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Discrepancy inequalities of Erdös-Turán and of LeVeque

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1 Introduction

In [1],Erdös and Turán obtained an upper bound of the discrepancy in terms of the exponential sums. In [3] LeVeque obtained another upper bound different from Erdös-Turán inequality. In [4], Nakajima and Ohkubo obtained an upper bound of the generalized discrepancies using a new Weyl's criterion for general distribution (mod 1).

In this paper, we generalize the discrepancy of weighted uniform distributed sequence to that of the regular summation method. We obtain upper and lower bounds of the generalized discrepancy (called V-discrepancy) of a weighted uniform distributed sequence having a continuous distribution function and related results. Our methods mainly owe to [2:Theorem 1.4, 2.4 and 2.5 of Chapter 2] and [5].

2 Definitions and Notations

Let V_N be a regular summation method of the sequence $g(1), g(2), \cdots, g(N)$. Let μ be a Borel probability measure on an arbitrary compact metric space X.

We assume that $X = [0,1]$ and $F(x) = \mu([0,x))$ is a continuous function.

Definition 1. *Let $(g(n))$ be a sequence and $V_N(g(n))$ be a summation method of $g(1), \cdots, g(N)$. If $\lim_{N\to\infty} V_N(g(n)) = \sigma$, then $(g(n))$ is said to be V-summable to σ .*

Definition 2. *The sequence $(g(n))$ is said to be (V,μ)-u.d. mod 1 if for all intervals $J \in X$, we have*

$$\lim_{N\to\infty} V_N(C_J(g(n))) = \int_X C_J d\mu,$$

where C_J denotes the characteristic function of J .

Definition 3. *Let $(g(n))$ be a sequence of real numbers and $J = [\alpha, \beta) \subseteq [0,1]$. The numbers*

$$D_N = \sup_J |V_N(C_J(g(n))) - \int_X C_J d\mu|,$$

and

$$D_N^* = \sup_{0\le\alpha<1} |V_N(C_{[0,\alpha)}(g(n))) - \int_X C_{[0,\alpha)} d\mu|,$$

are called the (V,μ) -discrepancy and the (V,μ) - discrepancy respectively.*

Setting $V_N(\bullet) = \dfrac{1}{s(N)}\sum_{n=1}^{N} p(n)\bullet$, we have the ordinary weighted uniform distribution i.e. $(p(n),\mu) - u.d.$.

Throughout in this paper we use the following notations :

$$f(x) \ll g(x) \ \text{ means } \ g(x) > 0, \limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < +\infty.$$

$$f(x) = \Omega(g(x)) \ \text{ means } \ g(x) > 0, \limsup_{x\to\infty} \frac{f(x)}{g(x)} > 0.$$

$$f(x) \asymp g(x) \ \text{ means } \ 0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \le \limsup_{x\to\infty} \frac{f(x)}{g(x)} < +\infty.$$

$$\{x\} \ \text{ means the fractional part of } \ x.$$

3 Theorems

By Definition 3, we obtain the following .

Theorem 1. *The sequence $(g(n))$ is (V,μ) -u.d. mod 1 if and only if*

$$\lim_{N\to\infty} D_N = 0.$$

Remark. $D_N^* \le D_N \le 2D_N^*$.

Theorem 2. *If we set, for $g(n) \in [0,1]$, $n = 1, 2, \cdots$,*

$$\Delta_N(y) = V_N(C_{[0,y)}(g(n))) - F(y),$$

then

$$\int_0^1 \Delta_N(y)^2 dy = (V_N(g(n) - G))^2 + \frac{1}{2\pi^2}\sum_{h=1}^{\infty} \frac{1}{h^2} |(V_N(e^{-2\pi i h g(n)}) - \int_0^1 e^{-2\pi i h y} dF(y)|^2,$$

where $G = \int_0^1 y dF(y)$.

Proof. We remark that $\Delta_N(y)$ is a piecewise continuous function in $[0,1]$ with finitely many discontinuities at $y = g(1), \cdots, g(N)$. Moreover, we have $\Delta_N(0) = \Delta_N(1)$.

We expand $\Delta_N(y)$ into a Fourie series $\sum_{h=-\infty}^{\infty} a_h e^{2\pi i h x}$ which will represent $\Delta_N(y)$ apart from finitely many points. So we have

$$a_h = \int_0^1 \Delta_N(y) e^{-2\pi i h y} dy,$$

and

$$a_0 = \int_0^1 \Delta_N(y)dy = V_N\left(\int_0^1 C_{[0,y)}(g(n))\right) - \int_0^1 F(y)dy$$
$$= V_N(1 - g(n)) - (1 - G) = -V_N(g(n) - G),$$

where $G = \int_0^1 ydF(y)$.

For $h \neq 0$, we obtain

$$a_h = \int_0^1 \Delta_N(y)e^{-2\pi i h y} = V_N\left(\int_0^1 C_{[0,y)}(g(n))e^{-2\pi i h y}dy\right) - \int_0^1 F(y)e^{-2\pi i h y}dy$$
$$= V_N\left(\int_{g(n)}^1 e^{-2\pi i h y}dy\right) - \left(\left[\frac{F(y)}{-2\pi i h}e^{-2\pi i h y}\right]_0^1 + \int_0^1 \frac{e^{-2\pi i h y}}{2\pi i h}dF(y)\right)$$
$$= \frac{1}{2\pi i h}\left(V_N(e^{-2\pi h g(n)}) - \int_0^1 e^{-2\pi i h y}dF(y)\right).$$

By Parseval's identity, we have

$$\int_0^1 \Delta_N^2(y) = a_0^2 + 2\sum_{h=1}^{\infty} |a_h|^2,$$

and the desired result follows immediately.

Theorem 3 *For any (V,μ) -u.d. sequence $(g(n))$ in X, where $\mu([0,y)) = y$, we have*

$$D_N \leq \frac{6}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h} - \frac{1}{m+1}\right)|V_N(e^{2\pi i h g(n)})|.$$

Proof. We set for $F(y) = y$,

$$\Delta_N(y) = R_N(y) = V_N(C_{[0,y)}(g(n))) - F(y) \quad \text{for} \quad 0 \leq y \leq 1,$$

and extend this equation with period 1 to $\mathbf{R}$.

We consider first a sequence $g(1), \cdots, g(N)$ in $[0,1]$ for which

$$\int_0^1 \Delta_N(y)dy = 0. \tag{1}$$

We put

$$S_h = V_N(e^{2\pi i h g(n)} - \int_0^1 e^{2\pi i h y}dF(y)).$$

Then we obtain for $h \neq 0$,

$$\frac{S_h}{-2\pi i h} = \frac{1}{-2\pi i h}V_N(e^{2\pi i h g(n)} - \int_0^1 e^{2\pi i h y}dF(y)) = a_{-h} = \int_0^1 \Delta_N(y)e^{2\pi i h y}dy. \tag{2}$$

Choose a positive integer m , and let a be a real number to be determined later. From (1) and (2), it follows that

$$\sum_{h=-m}^{m}{}' (m+1-|h|)e^{-2\pi i h a}\frac{S_h}{-2\pi i h} = \sum_{h=-m}^{m}{}' (m+1-|h|)e^{-2\pi i h a}\int_0^1 \Delta_N(y)e^{2\pi i h y}dy$$

$$= \int_0^1 \Delta_N(y) \sum_{h=-m}^{m\ \prime} (m+1-|h|)e^{2\pi ih(y-a)}dy$$

$$= \int_{-a}^{1-a} \Delta_N(y+a) \sum_{h=-m}^{m\ \prime} (m+1-|h|)e^{2\pi ihy}dy, \tag{3}$$

where the dash indicates that $h = 0$ is deleted from the range of summation.

By (3),

$$\sum_{h=-m}^{m'} (m+1-|h|)e^{2\pi ihy} = \begin{cases} \sin^2(m+1)\pi y/\sin^2\pi y & \text{if } y \text{ is not an integer} \\ (m+1)^2 & \text{otherwise,} \end{cases}$$

due to the periodicity of the integrand in (3) we confine the range of the last integral over $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Thus we obtain

$$\left| \int_0^1 \Delta_N(y) \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy \right|$$

$$= \left| \int_0^1 \Delta_N(y) \sum_{h=-m}^{m'} (m+1-|h|)e^{2\pi ihy}dy \cdot e^{-2\pi iha} \right| = \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \Delta_N(y+a) \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy \right|$$

$$\leq \frac{1}{2\pi} \sum_{h=-m}^{m\ \prime} (m+1-|h|) \frac{|S_h|}{|h|} = \frac{1}{\pi} \sum_{h=1}^{m} (m+1-h) \frac{|S_h|}{h}. \tag{4}$$

We have either

$$\Delta_N(b) = -D_N^* \qquad \text{or} \qquad \Delta_N(b+0) = D_N^* \qquad \text{for some} \qquad b \in [0,1]$$

We treat only the second alternative, the first one being treated quite similarly.

For $b < t < b + D_N^*$, we have

$$\Delta_N(t) = D_N^* + \Delta_N(t) - \Delta_N(b+0) \geq D_N^* + b - t,$$

because

$$\Delta_N(t) - \Delta_N(b+0) = V_N(C_{[0,t)}(g(n)) - C_{[0,b+0)}(g(n))) - (t-b) \geq b - t.$$

Now choose $a = b + \frac{1}{2}D_N^*$. Then

$$\Delta_N(y+a) \geq D_N^* + b - (y+a) = \frac{1}{2}D_N^* - y \quad \text{for} \quad |y| < \frac{1}{2}D_N^*.$$

Consequently, we obtain

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \Delta_N(y+a) \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy = \left(\int_{-\frac{1}{2}D_N^*}^{\frac{1}{2}D_N^*} + \int_{-\frac{1}{2}}^{-\frac{1}{2}D_N^*} + \int_{\frac{1}{2}D_N^*}^{\frac{1}{2}}\right) \Delta_N(y+a) \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy$$

$$\geq \int_{-\frac{1}{2}D_N^*}^{\frac{1}{2}D_N^*} \left(\frac{1}{2}D_N^* - y\right) \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy - D_N^* \int_{-\frac{1}{2}}^{-\frac{1}{2}D_N^*} \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy$$

$$-D_N^* \int_{\frac{1}{2}D_N^*}^{\frac{1}{2}} \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy$$

$$= D_N^* \int_0^{\frac{1}{2}D_N^*} \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy - 2D_N^* \int_{\frac{1}{2}D_N^*}^{\frac{1}{2}} \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy.$$

Since $\dfrac{\sin^2(m+1)\pi y}{\sin^2 \pi y}$ is even and the definite integral of this function over $[0, \dfrac{1}{2}]$ being $\dfrac{m+1}{2}$,we get

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \Delta_N(y+a) \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy \geq D_N^* \int_0^{\frac{1}{2}} \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy - 3D_N^* \int_{\frac{1}{2}D_N^*}^{\frac{1}{2}} \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy$$

$$\geq \frac{m+1}{2} D_N^* - 3D_N^* \int_{\frac{1}{2}D_N^*}^{\frac{1}{2}} \frac{dy}{4y^2} \geq \frac{m+4}{2} D_N^* - \frac{3}{2} \geq \frac{m+1}{2} D_N^* - \frac{3}{2}.$$

Hence we have , by (4)

$$\frac{m+1}{2} D_N^* - \frac{3}{2} \leq \int_{-\frac{1}{2}}^{\frac{1}{2}} \Delta_N(y+a) \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy$$

$$\leq \left| \int_{-\frac{1}{2}}^{\frac{1}{2}} \Delta_N(y+a) \frac{\sin^2(m+1)\pi y}{\sin^2 \pi y} dy \right| \leq \frac{1}{\pi} \sum_{h=1}^{m} (m+1-h) \frac{|S_h|}{h}.$$

Thus we have

$$D_N^* \leq \frac{3}{m+1} + \frac{2}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1}\right) |S_h|,$$

where $S_h = V_N(e^{2\pi i h g(n)})$ because of $F(y) = y$.

If $\widetilde{D}_N$ denotes the discrepancy extended over all half-open intervals mod 1, then we have, in case of $\int_0^1 \Delta_N(y)dy = 0$,

$$\widetilde{D}_N \leq \frac{6}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1}\right) |S_h|. \tag{5}$$

We shall show that for any finite sequence $g(1), \cdots, g(N)$ in I , there exists a $c \in I$ such that the shifted sequence $\{g(1)+c\}, \cdots, \{g(N)+c\}$ satisfies $\int_0^1 \Delta_N(y)dy = 0$.

This proves the Theorem, since both sides of (5) are invariant under the transition from $g(1), \cdots, g(N)$ to the shifted sequence. We have

$$V_N(1-g(n)) = \frac{1}{2}, \text{ because } V_N\left(\int_0^1 C_{[0,y)}(g(n))dy\right) = \int_0^1 F(y)dy, \ i.e.\ 0 = \int_0^1 \Delta_N(y)dy.$$

Therefore we have to prove the existence of a $c \in I$ for which

$$V_N(\{g(n)+c\}) = \frac{1}{2}.$$

For any $c \in I$, we have

$$V_N(\{g(n)+c\} - g(n)) = cV_N(C_{[0,1-c)}(g(n))) + V_N(C_{[1-c,1)}(g(n)))(c-1)$$

$$= c - V_N(C_{[1-c,1)}(g(n))) = V_N(1 - C_{[1-c,1)}(g(n))) - (1-c)$$
$$= V_N(C_{[0,1-c)}(g(n))) - (1-c) = \Delta_N(1-c)$$

Therefore it remains to show that

$$\Delta_N(1-c) = \frac{1}{2} - V_N(g(n)) = s,$$

say, for some $c \in I$. We consider only the case $s > 0$, the case $s < 0$ being similar.

Since $\int_0^1 \Delta_N(y)dy = s$,we have $\Delta_N(y) \geq s$ for some $y \in (0,1)$. But since $\Delta_N(1) = 0$ and $\Delta_N(y)$ is linear with positive jumps at $y = \{g(n)\}, n = 1, 2, \cdots, N$, the function Δ_N must take the value s in the interval $(y, 1)$. This completes the proof.

Theorem 4. *If $(g(n))$ in $[0,1]$ is (V,μ) -u.d. , then we have*

$$D_N \leq \frac{6}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}(\frac{1}{h} - \frac{1}{m+1})|V_N(e^{2\pi i h F(g(n))})|,$$

where $F(x) = \mu([0,x))$ is continuous on $0 \leq x \leq 1$ and $F(x)$ is extended over R with period 1.

Proof. If $(g(n))$ is (V,μ) -u.d. ,then we have the $(V,1)$ -u.d. sequence $F(g(n))$. Thus we have the Theorem by Theorem 3.

Theorem 5. *Let F be a continuous function. If $(g(n))$ is (V,μ) -u.d. ,then we have*

$$D_N \leq \frac{6}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}(\frac{1}{h} - \frac{1}{m+1})|V_N(e^{2\pi i h g(n)})| + \frac{4}{m+1}\int_0^1 (F(y) - y)\frac{\sin^2(m+1)\pi y}{\sin^2 \pi y}dy.$$

Proof. Let $y \in [0,1]$ be the value which takes the discrepancy D_N^{1*} in (V,μ) -u.d. , and $y\prime \in [0,1]$ be the value which takes the discrepancy of D_N^{2*} in $(V,1)$ u.d. . We have

$$D_N^{1*} = \sup_{\beta} |V_N(C_{[0,\beta)}(g(n))) - \mu([0,\beta))|.$$

Since $g(n) \in [0,1]$, $Prob.(F(g(n)) < x) = x$ for all $x \in [0,1]$. So if we set $0 \leq g(n) < \beta, F(\beta) = \beta'$, then $0 = F(0) \leq F(g(n)) < F(\beta) = \beta'$. Thus

$$D_N^{1*} = \sup_{\beta'} |V_N(C_{[0,\beta')}(g(n))) - \beta'| \leq D_N^{2*}.$$

Conversely, $D_N^{2*} \geq D_N^{1*}$. Therefore we obtain $D_N^{1*} = D_N^{2*}$. Hence we have $F(y) = y\prime$.

By the similar calculation (3) of Theorem 3, we have under $\int_0^1 \Delta_N(y)dy = 0,$

$$|\sum_{h=-m}^{m}{}' (m+1-|h|)e^{-2\pi i h a}\frac{S_h}{-2\pi i h}| = |\int_0^1 \Delta_N(y) \sum_{h=-m}^{m} (m+1-|h|)e^{2\pi i h(y-a)}dy|$$

$$= |\int_a^{1-a} \Delta_N(y+a) \sum_{h=-m}^{m} (m+1-|h|)e^{2\pi i h y}dy| = |\int_{-\frac{1}{2}}^{\frac{1}{2}} \Delta_N(y+a)\frac{\sin^2(m+1)\pi y}{\sin^2 \pi y}dy|$$

$$\leq \frac{1}{\pi}\sum_{h=1}^{m}(m+1-h)\frac{|S_h|}{h}, \tag{6}$$

where

$$\Delta_N(y) = V_N(C_{[0,y)}(g(n))) - F(y), \qquad S_h = V_N(e^{2\pi i h g(n)} - \int_0^1 e^{2\pi i h y}dF(y)).$$

We set

$$\Delta'_N(y) = \Delta_N(y) + F(y) - y = V_N(C_{[0,y)}(g(n))) - y.$$

Replacing Δ_N by Δ'_N in Theorem 3 , we have

$$\int_0^1 \Delta'_N(y)\frac{\sin^2(m+1)\pi y}{\sin^2 \pi y}dy \geq \frac{m+4}{2}D^*_N - \frac{3}{2} \geq \frac{m+1}{2}D^*_N - \frac{3}{2}, \tag{7}$$

and

$$\int_0^1 \Delta'_N(y)\frac{\sin^2(m+1)\pi y}{\sin^2 \pi y}dy = \int_0^1 \Delta_N(y)\frac{\sin^2(m+1)\pi y}{\sin^2 \pi y}dy + \int_0^1 (F(y)-y)\frac{\sin^2(m+1)\pi y}{\sin^2 \pi y}dy.$$

Therefore we have from (4),(6) and (7),

$$\frac{m+1}{2}D^*_N - \frac{3}{2} \leq \frac{1}{\pi}\sum_{h=1}^{m}(m+1-h)\frac{|S_h|}{h} + \int_0^1 (F(y)-y)\frac{\sin^2(m+1)\pi y}{\sin^2 \pi y}dy.$$

Thus we obtain , under $\int_0^1 \Delta_N(y)dy = 0$,

$$D_N \leq \frac{6}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}(\frac{1}{h} - \frac{1}{m+1})|S_h| + \frac{4}{m+1}\int_0^1 (F(y)-y)\frac{\sin^2(m+1)\pi y}{\sin^2 \pi y}dy.$$

We shall show that for any finite sequence $g(1), \cdots, g(N)$ in I , there exists a $c \in I$ such that the shifted sequence $\{g(1)+c\}, \cdots \{g(N)+c\}$ satisfies $\int_0^1 \Delta_N(y)dy = 0$.

Since

$$(0=)a_0 = \int_0^1 \Delta_N(y)dy = V_N(\int_0^1 C_{[0,y)}(g(n))dy) - \int_0^1 F(y)dy$$
$$= V_N(1-g(n)) - 1 + \int_0^1 ydF(y) = -V_N(g(n) - G),$$

where $G = \int_0^1 ydF(y)$, we have to prove the existence of a $c \in I$ for which $V_N(\{g(n)+c\}) = G$. For any $c \in I$, we have

$$V_N(\{g(n)+c\} - g(n)) = V_N(cC_{[0,1-c)}(g(n))) + V_N((c-1)C_{[1-c,1)}(g(n)))$$
$$= c - V_N(C_{[1-c,1)}(g(n))) = c - 1 + V_N(C_{[0,1-c)}(g(n))) = c - 1 + F(1-c) + \Delta_N(1-c).$$

Therefore, it remains to show that

$$\Delta_N(1-c) = 1 - c - F(1-c) + G - V_N(g(n)) = s,$$

say, for some $c \in I$. We consider only the case $s > 0$, the case $s < 0$ being completely analoguous.

Since $\int_0^1 \Delta_N(y)dy = s$,we have $\Delta_N(y) \geq s$ for some $y \in (0,1)$.

Since $\Delta_N(1) = 0$ and $\Delta_N(y)$ is continuous except at $(g(n))_1^N$ with positive jumps, the function Δ_N must take the value s in the interval $(y,1)$. Thus it completes the proof.

The following is an analogue of LeVeque's Inequality (cf. 2:Chap.2 Th.2.4).

Theorem 6. *Under the $F(y) = \mu([0,y)) = y$, we have*

$$D_N \leq \left(\frac{6}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}|V_N(e^{2\pi i h g(n)})|^2\right)^{\frac{1}{3}}.$$

Corollary. *If $(g(n)), g(n) \in [0,1]$ is a (V,μ) -u.d. ,then*

$$D_N \leq \left(\frac{6}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}|V_N(e^{2\pi i h F(g(n))})|^2\right)^{\frac{1}{3}},$$

where $F(y) = \mu([0,y))$ is continuous on $0 \leq y \leq 1$.

Theorem 7. *Under the condition such that $F'(y) < \infty$ exists for $0 \leq y \leq 1$,*

$$D_N \leq \left(||F'||\frac{6}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h^2}|V_N(e^{2\pi i h g(n)} - \int_0^1 e^{2\pi i h y}dF(y))|^2\right)^{\frac{1}{3}},$$

where $||F'||$ is a supremum norm of F'.

Remark. There exists a distribution function $F(x)$ such that $F'(a) = \infty$, e.g.

$$F(x) = \begin{cases} a - 1/\sqrt{\log b/(a-x)} & \text{if } 0 \leq x < a \\ a & \text{if } x = a \\ a + 1/\sqrt{\log b/(x-a)} & \text{if } a < x \leq 1, \end{cases}$$

where $a = 1/2, b = e^4/2$, satisfies that $F(0) = 0, F(1) = 1, F'(x) < \infty$ except at a , and $F'(a) = \infty$. Thus $||F'|| = \infty$.

Lemma 1. *Let $f(t)$ be continuous and of bounded variation on $[0,1]$.*
If $E_N(g) = V_N(f(g(n)) - \int_0^1 f(x)dF(x)$, then

$$E_N(g) = -\int_0^1 R_N(t)df(t), \tag{8}$$

where

$$R_N(t) = V_N(C_{[0,t)}(g(n)) - \int_X C_{[0,t)}dF(x).$$

Proof.

$$\int_0^1 R_N(t)df(t) = \int_0^1 V_N(C_{[0,t)}(g(n))df(t) - \int_0^1 \int_X C_{[0,t)}dF(x)\, df(t)$$

$$= V_N\left(\int_0^1 C_{[0,t)}(g(n))df(t)\right) - \left[\int_X C_{[0,t)}d\mu f(t)\right]_0^1 + \int_0^1 f(t)dF(t)$$

$$= V_N(-f(g(n))) + \int_0^1 f(t)dF(t) = -E_N(g).$$

Lemma 2. *Under the same condition of Lemma 1,we have*

$$|E_N(a)| \leq W(f)D_N^*,$$

where $W(f)$ denotes the total bounded variation of f.

Proof. By (12),

$$|E_N(g)| \leq \int_0^1 |R_N(t)||df(t)| \leq \int_0^1 D_N^*|df(t)| = D_N^* W(f).$$

Lemma 3. *Under the same condition of Lemma 1, we have*

$$|V_N(e^{2\pi i g(n)})| \leq 4D_N^* + \int_0^1 |d(F(t) - t)|,$$

where $D_N^* = \sup_{J=[0,x)} |V_N(C_J(g(n)) - \int_X C_J d\mu|$.

Proof. We set $S_N = V_N(e^{2\pi i g(n)})$. There exists a $\theta \in [0,1]$ such that $S_N = |S_N|e^{2\pi i\theta}$. Then

$$|S_N| = S_N e^{-2\pi i\theta} = V_N(e^{2\pi i(g(n)-\theta)}).$$

Since $|S_N|$ is a real number, we have

$$|S_N| = V_N(\cos(2\pi(g(n) - \theta))). \tag{9}$$

Now we put $f(t) = \cos(2\pi(t-\theta))$ on $t \in [0,1]$.
Since $f(t)$ is absolutely continuous, we have,

$$W(f) := \int_0^1 |f'(t)|dt = 2\pi \int_0^1 |\sin 2\pi(t-\theta)|d\theta$$

$$= 2\pi \int_{-\theta}^{1-\theta} |\sin 2\pi u|du = 2\pi \int_0^1 |\sin 2\pi u|du = 4.$$

On the other hand,

$$\int_0^1 f(t)dt = \int_0^1 \cos 2\pi(t-\theta)dt = 0.$$

By (13) and Lemma 2, we obtain

$$|S_N| = V_N(f(g(n))) - \int_0^1 f(t)dF(t) + \int_0^1 f(t)dF(t) - \int_0^1 f(t)dt$$

$$= E_N(g) + \int_0^1 f(t)d(F(t)-t) \leq |E_N(g)| + |\int_0^1 f(t)d(F(t)-t)|$$
$$\leq 4D_N^* + |\int_0^1 f(t)d(F(t)-t)| \leq 4D_N^* + \int_0^1 |d(F(t)-t)|.$$

This completes the proof.

By Lemma 3, Theorem 5 and due to the fact $D_N^* \leq D_N$, we have the following:

Theorem 8. *Let $F(x) = \mu([0,x))$ be a continuous function. If $(g(n))$ on $[0,1]$ is $(V,\mu)-u.d.$, then*

$$\frac{1}{4}|V_N(e^{2\pi i g(n)})| - \frac{1}{4}\int_0^1 |d(F(t)-t)| \leq D_N \leq$$
$$\leq \frac{6}{m+1} + \frac{4}{\pi}\sum_{h=1}^{m}(\frac{1}{h} - \frac{1}{m+1})|V_N(e^{2\pi i h g(n)})| + \frac{4}{m+1}\int_0^1 (F(y)-y)\frac{\sin^2(m+1)\pi y}{\sin^2 \pi y}dy.$$

Next theorem is an application of Theorem 8.

Theorem 9. *Let $g(t) \in C^2(1,\infty)$ be a positive strictly monotone function satisfying the following conditions: for some constant A, $0 \leq A < 1$ such that*

$$g(t) \to \infty, \quad 0 < g'(t) < 1 \quad \textit{for sufficiently large} \quad t,$$
$$g'(t) \to A, \quad \textit{monotonically decreasing as } t \to \infty,$$
$$\sum_{n=1}^{N} g'(n) \to \infty \textit{ as } N \to \infty, \quad \textit{and} \quad \int_1^N g'(t)^2 dt < +\infty \quad \textit{as} \quad N \to \infty.$$

Let $(g(n))$ on $[0,1]$ be (V,μ) -u.d. with a continuous distribution function $F(x)$ and let D_N be the discrepancy of $(g(n))$ with weight $g'(n)$ .Then for all sufficiently large N, we have

$$D_N \ll \frac{1}{s(N)},$$

on the other hand, we have the reversed inequality

$$D_N \gg \frac{1}{s(N)}$$

for infinityly many N.

Proof. To prove the theorem we need the following two results :

Theorem A. [7 : *p.226*] *If $g'(t)$ is monotone and $|g'(t)| \leq 1-\delta$ for some $\delta > 0$ in (a,b) and $p(t)$ is monotonically decreasing and differentiable, then*

$$|\sum_{a<n\leq b} p(n)e^{2\pi i g(n)} - \int_a^b p(t)e^{2\pi i g(t)}dt| \leq A_\delta \max_{a\leq t\leq b} p(t).$$

Theorem B. [6:Theorem 2.] *Let $g(t) \in C^2[1,\infty)$ be a positive strictly increasing function satisfying the following condition :*

$$g(t) \to \infty \ \text{as } t \to \infty, \quad g'(t) \to A \ , \ \text{monotonically as } t \to \infty,$$

$$\frac{g'(t)}{p(t)} \ \text{is monotone for } \ t \geq 1,$$

then the discrepancy of $(g(n))$ with weight $p(n)$ and distribution function $F(x) = x$ is

$$D_N \ll \frac{1}{s(N)} \int_1^N p(t)g'(t)dt + \frac{p(N)}{s(N)g'(N)},$$

where $s(N) = \sum_{n=1}^{N} p(n)$.

We set $p(t) = g'(t)$. By Euler summation formula(cf.[2:Chap.1, Example 2.4]), we have

$$\sum_{n=1}^{N} g'(n)e^{2\pi i h g(n)} = \int_1^N g'(t)e^{2\pi i h g(t)}dt + \frac{1}{2}(g'(1)e^{2\pi i h g(1)} + g'(N)e^{2\pi i h g(N)}) +$$

$$+ \int_1^N (\{t\} - \frac{1}{2}) \left(g'(t)e^{2\pi i h g(t)}\right)' dt.$$

Since $\sum_{n=1}^{N} g'(n) \to \infty$ and $\int_1^N g'(t)^2 dt < \infty$, we obtain

$$\lim_{N\to\infty} \frac{1}{\sum_{n=1}^{N} g'(n)} \sum_{n=1}^{N} g'(n)e^{2\pi i h g(n)} = \lim_{N\to\infty} \frac{1}{\sum_{n=1}^{N} g'(n)} \int_1^N g'(t)e^{2\pi i h g(t)}dt = 0 = \int_0^1 e^{2\pi i h y}dy.$$

Thus we have the distribution function $F(y) = y$. Now we apply Theorem 8 and Theorem B with $F(y) = y$, and we obtain,

$$D_N \ll \frac{1}{s(N)} \int_1^N g'(t)^2 dt + \frac{1}{s(N)} \ll \frac{1}{s(N)}.$$

Moreover we have from the left hand side of Theorem 8 by using Euler's summation formula,

$$4s(N)D(N) \geq |\sum_{n=1}^{N} g'(n)e^{2\pi i g(n)}| \geq \Omega(1).$$

Since g' is monotonely decreasing and $g(n) \to \infty$, $s(N) = \sum_{n=1}^{N} g'(n) \sim g(N)$, which proves Theorem 9.

Example. $g(n) = \log n$ satisfies the conditions of Theorem 9. Thus the order of magnitude of D_N of the sequence $(\log n)$ with weight $\frac{1}{n}$ is $\frac{1}{\log N}$.

Lemma 4. *Let* $0 < K(t) = \frac{g'(t)}{p(t)}$ *be monotone and differentiable* $\uparrow$ *or* $\downarrow$ $K > 0$, $g'(t)$ *be monotonically decreasing,* $|g'(t)| \le 1-\delta$ *and* $p(t)$ *be monotonicallly decreasing to zero and differentiable. Then*

$$\sum_{n=1}^{N} p(n)e^{2\pi i g(n)} = \Omega(1)$$

Proof. Since $K(t)$ is monotone and not identically $\sum_{n=1}^{N} p(n)e^{2\pi i g(n)} = 0$ for sufficiently large N , we have for any $\epsilon > 0$, there exists an N_0 such that for any $t \ge N_0$

$$\left|\frac{1}{K(t)} - \frac{1}{K}\right| < \epsilon, \quad \sum_{n=1}^{N_0-1} p(n)e^{2\pi i g(n)} \ne 0, \quad p(N_0) \le \frac{\epsilon}{A_\delta 2\pi}.$$

Applying Theorem A,

$$\sum_{n=1}^{N} p(n)e^{2\pi i g(n)} = \sum_{n=1}^{N_0-1} p(n)e^{2\pi i g(n)} + \int_{N_0}^{N} p(t)e^{2\pi i g(t)}dt + A_\delta \max_{N_0 \le t \le N} p(t).$$

Noticing

$$\int_{N_0}^{N} p(t)e^{2\pi i g(t)}dt = \int_{N_0}^{N} \frac{g'(t)}{K(t)} e^{2\pi i g(t)}dt,$$

we obtain

$$\int_{N_0}^{N} \left(\frac{g'(t)}{K(t)} - \frac{g'(t)}{K}\right) e^{2\pi i g(t)}dt = \int_{N_0}^{N} \left(\frac{1}{K(t)} - \frac{1}{K}\right) d\left(\frac{e^{2\pi i g(t)}}{2\pi i}\right)$$

$$= \left(\frac{1}{K(N)} - \frac{1}{K}\right)\frac{e^{2\pi i g(N)}}{2\pi i} - \left(\frac{1}{K(N_0)} - \frac{1}{K}\right)\frac{e^{2\pi i g(N_0)}}{2\pi i} + A_N,$$

where

$$|A_N| \le \int_{N_0}^{N} \frac{1}{2\pi}\left|d\left(\frac{1}{K(t)}\right)\right| = \frac{1}{2\pi}\left|\frac{1}{K(t)} - \frac{1}{K(N_0)}\right| < \frac{\epsilon}{2\pi}.$$

Thus

$$\left|\int_{N_0}^{N} \left(\frac{g'(t)}{K(t)} - \frac{g'(t)}{K}\right) e^{2\pi i g(t)}dt\right| \le \frac{3}{2\pi}\epsilon,$$

and

$$\frac{1}{K}\int_{N_0}^{N} g'(t)e^{2\pi i g(t)}dt = \frac{1}{K}\frac{1}{2\pi i}(e^{2\pi i g(N)} - e^{2\pi i g(N_0)}).$$

Therefore

$$\sum_{n=1}^{N} p(n)e^{2\pi i g(n)} = \sum_{n=1}^{N_0-1} p(n)e^{2\pi i g(n)} + \frac{1}{K}\frac{1}{2\pi i}(e^{2\pi i g(N)} - e^{2\pi i g(N_0)}) + c, \quad |c| \le \frac{4}{2\pi}\epsilon$$

$$= \Omega(1).$$

We obtain the following theorem by Lemma 4 and Theorem B :

Theorem 10. *Let* $g(t) \in C^2, g'(t) \to$ *Constant* < 1, *monotonically as* $t \to \infty$, $0 < c \leq \dfrac{g'(t)}{p(t)} < \infty$ *monotone and differeitiable for* $t \geq 1$, *some constant* c , $p(t)$ *be monotonically decreasing and differentiable and* $\int_1^N p(t)g'(t)dt < \infty$.

Let $(g(n))$ *on* $[0,1)$ *be* $(V,\mu)-$ *u.d. with a distribution function* $F(x) = x$ *and let* D_N *be the discrepancy of* $(g(n))$ *with weight* $p(n)$. *Then we have for all sufficiently large* N, *we have*

$$D_N \ll \frac{1}{s(N)},$$

on the other hand, we have the reversed inequality

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for infinityly many N .

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