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On the mean square of the product of the zeta- and L- functions

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§1. Introduction.

In our recent paper [1] we have studied the fourth power moment of the Riemann zeta-function, and established an explicit formula for the expression

$$I_2(T,G)=(\pi\sqrt{G})^{-1}\int_{-\infty}^{\infty}|\zeta(\tfrac{1}{2}+i(t+T))|^4\exp(-(\frac{t}{G})^2)dt,$$

where T and G are arbitrary positive numbers. Our formula revealed, among other things, a close relationship between the zeta-function and the automorphic L-functions over the full modular group.

The aim of our talk is to indicate that it is possible to extend such a relationship to the situations involving Dirichlet L-functions. This time, as may be expected, the underlying group is not the full modular group but a congruence subgroup whose characterization depends on how to incorporate Dirichlet L-functions into our consideration. To be more precise we introduce two typical extensions of $I_2(T,G)$:

$$J(T,G;\chi)=(\pi\sqrt{G})^{-1}\int_{-\infty}^{\infty}|\zeta(\tfrac{1}{2}+i(t+T))L(\tfrac{1}{2}+i(t+T),\chi)|^2\exp(-(\frac{t}{G})^2)dt,$$

$$I_2(T,G;\chi)=(\pi\sqrt{G})^{-1}\int_{-\infty}^{\infty}|L(\tfrac{1}{2}+i(t+T),\chi)|^4\exp(-(\frac{t}{G})^2)dt.$$

Then we can show that if χ is a primitive character mod q the mean $J(T,G;\chi)$ admits an expansion in terms of automorphic L-functions over the congruence subgroup $\Gamma_0(q)$, and the mean $I_2(T,G;\chi)$ is controlled by the principal congruence subgroup $\Gamma_1(q)$. We note that $J(T,G;\chi)$ contains the important case of the mean squares of the Dedekind zeta-functions of quadratic number fields.

Here we shall make the statement on $J(T,G;\chi)$ explicit on the technical assumption that q *is an odd prime number*. This is to avoid unnecessary complexity, and in fact its elimination is by no means difficult. On the other hand the above statement on $I_2(T,G;\chi)$ is provisional, for we have not yet finished all details. The difficulty lies mainly in the geometrical structure of the fundamental region of the group $\Gamma_1(q)$, which can be highly complicated; and thus the contribution coming from the continuous spectrum is rather hard to manage. The same, but in a much lesser extent, can be said about $J(T,G;\chi)$ when q has many prime factors; thus the above assumption has been introduced.

§2. Definitions.

To state our result we have to introduce some rudiments from the theory of automorphic forms. We stress that q is an odd prime, and all implicit constants in the formulas below are possibly dependent on q.

First, let $\mathcal{F}_0$ be the traditional fundamental region of the full modular group in the upper half plane, and let $\mathcal{F}$ be the fundamental region of $\Gamma = \Gamma_0(q)$, which is composed of images of $\mathcal{F}_0$ in the following way:

(1) $$\mathcal{F} = \mathcal{F}_0 \cup \bigcup_{j=-r}^{r} ST^j(\mathcal{F}_0),$$

where $r = (q-1)/2$, and

$$S = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}.$$

We denote by $\mathcal{H}$ the Hilbert space spanned by all Γ-invariant functions over the upper half plane that are square integrable over $\mathcal{F}$ with respect to the Poincaré metric. The non-Euclidean Laplacian

$$\mathcal{L} = -y^2((\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2)$$

induces the orthogonal decomposition

(2) $$\mathcal{H} = \mathbb{C} + \mathcal{H}_{cnt} + \mathcal{H}_{csp},$$

where $\mathcal{H}_{cnt}$ and $\mathcal{H}_{csp}$ correspond to the continuous and the discrete spectrum of $\mathcal{L}$, respectively. Since $\mathcal{F}$ has the two inequivalent cusps at $i\infty$ and 0, the two Eisenstein series

$$E_\infty(z,s) = \sum_{g \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}(gz))^s \quad \text{and} \quad E_0(z,s) = \sum_{g \in \Gamma_0 \backslash \Gamma} (\mathrm{Im}(Sgz))^s$$

are to be used to describe the nature of $\mathcal{H}_{cnt}$. Here Γ_∞ and Γ_0 are the stabilizers of the points $i\infty$ and 0, respectively. As for the discrete spectrum we denote it by

$$\mathcal{S} = \{\lambda_j = \kappa_j^2 + \tfrac{1}{4};\, j \ge 1\},$$

in which we have Selberg's lower bound $\lambda_j \ge \frac{3}{8}$. Then the subspace $\mathcal{H}_{csp}$ has the orthonormal base

$$\{\varphi_j;\ j \ge 1\}$$

such that each form φ_j satisfies $\mathcal{L}\varphi_j = \lambda_j \varphi_j$.

The fact (2) is now expressed more precisely as the spectral expansion formula: For each element f of $\mathcal{H}$ we have the L^2-identity

(3) $$f(z) = \sum_{j \ge 0} a_j \varphi_j(z) + \frac{1}{4\pi}\int_{-\infty}^{\infty} a_\infty(t) E_\infty(z, \tfrac{1}{2} + it)dt + \frac{1}{4q\pi}\int_{-\infty}^{\infty} a_0(t) E_0(z, \tfrac{1}{2} + it)dt,$$

where φ_0 is a constant function, and

$$a_j = \int_{\mathcal{F}} f(z)\overline{\varphi_j(z)}d\mu(z), \quad a_\alpha(t) = \int_{\mathcal{F}} f(z)\overline{E_\alpha(z, \tfrac{1}{2} + it)}d\mu(z).$$

As usual we should take into account the action of Hecke operators $T(n)$ over φ_j: Then we may assume that for all j and n, $(n.q) = 1$, there exists a real number $t_j(n)$ such that

(4) $$(T(n)\varphi_j)(z)=\frac{1}{\sqrt{n}}\sum_{ad=n}\sum_{b=1}^{d}\varphi_j(\frac{az+b}{d})=t_j(n)\varphi_j(z).$$

Also, by the symmetry of $\mathcal{F}$ that is visible in (1), we can assume that

(5) $$(T_{-1}\varphi_j)(z)=\varphi_j(-\bar{z})=\varepsilon_j\varphi_j(z)$$

with $\varepsilon_j=\pm1$. We should remark that an elementary reasoning yields the bound

$$t_j(n)\ll n.$$

Though the inequality (7) below can imply a better bound, this is sufficient for our purpose.

We then consider the Fourier expansion

$$\varphi_j(x+iy)=\sqrt{y}\sum_{n\neq 0}\rho_j(n)K_{i\kappa_j}(2\pi|n|y)e(nx),$$

where K_ν is the K-Bessel function and $e(x)=\exp(2\pi ix)$. The relations (4) and (5) imply

(6) $$\rho_j(n)=t_j(n_1)\rho_j(n_2) \quad\text{and}\quad \rho_j(-n)=\varepsilon_j\rho_j(n),$$

where $n=n_1n_2$ with $(n_1,q)=1$, $n_2 | q^\infty$. The automorphic L-function $L_j(s)$ attached to φ_j is defined by

$$L_j(s)=\sum_{n\ge1}\rho_j(n)n^{-s},$$

which is absolutely convergent for e.g., $\mathrm{Re}(s)>\frac{5}{4}$, since it is known that for any fixed $\eta>0$

(7) $$\rho_j(n)\ll\exp(\tfrac{\pi}{2}\kappa_j)m^{\frac{1}{4}+\eta}.$$

It can be shown that $L_j(s)$ is in fact an integral function and satisfies the functional equation

(8) $$L_j(s)=\frac{1}{\pi}\left(\frac{\sqrt{q}}{2\pi}\right)^{1-2s}\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j)(\varepsilon_j\cosh(\pi\kappa_j)-\cos(\pi s))L_j^*(1-s).$$

Here $L_j^*(s)$ is the L-function attached to the cusp form $\varphi_j^*(z)=\varphi_j(-1/(qz))$. Obviously we have $\mathcal{L}\varphi_j^*=\kappa_j\varphi_j^*$ and $\|\varphi_j^*\|\ll1$, and thus each Fourier coefficient of φ_j^* satisfies the same inequality as (7). This implies that $L_j^*(s)$ is bounded by $\exp(\frac{\pi}{2}\kappa_j)$ in the region of absolute convergence. Hence the identity (8) yields the assertion that if $\mathrm{Re}(s)$ is bounded we have

(9) $$L_j(s)\ll(|s|\kappa_j)^c\exp(\tfrac{\pi}{2}\kappa_j)$$

with a certain positive constant c depending only on $\mathrm{Re}(s)$.

We also need to know a little about the χ-twist of the Hecke series attached to φ_j:

$$H_j(s,\chi)=\sum_{n\ge1}\chi(n)t_j(n)n^{-s}.$$

It converges to an integral function satisfying the functional equation

$$H_j(s)=\frac{\tau(\chi)}{\pi\overline{\tau(\chi)}}\left(\frac{q}{2\pi}\right)^{1-2s}\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j)$$
$$\times(\varepsilon_j\chi(-1)\cosh(\pi\kappa_j)-\cos(\pi s))H_j(1-s,\overline{\chi}),$$

where $\tau(\chi)$ is the Gauss sum for χ. In particular, if $\mathrm{Re}(s)$ is bounded, we have

(10) $$H_j(s,\chi)\ll(|s|\kappa_j)^c$$

with c being as in (9).

The relation (6) and the multiplicative property of Hecke eigenvalues yield the following identity, which is an essential tool in our argument: In the region of absolute convergence we have

(11) $$\sum_{n\geq 1}\sigma_a(n,\chi)\rho_j(n)n^{-s} = H_j(s-a,\chi)L_j(s)/L(2s-a,\chi),$$

where

$$\sigma_a(n,\chi)=\sum_{d|n}\chi(d)d^a.$$

Next we move to the elements of the theory of holomorphic cusp forms over Γ. Thus let

$$\{\varphi_{j,k};j\leq \vartheta_k(q)\}$$

be the orthonormal base of the Petersson unitary space composed of holomorphic cusp forms of the even weight $2k$ with respect to the group Γ. We may assume that every $\varphi_{j,k}$ is an eigenfunction of all Hecke operators $T_k(n)$ so that for $(n,q)=1$

(12) $$(T_k(n)\varphi_{j,k})(z)=\frac{1}{\sqrt{n}}\sum_{ad=n}\left(\frac{a}{d}\right)^k\sum_{b=1}^{d}\varphi_{j,k}(\frac{az+b}{d})=t_{j,k}(n)\varphi_{j,k}(z)$$

with a certain real number $t_{j,k}(n)$. The Fourier coefficients $\rho_{j,k}(n)$ of $\varphi_{j,k}$ is defined by the expansion

$$\varphi_{j,k}(z)=\sum_{n\geq 1}\rho_{j,k}(n)n^{k-\frac{1}{2}}e(nz).$$

Then the relation (12) implies

$$\rho_{j,k}(n)=t_{j,k}(n_1)\rho_{j,k}(n_2),$$

where n_1 and n_2 are as in (6). The L-function $L_{j,k}(s)$ attached to $\varphi_{j,k}$ is defined by

$$L_{j,k}(s)=\sum_{n\geq 1}\rho_{j,k}(n)n^{-s},$$

which is absolutely convergent for e.g., $\mathrm{Re}(s)>\frac{5}{4}$, since it is known that for any fixed $\eta>0$

$$\rho_{j,k}(n)\ll p(k)m^{\frac{1}{4}+\eta},$$

where

$$p(k)=(4\pi)^k((2k-2)!)^{-\frac{1}{2}}.$$

The function $L_j(s)$ is entire and satisfies the functional equation

$$L_{j,k}(s)=(-1)^k\left(\frac{\sqrt{q}}{2\pi}\right)^{1-2s}\frac{\Gamma(k+\frac{1}{2}-s)}{\Gamma(k-\frac{1}{2}+s)}L_{j,k}^*(1-s).$$

Here $L_{j,k}^*(s)$ is the L-function attached to the cusp form $\varphi_{j,k}^*(z)=\varphi_{j,k}(-1/(qz))$. As before it entails the assertion that if $\mathrm{Re}(s)$ is bounded

(13) $$L_{j,k}(s)\ll p(k)(|s|k)^c$$

with c being as in (9).

We need again to know some facts about the χ-twist of the Hecke series attached to $\varphi_{j,k}$:

$$H_{j,k}(s,\chi)=\sum_{n\ge 1}\chi(n)t_{j,k}(n)n^{-s}.$$

It is an integral function satisfying the functional equation

$$H_{j,k}(s)=(-1)^k\frac{\tau(\chi)}{\tau(\overline{\chi})}\left(\frac{q}{2\pi}\right)^{1-2s}\frac{\Gamma(k+\frac{1}{2}-s)}{\Gamma(k-\frac{1}{2}+s)}H_{j,k}(1-s,\chi).$$

This and an elementary bound for $t_{j,k}(n)$ yield that if $\mathrm{Re}(s)$ is bounded

(14) $$H_{j,k}(s,\chi)\ll(|s|k)^c$$

with c being as in (9).

Finally we have the following analogue of the identity (11): In the region of absolute convergence

$$\sum_{n\ge 1}\sigma_a(n,\chi)\rho_{j,k}(n)n^{-s}=H_{j,k}(s-a,\chi)L_{j,k}(s)/L(2s-a,\chi).$$

§3. The result.

We are now ready to state our main result on the mean $J(T,G;\chi)$. To this end let us first put

$$r(k)=(2k-1)!\,2^{-4k+1}\pi^{-2k-1},$$

$$\Theta^{(j)}(\xi;T,G;\chi)=\left\{\varepsilon_j(e^{\pi\xi}-\chi(-1)e^{-\pi\xi})+i(1+\chi(-1))\right\}\frac{\Lambda(i\xi;T,G)}{\sinh(\pi\xi)},$$

and

$$\Lambda(\xi;T,G)=\frac{\Gamma(\frac{1}{2}+\xi)^2}{\Gamma(1+2\xi)}\int_0^\infty x^{-1-\xi-iT}(1+x)^{-\frac{1}{2}+iT}$$
$$\times F(\tfrac{1}{2}+\xi,\tfrac{1}{2}+\xi;1+2\xi;-\tfrac{1}{x})\exp(-(\tfrac{G}{2}\log(1+\tfrac{1}{x}))^2)dx$$

with F standing for the hypergeometric function. We need also to introduce the convention $\varepsilon_0=1$.

Theorem. *If χ is a non-trivial character mod q an odd prime number, then for any T and G such that $1\le G\le T(\log qT)^{-1}$ we have*

$$J(T,G;\chi)=$$
$$\frac{1}{4}\mathrm{Re}\left[\tau(\overline{\chi})\sum_{j=1}^{\infty}\frac{1}{\cosh(\pi\kappa_j)}|L_j(\tfrac{1}{2})|^2H_j(\tfrac{1}{2},\chi)\{\Theta^{(j)}(\kappa_j;T,G;\chi)+\Theta^{(j)}(-\kappa_j;T,G;\chi)\}\right]$$
$$+(1+\chi(-1))\,\mathrm{Re}\left[\tau(\overline{\chi})\sum_{k=1}^{\infty}\sum_{j=1}^{\vartheta_k(q)}(-1)^k r(k)|L_{j,k}(\tfrac{1}{2})|^2H_{j,k}(\tfrac{1}{2},\chi)\Lambda(k-\tfrac{1}{2};T,G;\chi)\right]$$
$$+\frac{1}{\pi}\mathrm{Re}\left[\tau(\overline{\chi})\int_{-\infty}^{\infty}\frac{|\zeta(\frac{1}{2}+it)|^4L(\frac{1}{2}+it,\chi)L(\frac{1}{2}-it,\chi)}{|\zeta(1+2it)|^2|1+q^{\frac{1}{2}+it}|^2}\Theta^{(0)}(t;T,G;\chi)dt\right]$$
$$+O((\log qT)^2).$$

Here the implied constant is absolute.

Because of the rapid decay of $\Lambda(\xi;T,G)$, which is shown in [1], and in view of the assertions (9), (10), (13) and (14) the series and the integral in the above are all absolutely convergent.

We stress that with an extra effort one can make the O-term explicit. Then it will turn out to be a quadratic polynomial of $\log T$, the coefficients of which are related to $L(1,\chi)$. We should remark also that in the special case where $\chi(-1)=-1$ there are no contributions from the holomorphic cusp forms. This fact can be generalized to any composite modulus, and we can say exactly the same on the mean square of the Dedekind zeta-functions of *imaginary* quadratic number fields. It appears to us that this peculiar fact may probably have a relation with the spectral theory of the Hilbert modular forms over relevant number fields.

It is also possible to deduce from our formula an asymptotic result that has a feature similar to our former result given in [1 ,Corollary to the Theorem]. Then the issue concerning the exceptional eigenvalues will become prominent, and there is a possibility that our result could be used to show an interaction between the lower bound of the eigenvalues λ_j and the sizes of $L(1,\chi)$ and $L(\frac{1}{2}+it,\chi)$.

Further we note that a naive comparison of the cusp form contributions and that of the continuous spectrum leads us to the following:

Conjecture. *For each fixed j, k and* $\eta>0$

$$L_j(\tfrac{1}{2})\ll q^{-\frac{1}{2}+\eta},$$

$$L_{j,k}(\tfrac{1}{2})\ll q^{-\frac{1}{2}+\eta}.$$

As for the detailed proof as well as a further discussion that are to be developed elsewhere, we note only that the main frame is essentially the same as that of [1]. It depends on an extension of Kuznetsov's trace formula, which is a consequence of (3), to forms over the group $\Gamma_0(q)$ and on those facts given in the second section.

Reference

[1] Y. Motohashi: An explicit formula for the fourth power mean of the Riemann zeta-function (to appear in Acta Mathematica).