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Numerical Analysis for Motion of a Surface by Its Mean Curvature

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In the research fields of applied sciences like physics, engineering and biology, it is important to track the evolution (motion) of a surface, such as the interface between two kind of materials or two different phases of a certain kind of material. The problem how to track and compute the motion of a surface with a curvature-dependent speed is usually a key point in the studies. Here, we introduce a class of difference schemes for computing the evolution of a surface moved by its mean curvature, with the *level surface approach* via *viscosity solutions.* The difference scheme is proved to be stable with respect to the maximum norm, which follows from the maximum principle. The method also applies to a generalized model, so-called generalized mean curvature flow equations (see [CGG1]).

§1. A stable difference scheme for the mean curvature flow equation.

We consider the Cauchy problem of the mean curvature flow equation

$$\text{(E)} \qquad u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right), \quad (t,x) \in Q = (0,\infty) \times \mathbb{R}^N$$

$$\text{(IV)} \qquad u(0,x) = u_0, \quad x \in \mathbb{R}^N.$$

Let $u = u(t,x)$ be a continuous viscosity solution of (E)–(IV) which takes a negative constant for large $|x|$. If for each $t \in (0,\infty)$ there is a bounded open set $D(t)$ in $\mathbb{R}^N$ such that $u(t,x) > 0$ for $x \in D(t)$ and $u(t,x) < 0$ for $x \notin \overline{D(t)}$, then the 0-level set $\Gamma(t) = \{x;\ u(t,x) = 0\}$ of $u(t,x)$ determines a closed surface which moves with a speed $\mathbf{V} = (n-1)\mathbf{H}$ at each point $x \in \Gamma(t)$ where $\mathbf{H}(t,x)$ is the mean curvature vector at $x \in \Gamma(t)$, provided that $\nabla u \neq 0$ on $\Gamma(t)$. The global existence and uniqueness of the viscosity solution to (E)–(IV) have been proved by Chen, Giga & Goto [CGG1] and Evans & Spruck [ES]. And more important thing is that the level set $\Gamma(t)$ is uniquely determined by its initial data $\Gamma(0)$ which is independent of the choice of its *defining function* u_0, provided that u_0 is bounded, continuous and $u_0 > 0$ for $x \in D(0)$, $u_0 < 0$ for $x \notin \overline{D(0)}$ and $\Gamma(0) = \{x;\ u_0(x) = 0\}$. Moreover, in [CGG1] these results are proved for a general model (generalized mean curvature equations).

Here, we discuss the difference methods for computing $u(t,x)$, the viscosity solution of (E)–(IV). To overcome the difficulty of taking 0 value in the denominator which will cause errors in computers and stop the computation, we introduce a parameter $\delta > 0$ and consider the difference approximation of a modified equation

$$(\mathrm{E}_\delta) \qquad u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{(|\nabla u|^\sigma + \delta)^{1/\sigma}}\right), \quad (t,x) \in Q = (0,\infty) \times \mathbb{R}^N$$

with the same initial value (IV). Here, $\sigma \geqslant 1$ is fixed. We can show that the viscosity solution of (E_δ) with (IV) tends to that of (E) when $\delta \to 0$. Thus, it is reasonable to deal with the computation of the solution of (E_δ) as an approximation of the solution of (E) with the same initial value (IV), for a sufficiently small $\delta > 0$ (say, $\delta = 10^{-50}$).

Now we introduce our difference scheme for (E_δ), and for simplicity we interpret the scheme here for the case $N = 2$. Our difference equation for (E_δ) is given by

$$(1) \qquad \begin{aligned} &\frac{u_{jk}^{n+1} - u_{jk}^n}{\tau} = g_{jk}^n \sum_{i=1}^{2} \mathrm{D}_i \left(\frac{\mathrm{D}_i u_{jk}^{n+\theta}}{((g_{jk}^n)^\sigma + \delta)^{1/\sigma}}\right), \\ &\qquad\qquad j,k = 0, \pm 1, \pm 2, \cdots;\ n = 0,1,2,\cdots; \\ &u_{jk}^0 = u_0(x_j, y_k), \quad j,k = 0, \pm 1, \pm 2, \cdots. \end{aligned}$$

Here, several notations have been introduced as below. Denoting by x and y the spatial variables in $\mathbb{R}^2$, we use x_j and y_k for the spatial coordinates of the net points.

Notation:

$\tau \equiv \Delta t > 0$: increment of the time variable t;

$t_n = n\tau$: nth time step;

$h_1,\ h_2$: mesh sizes of x and y directions, respectively;

$(x_j, y_k) = (jh_1, kh_2)$: net point in $\mathbb{R}^2$, $j,k = 0, \pm 1, \pm 2, \cdots$;

u_{jk}^n: value of the difference solution approximate to $u(t_n, x_j, y_k)$;

$D_1 u^n_{jk} = (u^n_{j+\frac12,k} - u^n_{j-\frac12,k})/h_1$, $D_2 u^n_{jk} = (u^n_{j,k+\frac12} - u^n_{j,k-\frac12})/h_2$: the approximations to $u_x(t_n, x_j, y_k)$ and $u_y(t_n, x_j, y_k)$ by central difference approach, respectively;

$g^n_{jk} \equiv g(D\, u^n_{jk})$: discretization of $|\nabla u|$ at (t_n, x_j, y_k), which is chosen positive definite for $\{D^{\pm}_i u^n_{jk};\ i = 1, \dots, N\}$, where D^+ and D^- denote the standard forward and backward differences, respectively. For instance, we may take $g^n_{jk} = \big(\frac14 \sum_i^N \big(|D^+_i u^n_{jk}| + |D^-_i u^n_{jk}|\big)^2\big)^{1/2}$, or $g^n_{jk} = \max_{1 \leqslant i \leqslant N} \{|D^+_i u^n_{jk}|, |D^-_i u^n_{jk}|\}$.

The notation u^θ denotes $\theta u^{n+1} + (1-\theta)u^n$ for a fixed parameter $\theta \in [0,1]$, and the difference equation (1) is explicit for u^{n+1} if $\theta = 0$, while implicit if $0 < \theta \leqslant 1$.

We can prove a sufficient condition for the L^∞ stability of maximum principle type for the difference scheme (1), as the following

Theorem 1. *The difference scheme (1) is stable in the sense of $\|u^n\|_\infty \leqslant \|u^0\|_\infty$ if either $\theta = 1$ or $4\tau(1/h_1^2 + 1/h_2^2) \leqslant 1/(1-\theta)$ when $0 \leqslant \theta < 1$, where $\|u^n\|_\infty = \sup_{j,k} |u^n_{jk}|$.*

It is sometimes convenient and economic to deal with a surface of rotation in a lower dimension space. Here, to compute the motion of a surface with axisymmetry in $\mathbb{R}^3$, i.e., surfaces of rotation, we rewrite (E) into

$$(\mathrm{E}_r) \qquad u_t = |\tilde\nabla u|\,\widetilde{\mathrm{div}}\Big(\frac{\tilde\nabla u}{|\tilde\nabla u|}\Big) + \frac{1}{r}u_r, \quad (t,r,z) \in Q = (0,\infty) \times (0,\infty) \times \mathbb{R}$$

where $r = \sqrt{x^2 + y^2}$ for $(x,y,z) \in \mathbb{R}^3$, and the differential operators $\tilde\nabla$ and $\widetilde{\mathrm{div}}$ are those with respect to $(r,z) \in \mathbb{R}^2$.

For this equation, our difference scheme is constructed as

$$(2) \qquad \begin{aligned}
&\frac{u^{n+1}_{0k} - u^n_{0k}}{\tau} = g^n_{0k} \sum_{p,q=1}^{2} D_i \left(\frac{D_i u^{n+\theta}_{0k}}{((g^n_{0k})^\sigma + \delta)^{1/\sigma}} \right) + \frac{2(u^{n+\theta}_{1k} - u^{n+\theta}_{0k})}{h_1^2},\\
&\qquad k = 0, \pm1, \pm2, \cdots;\ n = 0, 1, 2, \cdots;\\
&\frac{u^{n+1}_{jk} - u^n_{jk}}{\tau} = g^n_{jk} \sum_{p,q=1}^{2} D_i \left(\frac{D_i u^{n+\theta}_{jk}}{((g^n_{jk})^\sigma + \delta)^{1/\sigma}} \right) + \frac{1}{r_j} \cdot \frac{u^{n+\theta}_{j+1,k} - u^{n+\theta}_{jk}}{h_1},\\
&\qquad k = 0, \pm1, \pm2, \cdots;\ j = 1, 2, \cdots;\ n = 0, 1, 2, \cdots;\\
&u^0_{jk} = u_0(x_j, y_k), \quad k = 0, \pm1, \pm2, \cdots;\ j = 0, 1, 2, \cdots.
\end{aligned}$$

where u^n_{jk} is the approximation of $u(t_n, r_j, z_k)$, $(r_j, z_k) = (jh_1, kh_2)$, and D_1 and D_2 denote the difference operators for ∂_r and ∂_z. In this case, the stability condition is given by

Theorem 2. *The difference equation (2) is stable if either*

$$\theta = 1 \quad \text{or} \quad 6\frac{\tau}{h_1^2} + 4\frac{\tau}{h_2^2} \leqslant \frac{1}{1-\theta} \quad \text{when} \quad 0 \leqslant \theta < 1.$$

§2. A stable difference scheme for the generalized mean curvature flow equation.

With the above-mentioned methods, we can construct a stable difference scheme for the so-called generalized mean curvature flow equation

$$(\mathrm{E}') \qquad u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) + \nu|\nabla u|, \quad (t,x) \in Q = (0,\infty) \times \mathbb{R}^N$$

where ν is a constant (see [CGG1]).

The difference scheme for (E′) is constructed by the following two parts:

(1) the first part of the scheme is constructed as that for (E) in the previous section;

(2) the second part of the scheme is constructed in the way of any kind of stable difference scheme with monotonicity for the Hamilton-Jacobi equation $u_t = \nu|\nabla u|$, such as Lax-Friedrichs scheme, Godunov scheme, etc. (see, for example [CL]).

Then, we can show that the obtained difference scheme is stable if the value of τ/h_i^2 $(i = 1, 2)$ are taken sufficiently small ([CGH]).

§3. Remarks.

1. It is important to note that the stability conditions do not depend on $\delta > 0$ and $\sigma \geqslant 1$.

2. If g^n_{jk} is not positive definite for $\{\mathrm{D}^{\pm} u^n_{jk}\}$, then the difference solution may not converge to the solution of (E_δ), nor to that of (E) when $\delta \to 0$.

3. Osher and Sethian discussed some difference schemes, constructed in a different way, with level surface approach ([OS], [S]). They computed several interesting examples including the torus and dumbbells without discussing the fundamental theory such as stability, etc. In [S], an example of unstable computation of a torus was presented with a quite large Δt but no condition for the stability was given there.

4. In [OS] and [S], the axisymmetric surfaces are computed under the rectilinear coordinates instead of the cylindrical coordinates.

With our stable difference schemes and level surface approach, we investigated motions of several typical surfaces, including the shrink of a torus (surface of a doughnut) and the break of a dumbbell. With this method we can track motions of a surface even after the time when a singularity occurs.

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