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SOME TOPICS IN INVERSE PROBLEMS.

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ABSTRACT

Some statements of inverse problems for differential equations are considered. Results of investigation of inverse kinematic problem, problem of acoustic waves propagation velocity finding, determining of a density, elastic modules for an elastisity halfspace and of electromagnetic parameters in the Earth's crust are described.

1. To the present time it is known many applications of inverse problems to different field of the sciences and technology. Ones of the most interesting inverse problems are problems of variable coefficients finding for differential equations. This interest is related to the fact, that the coefficients of the differential equations contain an information about properties a medium in which taken place some physical processes. Sufficiently often one can find the coefficients from observations this processes in an available domain. As the interesting parameters of a medium can be its density, elasticity modules, conductivity, electric and magnetic permeabilities. With mathematical point of view the inverse problems of finding the coefficients of differential equations are nonlinear problems as rule. But sometimes linear problems arise in applications. Such problem consist in determining of some models of external sources, which unknown, from knowledge some information about a solution of the differential equation. Finding of a density of radioactive elements in Earth's crust, sources of electromagnetic, acoustic and seismic waves are the same problems. With mathematical point of view this problems are the problems of right-hand sides of differential equations determining.

What kind of questions are interested under investigation of inverse problems? At first, what observations have we do in order to find the interested parameters uniquely. At second, how one can reconstructs the parameters from data and what is stability estimate of this reconstruction. And at third, what are necessary and sufficient conditions on data for a solvability of the inverse problem.

2. With historical point of view one of the earlist inverse problems is the inverse kinematic problem. It arose in geophysics. The base problem of geophysics is study of Earth's inner structure from observations on its surface for different geophysical fields. If one assumes that Earth is an elastic body, then velocities of elastic waves are very interesting object for studying. Seismic stations measure oscillations of points on the surface. Using the measurements one can describes the picture of seismic waves propagation along Earth's surface from a localized source. Therefore one can finds time-travels of the seismic waves between the point of the source application and any point of the surface. The problem is: find the velocities of the seismic waves inside of Earth from this data. Thus one comes to the following mathematical formulation of the inverse problem.

Inverse kinematic problem. Let D is domain boundered by $S = \{x| \ x \in \mathbf{R}^3, \ |x| = 1\}$ and inside D are propageted signals with a velocity $c(x) > 0$, given time-travels $\tau(x, x^0)$ of the signals for $\forall x^0 \in S, \ \forall x \in S$, find $c(x)$ in D.

The geometrical formulation the problem is: in domain D is given Riemann's metric by formula

$$d\tau = n(x) \left(\sum_{i=1}^{3} dx_i^2 \right)^{1/2}, \tag{1}$$

where $d\tau$ is an element of a length in Riemann's metric and $n(x) = 1/c(x)$, given distances in the metric (1) between arbitrary points $x, x^0 \in S$, find the metric (1).

It is well known that $\tau(x, x^0)$ satisfy to eiconal equation

$$|\nabla_x \tau|^2 = n^2(x) \ , x \in D, \tag{2}$$

and the condition

$$\tau(x, x^0) = O(|x - x^0|) \ as \ x \to x^0. \tag{3}$$

With the other side $\tau(x, x^0)$ solves the problem of the minimizing of time-travels between points x, x^0, that is

$$\tau(x, x^0) = \inf_L \int_{L(x,x^0)} n ds = \int_{\Gamma(x,x^0)} n ds, \tag{4}$$

for any sufficiently smooth curves $L(x, x^0)$ joined the points x, x^0. The curve $\Gamma(x, x^0)$ which realized this infimum is called by the ray or the geodesic line corresponding to the points x, x^0.

The problem of finding $n(x)$ in D from data $\tau(x, x^0)$ for $\forall x, x^0 \in S$ is a nonlinear problem, since the geodesic lines $\Gamma(x, x^0)$ depend from $n(x)$.

Let us consider a linear problem related to the nonliner problem. Let the function $n(x)$ is represented in the form

$$n = n_0(x) \ + \ n_1(x), \tag{5}$$

where $n_0(x) > 0$ is a known function and $n_1(x)$ is a small function in the comparison with $n_0(x)$. If

$$\tau_0(x, x^0) = \inf_L \int_{L(x,x^0)} n_0 ds = \int_{\Gamma(x,x^0)} n_0 ds,$$

then one can easy shows that

$$\tau - \tau_0 := \tau_1(x, x^0) = \int_{\Gamma_0(x,x^0)} n_1 ds \tag{6}$$

with the accuracy to the term of the order n_1^2.

Indeed, the eiconal equation

$$|\nabla(\tau_0 + \tau_1)|^2 = (n_0 + n_1)^2, \tag{7}$$

by using the equation

$$|\nabla \tau_0|^2 = n_0^2, \tag{8}$$

and neglected by the terms n_1^2 , $|\nabla \tau_1|^2$ implies

$$\nabla \tau_0 \cdot \nabla \tau_1 = n_0 n_1. \tag{9}$$

Since $\nu_0 = \nabla \tau_0 / n_0$ is an unit vector of the tangent to $\Gamma_0(x, x^0)$ at the point x, the equation (9) can be written in the form

$$d\tau_1 / ds = n_1, \tag{10}$$

or, by integrating along the $\Gamma_0(x, x^0)$, in the form (6).

The problem of finding the function n_1 from data $\tau_1(x, x^0)$ for $\forall x, x^0 \in S$ is linear problem of integral geometry. If $n_0 = const$, then $\Gamma_0(x, x^0)$ are straight lines and this problem is well known problem of tomography.

Let us consider some results related to the inverse kinematic problem. In the beginning our century this problem was studied by G. Gerglotz and E. Wiechert in the assuming of the spherically symmetry of the physical model, that is $n = n_0(r), r = |x|$. Under the condition

$$\frac{d}{dr}(rn_0(r)) > 0, \tag{11}$$

they showed, that the function $n_0(r)$ can be determined uniquely from data $\tau(x, x^0)$ for $\forall x \in S$ and fixed $x^0 \in S$. A formula for $n_0(r)$ was obtained. 60 years later M. Gerver and V. Markushevich studied this problem without of the supposition (11) and described the set of solutions of the inverse kinematic problem [1]. In the last case the inverse problem is not solved uniquely.

First result related to multidimensional inverse kinematic problem was obtained by V. Romanov [2] in an linear approximation under assumption (5), where $n_0(x) = n_0(r)$, $r = |x|$, and the condition (11) was fulfilled. An uniqueness theorem and an algorithm of $n_1(x)$ reconstruction were obtained.For nonlinear problem an uniqueness theorem was prowed by Ju. Anikonov [3] in a class of analytic functions. In the paper [4] for the nonlinear problem was shown that if $n_0 = const$, then there exist a constant $\epsilon > 0$ such that any function $n(x)$ satisfied the condition

$$\|n - n_0\|_{C^4(D)} < \epsilon, \tag{12}$$

was determined uniquely from data $\tau(x, x^0)$ for $\forall\ x, x^0 \in S$.

R. Mukhometov proved [5] an uniqueness theorem for two-dimensional case under the assumption that the field of the rays regularly in D that is each pair points $x, x^0\ \in D$ can be jointed by one ray $\Gamma(x, x^0)$ only. Some later this result was established for three-dimensional case by R. Mukhometov, V. Romanov [6], I. Bernstein, M. Gerver [7] and G. Beilkin [8], independently. The stability estimates for the problem are

$$\|n^{(1)} - n^{(2)}\|^2_{L^2(D)} \leq \frac{1}{4\pi} \int\limits_S \int\limits_S |\nabla_{x,\xi}[\tau^{(1)}(x,\xi) - \tau^{(2)}(x,\xi)]|^2\ dS_x dS_\xi, \tag{13}$$

for the two-dimensional case and

$$\|n^{(1)} - n^{(2)}\|^2_{L^2(D)} \leq C \int\limits_S \int\limits_S \{ |\nabla_{x,\xi}[\ \tau^{(1)}(x,\xi) - \tau^{(2)}(x,\xi)\ ]|^2 +$$

$$+ |x-\xi|^2 \sum_{i,j=1}^{3} [\ \frac{d^2}{dx_i d\xi_j}(\tau^{(1)}(x,\xi) - \tau^2(x,\xi))\ ]^2 \} \frac{1}{|x-\xi|} dS_x dS_\xi \tag{14}$$

for the three-dimensional case. In the formulas (13),(14) the function $\tau^{(k)}$ corresponds to the function $n^{(k)}$, k=1,2. The estimates (13),(14) are differed sufficiently strongly.

There are some numerical algorithms for solving the inverse kinematic problem (see for example the papers [9], [10]).

3. The condition of the regularity of rays is not fulfilled in applications very often. Therefore it is interesting to consider more full dynamical information about the physical processes of the seismic waves propagation.

At first for simplicity we consider an inverse problem for the acoustic equation. Let the equation

$$n^2(x)\ u_{tt}\ = \Delta u,\ x \in \mathbf{R}^3_+, t\in \mathbf{R}, \tag{15}$$

is considered in the region $\mathbf{R}^3_+ = \{x \mid x_3 > 0\}$ under conditions

$$u|_{t<0} = 0, \frac{\partial u}{\partial x_3}|_{x_3=0} = \delta(x_1 - x_1^0, x_2 - x_2^0, t), \tag{16}$$

where $x^0 = (x_1^0, x_2^0, 0) \in S := \partial\mathbf{R}^3_+$. The solution of the problem (15), (16) is given for $x \in S$:

$$u|_S =\ f(x,t,x^0);\ x, x^0 \in S,\ t \in \mathbf{R}. \tag{17}$$

The inverse problem is: from data (17) determine $n(x)$.

If $n(x) = n(x_3)$ this problem studied by V. Romanov in the papers [11],[12]. It is proved that $n(x_3) \in C^2(\mathbf{R}^3_+)$ is determined from the function $f(x^0,t,x^0) := F(t)$, x^0 fixed, uniquely.

If the function $n(x)$ is represented in the form (5) and $n_0(x) = n_0 = const$, then the linearized inverse problem can be reduced to different problems of integral geometry. In particularly, if $x^0 \in S$ is a fixed point and data are

$$u|_S = g_1(x,t), \;\; x \in S, t \in \mathbf{R}, \tag{18}$$

then the problem of integral geometry is: find $n_1(x)$, if given the integrals:

$$\int\limits_{S(x,x^0,t)} |x^0 - \xi|^2 n_1(\xi) d\omega_\xi = G_1(x,t), \;\; x \in S, t \geq |x - x^0|/n_0, \tag{19}$$

where $S(x, x^0, t)$ are the ellipsoids

$$S(x,x^0,t) = \{\xi |\; |x^0 - \xi| + |\xi - x| = n_0 t\}$$

and $d\omega_\xi = sin\theta d\theta d\phi$ is an element of the unit sphere with the center at the point x^0. In the case if data are

$$u|_{x=x^0} = g_2(x^0,t), \;\; x^0 \in S, t \in \mathbf{R}, \tag{20}$$

and x^0 runs S then the inverse problem can be reduce to the problem of finding the function $n_1(x)$ from knowledge the integrals

$$\int\limits_{|\xi - x^0| = r} n_1(\xi) d\omega_\xi = G_2(x^0, r), \;\; x^0 \in S, r > 0, \tag{21}$$

for various spheres with the center at the points x^0 and the different radiuses.

The functions G_1 and G_2 in the formulas (19), (21) can be expressed from g_1 and g_2 correspondly.

In the both cases the function $n_1(x) \in C(\mathbf{R}^3_+)$ is determined from data uniquely (see for example the book [13]). There are some algorithms for solving this problem and stability estimates.

For nonlinear inverse problem an uniqueness theorem can be obtained in the special class of functions $n(x)$ represented in the form

$$n(x) = \sum_{k=1}^{N} \phi_k(x_1, x_2) \psi_k(x_3), \tag{22}$$

where $\phi_k, \psi_k,\; k = \; 1, 2, ..., N$, are some smooth functions of their variables, N is a finite number. There is also a local theorem of the uniqueness and solvability in the class of analytic functions with respect to variables x_1, x_2 and smooth in x_3 [14].

An interesting algorithm for solving this problem in the general case was developed by M. Belishev in the papers [15], [16].

4. Let us formulate an inverse problem for the elastic body equations. Let the region $\mathbf{R}^3_+ = \{x|\ x_3 > 0\}$ is filled by an isotropic elastic medium with a density ρ and elastic modules λ, μ. The system equations can be written in the following form

$$\rho \frac{\partial^2 u_k}{\partial t^2} = \sum_{j=1}^{3} \frac{\partial \sigma_{kj}}{\partial x_j}, k = 1,2,3,\ x \in \mathbf{R}^3_+, t \in \mathbf{R}, \tag{23}$$

where $u_k,\ k = 1,2,3,$ are the components of the displacements vector $\mathbf{u}$ and $\sigma_{kj},\ k,j = 1,2,3,$ are components of the strength tensor which are related to the components of the vector $\mathbf{u}$ by the formula

$$\sigma_{kj} = \delta_{kj} \lambda div \mathbf{u} + \ \mu(\frac{\partial u_k}{\partial x_j} + \ \frac{\partial u_j}{\partial x_k}). \tag{24}$$

Here δ_{kj} is Kronecker's symbol. Let us define the equations (23), (24) by the following initial and boundary conditions:

$$\mathbf{u}|_{t<0} = 0, \tag{25}$$

$$\sigma_{3j}|_{x_3=0} = f_j \delta(x_1 - x_1^0, x_2 - x_2^0, t),\ j = 1,2,3, \tag{26}$$

where $(x_1^0, x_2^0, 0) := x^0 \in S,\ f_j \neq 0,\ j = 1,2,3,$ are fixed numbers.

The inverse problem is: given the functions $u_k(x,t,x^0), k = 1,2,3,$ for $\forall x, x^0 \in S = \{x|\ x_3 = 0\}$ and $t \in \mathbf{R}$; find the coefficients ρ, λ, μ.

At the first time this problem was considered by A. Alekseev [17] for the case when ρ, λ, μ were depended from one variable x_3 only. He reduced the problem by using Fourier-Bessel transform to some spectral inverse problems for ordinary differential equations of a second order. A some later the one-dimensional problem was studied by A. Blagoveschenskii [18] in a dynamical variant by reducing this problem to some inverse problems for scalar partial differential equations of the hyperbolic type. He proved a local existence and uniqueness theorem and proposed an algorithm for its solution. The three-dimensional inverse problem was considered by V. Romanov [19],[13] in the linear approximation. It was proved that data of the inverse problem are determined the coefficients ρ, λ, μ uniquely, if they are represented in the form:

$$\rho = \rho_0(x_3) + \rho_1(x),\ \ \lambda = \lambda_0(x_3) + \lambda_1(x),\ \ \mu = \mu_0(x_3) + \mu_1(x),$$

where $\rho_0(x_3), \lambda_0(x_3), \mu_0(x_3)$ were given functions and $\rho_1(x), \lambda_1(x), \mu_1(x)$ were small functions. Under the condition of the rays regularity the nonlinear problem was studied by V. Yakhno [20] by using of the inverse kinematic problem for longitudinal and transversal waves velocities determinating and using some dynamical information for finding ρ. Recently an interesting result was obtained by G. Nakamura and G. Ulman [21]. They proved an uniqueness theorem in the small, when unknown coefficients ρ, λ, μ were near the coefficients ρ_0, λ_0, μ_0, correspondly.

5. There is interesting problems in the geoelectrics. The process of electromagnetic waves propagation is described by Maxwell's system equations

$$\nabla \times H = \epsilon\frac{\partial E}{\partial t} + \sigma E,\ \nabla \times E = -\mu\frac{\partial H}{\partial t},\ x \in \mathbf{R^3},\ t \in \mathbf{R}. \tag{27}$$

If one assumes a geophysical model in which Earth's surface is the plane $S = \{x|\ x_3 = 0\}$ and the region $\mathbf{R}^3_-$ is filled by air and the region $\mathbf{R}^3_+$ is filled by ground, then one can considers the electromagnetic field which is produced by the external current source applied at the point $x^0 \in S$ at the moment $t = 0$. It corresponds the following initial and boundary conditions:

$$(\ H,\ E\ )_{t<0} = 0, \tag{28}$$

$$[\ H_k\ ]_{x_3=0} = g_k\delta(x_1 - x_1^0, x_2 - x_2^0, t), [\ E_k\ ]_{x_3=0} = 0,\ \ k = 1, 2. \tag{29}$$

Here symbol $[\ \cdot\ ]_{x_3=0}$ means a difference of function limiting values computing at the points of S outwards the domain $\mathbf{R}^3_+$ and $\mathbf{R}^3_-$, respectevely. One can observes some components of the electromagnetic field on the interface S. For example let us given

$$E_k|_{x_3=0} = f_k(x, t, x^0),\ \ k = 1, 2,\ \ x,\ x^0 \in S,\ \ t \in \mathbf{R}. \tag{30}$$

The problem is: find ϵ, μ, σ in $\mathbf{R}^3_+$ from data (30), if $\epsilon = \epsilon^-, \mu = \mu^-, \sigma = \sigma^-$ in $\mathbf{R}^3_-$ and $\epsilon^- > 0,\ \ \mu^- > 0,\ \ \sigma^- \geq 0$ are given constants.

Some result which related to this problem are described by V. Romanov, S. Kabanikhin in the book [22]. It is shown that the one-dimensional, the linearized three-dimensional problems and some variants of the nonlinear problem have the uniqueness solution, were obtained same stability estimates and were given some numerical methods for solution this problems. It is interesting to remark that the inverse problem has not an uniqueness solution always. For example it is proved in the paper [23], if one considers the simplest model of an anisotropic medium which is described by the following diagonal matrixes

$$\epsilon = diag(\epsilon_1, \epsilon_1, \epsilon_2), \ \ \mu = diag(\mu_1, \mu_1, \mu_2) \tag{31}$$

and $\epsilon = \epsilon(x_3), \mu = \mu(x_3), \sigma = 0$, then the electromagnetic field on the interface S depends from three nonlinear combinations of the functions $\epsilon_1, \epsilon_2, \mu_1, \mu_2$ only. The combinations are

$$\frac{\epsilon_1}{\mu_1}|_{x_3=g(s)}, \ \epsilon_1\mu_2|_{x_3=g(s)}, \ \epsilon_2\mu_1|_{x_3=g(s)}, \tag{32}$$

where $g(s)$ is determined by the equation:

$$s = \int\limits_0^{g(s)} [\epsilon_1(z)\mu_1(z)]^{1/2} dz.$$

Thus it is impossible to find the four function $\epsilon_1, \epsilon_2, \mu_1, \mu_2$ from any observations on S. Nevertheless the combinations (32) can be determined from data (30) uniquely.

There are many interesting inverse problems in different applications: in the industry, mechanics, medicine, biology, ecology and the other fields.

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