensional unimodal case.

In this section we investigate fuzzy potentials of unimodal fuzzy numbers on $R_+ := R_+^1$ by applying the results in Section 3.

Definition. For a fuzzy number $\tilde{s} \in \mathcal{F}(R_+)$, \tilde{s} is called unimodal if its α -cuts \tilde{s}_α are bounded closed intervals, say $[\min \tilde{s}_\alpha, \max \tilde{s}_\alpha] \subset R_+$ for all $\alpha \in [0, 1]$.

Let $\mathcal{F}_u(R_+)$ be the set of all the unimodal fuzzy numbers on R_+ . In one-dimensional unimodal case, Assumption *A* and *B* for the fuzzy relation \tilde{q} are reduced to the following two conditions (*C1*) and (*C2*) :

$$(C1) \quad \tilde{q}(\cdot, 1) \in \mathcal{F}_u(R_+).$$

$$(C2) \quad \tilde{q}_\alpha(1) \subset [0, 1) \text{ for all } \alpha \in [0, 1].$$

From Condition *C1*, $\tilde{q}(\cdot, 1)$ is a bounded closed interval of R_+ ($\alpha \in [0, 1]$). We write $\tilde{q}_\alpha(1) = [\min \tilde{q}_\alpha(1), \max \tilde{q}_\alpha(1)]$ ($\alpha \in [0, 1]$) for convenience. Further from Condition *C2*, we obtain $0 \leq \min \tilde{q}_\alpha(1) \leq \max \tilde{q}_\alpha(1) < 1$ for all $\alpha \in [0, 1]$. Then we have the following lemma.

Lemma 4.1. For $\tilde{p} \in \mathcal{F}_u(R_+)$, it holds that $\tilde{q}(\tilde{p}) \in \mathcal{F}_u(R_+)$.

We shall obtain the following results, applying Theorem 3.2 and 3.3 to one-dimensional unimodal case.

Corollary 4.1. For $\tilde{p} \in \mathcal{F}_u(R_+)$, (i) and (ii) hold:

- (i) The potential $\tilde{u} := Q(\tilde{p}) \in \mathcal{F}_u(R_+)$.
- (ii) Its α -cut $\tilde{u}_\alpha = [\min \tilde{u}_\alpha, \max \tilde{u}_\alpha]$ is given by

$$\min \tilde{u}_\alpha = \frac{\min \tilde{p}_\alpha}{1 - \min \tilde{q}_\alpha(1)} \quad \text{and} \quad \max \tilde{u}_\alpha = \frac{\max \tilde{p}_\alpha}{1 - \max \tilde{q}_\alpha(1)} \quad \text{for } \alpha \in [0, 1]. \quad (4.1)$$

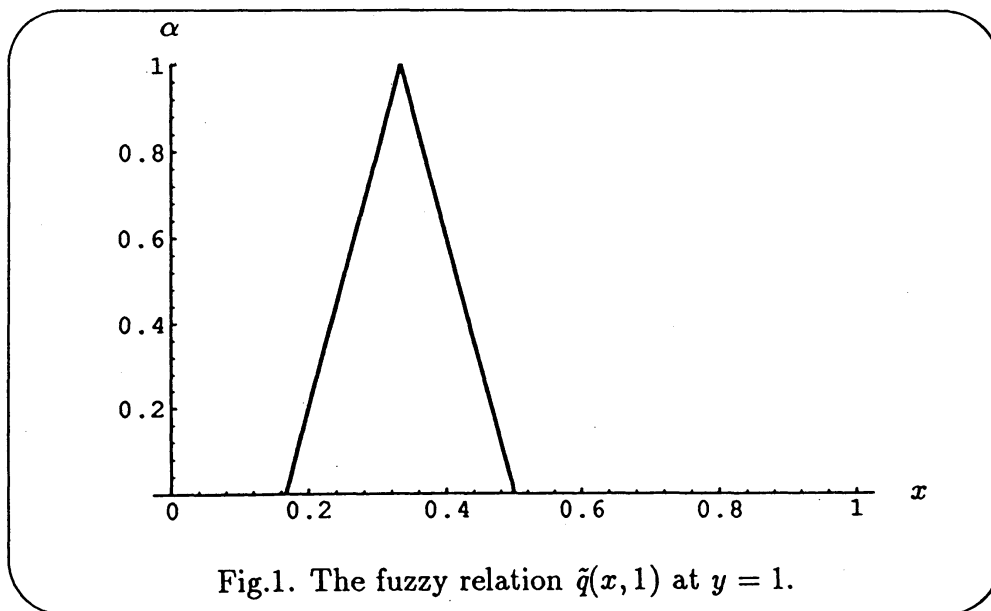
Finally we shall give a one-dimensional unimodal numerical example. Let the fuzzy set $\tilde{q}(\cdot, 1)$ given by

$$\tilde{q}(x, 1) = \begin{cases} 1 - 2|3x - 1|, & 1/6 \leq x \leq 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

Then, from Theorem 2.1, the fuzzy relation \tilde{q} on R_+ , which satisfies Assumption A and B, is given by

$$\tilde{q}(x, y) = \begin{cases} \tilde{q}(x/y, 1), & x \geq 0 \text{ and } y > 0 \\ I_{\{0\}}(x), & x \geq 0 \text{ and } y = 0. \end{cases}$$

The fuzzy set $\tilde{q}(x, 1)$ and the relation $\tilde{q}(x, y)$ are shown respectively in Figure 1 and 2.



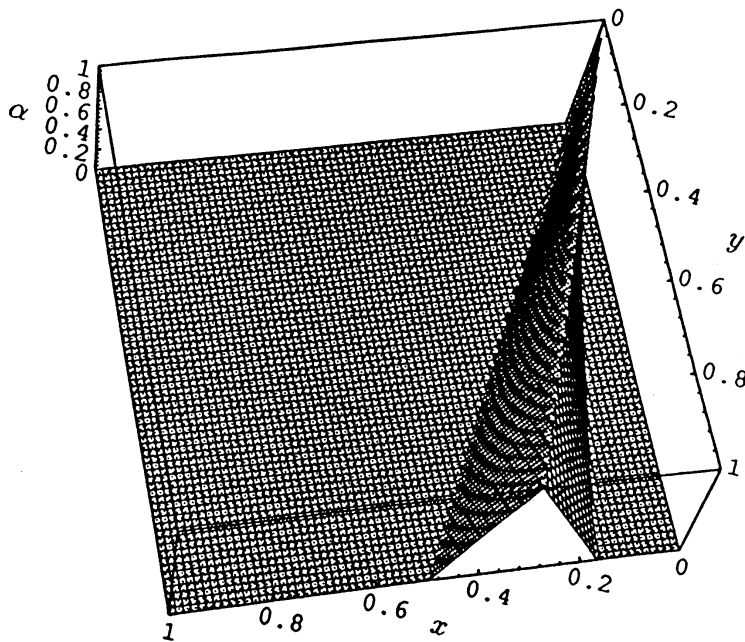


Fig.2. The fuzzy relation $\tilde{q}(x, y)$.

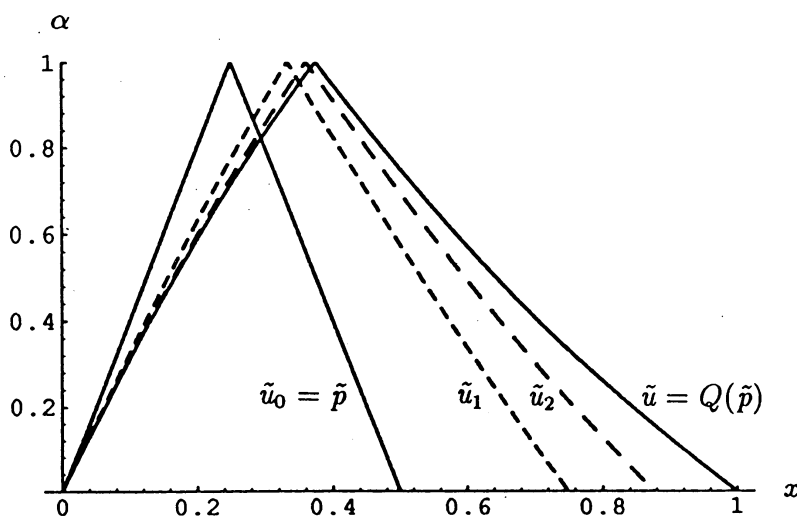


Fig.3. The sequence $\{\tilde{u}_i\}_{i=0}^{\infty}$ and the potential \tilde{u} .

Here we put a fuzzy set \tilde{p} by

$$\tilde{p}(x) = \begin{cases} 1 - |4x - 1|, & 0 \leq x \leq 1/2 \\ 0, & \text{otherwise.} \end{cases}$$

Calculating the fuzzy potential $\tilde{u} = Q(\tilde{p})$ of \tilde{p} by (4.1), we obtain

$$\tilde{u}(x) = \begin{cases} \min \{10x/(3+2x), -6(1+x)/(3+2x)\}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

We put $\tilde{u}_l := \sum_{k=0}^l \tilde{q}_k(\tilde{p}) = Q_l(\tilde{p})$ ($l = 0, 1, 2, \dots$). Figure 3 shows the convergence of the sequence of fuzzy states $\{\tilde{u}_l\}_{l=0}^{\infty}$ to the fuzzy potential \tilde{u} , which is the unique solution of (3.10).

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