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On arithmetical functions whose generating functions are of the form $\zeta(s)\zeta^{\alpha}(s+1)f(s+1)$

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1 Introduction

Our main motivation for considering the class of Dirichlet series in the title (where $\alpha \in \mathbb{C}$ and $f(s+1)$ is assumed to have a Dirichlet series expansion absolutely convergent in the half plane $\sigma > -\lambda$, for some $\lambda > 0$), is that generating functions of certain classical arithmetical functions have this form. For instance each of the sequences

$$\left\{\left(\frac{\sigma(n)}{n}\right)^{\alpha}\right\}_{n=1}^{\infty}, \quad \left\{\left(\frac{\phi(n)}{n}\right)^{-\alpha}\right\}_{n=1}^{\infty} \quad \text{and} \quad \left\{\left(\frac{\sigma(n)}{\phi(n)}\right)^{\alpha/2}\right\}_{n=1}^{\infty} \tag{1}$$

(where as usual σ and ϕ denote the sum-of-divisors and Euler's functions) is the sequence of coefficients $a(n)$ of such a series.

Our goal is to establish explicit expressions for P and E in

$$\sum_{n\leq x} a(n) = P(x) + E(x) = \text{ principal term + error term} \tag{2}$$

(Theorem 1 in Section 2 below), and then (Theorems 2 and 3) to obtain O and Ω-estimates for E in the case where α is a real number and a multiplicative (with some additional conditions).

In Theorems 4 and 5 we apply these results to the special cases where $\{a(n)\}$ is a sequence in (1). Our results cover all real values of α. For the two first sequences, apart from the cases $\alpha = \pm 1$ and $\alpha = 0$, they supersede what is known today (see [1], [7], [10], [12], [14], [15], [18] for the current records, and also [3], [5], [6], [9], [11], [13], [16], [19]). The third sequence was to our knowledge not studied in this context. We also deduce similar results for the sequences $\{\sigma^{\alpha}(n)\}$ and $\{\phi^{\alpha}(n)\}$ (Corollaries 1, 2 and 3).

The proofs will be published elsewhere [2]. In Section 3 below we briefly describe the methods we use.

2 Statement of the results

Theorem 1 . *Let $\{a(n)\}$ be a sequence of complex numbers satisfying*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \zeta(s)\zeta^{\alpha}(s+1)f(s+1)$$

for a complex α and $f(s+1)$ having a Dirichlet series expansion

$$f(s+1) = \sum_{n=1}^{\infty} \frac{b(n)}{n^{s+1}},$$

which is absolutely convergent in the half plane $\sigma > -\lambda$ for a $\lambda > 0$ (and thus with $|b(n)| << n^{\delta}$ for some $\delta < 1$). Let

$$\zeta^{\alpha}(s+1)f(s+1) = \sum_{n=1}^{\infty} \frac{v(n)}{n^s}.$$

Then there is a number b, $0 < b < 1$, such that

$$\sum_{n\le x} a(n) = \zeta^{\alpha}(2)f(2)x + \sum_{r=0}^{[\alpha_0]} B_r(\log x)^{\alpha-r} - \sum_{n\le y} v(n)\psi(\frac{x}{n}) + o(1)$$

where $y = x/\exp(\log^b x)$ and α_0 denotes the real part of α.

Theorem 2 . *Let $v_n = v(n)$ be a real multiplicative arithmetical function satisfying, for some real numbers $\alpha > 0$ and $\beta \ge 0$*

(h1) $$\sum_{n\le x} |v_n| = O(\log^{\alpha} x) \ ;$$

(h2) $$\sum_{n\le x} (nv_n)^2 = O(x\log^{\beta} x) \ ;$$

(h3) $p^k v(p^k)$ *is an ultimately monotonic function of p when $k = 1$ and $k = 2$, and is bounded for every $k \ge 1$.*

Then, if we set $y := x\exp(-(\log x)^b)$ for some positive number b, $t := \log x$, and $u := \log t = \log\log x$, we have

$$\sum_{n\le y} v_n\psi(\frac{x}{n}) = O(t^{2\alpha/3}u^{4\alpha/3}). \tag{3}$$

Theorem 3 . *Let $v_n = v(n)$ be a real multiplicative arithmetical function satisfying, for some real positive number α,*

(h1) $$\sum_{n\le x} |v_n| = O(\log^\alpha x) \;;$$

(h4) $v(p^j)$ *is of the same sign* $*$ *for all* p *and all* $j \ge 1$;

(h5) $$\sum_{i\ge 0} \frac{v(p^i)}{p^i} \ne 0 \quad \textit{for every } p \;.$$

Let P be the set of prime numbers if $* = +$ in (h4), and the set of primes $p \equiv 2(3)$ if $* = -$. Let m be a real positive unbounded variable, $0 < a < 1$, and define $A = A(m)$ and $x = x(m)$ as follows.

$$A := \prod_{\substack{p\le m\\ p\in P}} p =: \exp((\log x)^a) \;. \tag{4}$$

Finally let $y(X) := X\exp(-(\log X)^b)$ for some $b > a$, $b < 1$. Then there is a positive constant C such that for all sufficiently large m there are some numbers $X = X(m) \le (A+1)x$ and $X' = X'(m) \le (A+1)x$ satisfying

$$\sum_{n\le y(X)} v_n \psi(\frac{X}{n}) \ge C\Big(\prod_{\substack{p\le m\\ p\in P}} (1+|v_p|)\Big) + O(1) \tag{5}$$

and

$$\sum_{n\le y(X')} v_n \psi(\frac{X'}{n}) \le -C\Big(\prod_{\substack{p\le m\\ p\in P}} (1+|v_p|)\Big) + O(1). \tag{6}$$

2.1 Applications to the functions σ and ϕ

The sequences $\{a(n)\}$ in (1) satisfy the hypotheses of Theorem 1, and thus we can find a number b with $0 < b < 1$ such that for every real number α we have

$$\sum_{n\le x} \Big(\frac{\sigma(n)}{n}\Big)^\alpha = \zeta^\alpha(2) f_\alpha(2) x + \sum_{r=0}^{[\alpha]} a_r (\log x)^{\alpha - r} + e_{f_\alpha}(x) + o(1) \;, \tag{7}$$

where

$$e_{f_\alpha} := -\sum_{n\le y} v_{f_\alpha} \psi(\frac{x}{n}) \quad \text{and} \quad y := \exp(-(\log x)^b) \;,$$

where f_α and v_{f_α} are defined by

$$\begin{aligned}\sum_{n\le x}^{\infty} \frac{(\sigma(n)/n)^\alpha}{n^s} &= \zeta(s)\zeta^\alpha(s+1) f_\alpha(s+1) \quad \text{and} \\ \sum_{n=1}^{\infty} \frac{v_{f_\alpha}(n)}{n^s} &= \zeta^\alpha(s+1) f_\alpha(s+1) \;,\end{aligned}$$

and where the $a_r = a_r(\alpha)$ are certain real constants (the sum in which they appear being of course empty if $\alpha < 0$).

Similarly we have, with an obvious notation

$$\sum_{n\le x}\left(\frac{\phi(n)}{n}\right)^{\alpha} = \zeta^{-\alpha}(2)g_\alpha(2)x + \sum_{r=0}^{[-\alpha]} b_r(\log x)^{-\alpha-r} + e_{g_\alpha}(x) + o(1)\ , \tag{8}$$

and

$$\sum_{n\le x}\left(\frac{\phi(n)}{\sigma(n)}\right)^{\alpha/2} = \zeta^{\alpha}(2)k_\alpha(2)x + \sum_{r=0}^{[\alpha]} c_r(\log x)^{\alpha-r} + e_{k_\alpha}(x) + o(1)\ . \tag{9}$$

The following estimates for the error terms e_{f_α}, e_{g_α} and e_{k_α} of these summatory functions are consequences of Theorems 2 and 3.

Theorem 4 . *With the notation as just above we have, for each real number* α,

$$e_{h_\alpha} = O((\log x)^{\frac{2|\alpha|}{3}}(\log\log x)^{\frac{4|\alpha|}{3}})\ , \tag{10}$$

where h *denotes any of the symbols* f, g *and* k.

Theorem 5 . *On the other hand we have, also for each real number* α,

$$e_{h_\alpha} = \begin{cases} \Omega_\pm((\log\log x)^{|\alpha|}) & \text{if } h = f \text{ or } k \text{ and } \alpha \ge 0, \\ & \text{or } h = g \quad \text{and } \alpha \le 0; \\ \Omega_\pm((\log\log x)^{\frac{|\alpha|}{2}}) & \text{if } h = f \text{ or } k \text{ and } \alpha \le 0, \\ & \text{or } h = g \quad \text{and } \alpha \ge 0. \end{cases} \tag{11}$$

Comments. (1) For $\alpha = 1$ and $h = g$ Theorem 2 is due to Walfisz [18]; for $\alpha = 1$ and $h = f$ though, it is not as good as Walfisz' [18, (3.1.5)]: his proof exploits the monotonocity of $v_{f_1}(n) = 1/n$, and cannot be generalised to other values of α. For positive values of $\alpha \ne 1$ and $h = g$ Theorem 2 improves on Ilyasov's [7] and Sivaramasarma's [14]; for positive integral values of α it improves on Balakrishnan's [1]. As for the other cases there are to our knowledge no O-estimates in the literature.

(2) We believe Theorem 3 is new, except when $\alpha = 1$ and $h = f$ and when $\alpha = \pm 1$ and $h = g$. In these three cases it is Pétermann's [11], [12] and Montgomery's [10].

Corollary 1 . *If* $\beta > 0$ *we have*

$$\sum_{n\le x}\sigma^\beta(n) = \frac{\zeta^\beta(2)f_\beta(2)}{\beta+1}x^{\beta+1} + x^\beta\sum_{r=0}^{[\beta]} a'_r(\log x)^{\beta-r} + E_{f_\beta}(x) + o(x^\beta)\ , \tag{12}$$

where the $a'_r = a'_r(\beta)$ *are some real constants and*

$$E_{f_\beta}(x) = \begin{cases} O(x^\beta(\log x)^{2\beta/3}(\log\log x)^{4\beta/3}) \\ \Omega_\pm(x^\beta(\log\log x)^\beta)\ . \end{cases} \tag{13}$$

We also have

$$\sum_{n\le x}\phi^{\beta}(n)=\frac{\zeta^{-\beta}(2)g_{\beta}(2)}{\beta+1}x^{\beta+1}+E_{g_\beta}(x)+o(x^{\beta})\ , \tag{14}$$

with

$$E_{g_\beta}(x)=\begin{cases} O(x^{\beta}(\log x)^{2\beta/3}(\log\log x)^{4\beta/3}) \\ \Omega_{\pm}(x^{\beta}(\log\log x)^{\beta/2})\ . \end{cases} \tag{15}$$

Corollary 2 . *If* $-1\le\beta<0$ *we have*

$$\sum_{n\le x}\sigma^{\beta}(n)=\zeta^{\beta}(2)f_{\beta}(2)\ \times\begin{Bmatrix} \dfrac{x^{\beta+1}}{\beta+1} & \textit{if}\ \ -1<\beta<0 \\ \log x & \textit{if}\quad \beta=-1 \end{Bmatrix}+A+E_{f_\beta}(x)+o(x^{\beta})\ , \tag{16}$$

where $A=A(\beta)$ *is a constant and* $E_{f_\beta}(x)$ *satisfies*

$$E_{f_\beta}(x)=\begin{cases} O(x^{\beta}(\log x)^{2|\beta|/3}(\log\log x)^{4|\beta|/3}) \\ \Omega_{\pm}(x^{\beta}(\log\log x)^{|\beta|/2})\ . \end{cases} \tag{17}$$

We also have

$$\sum_{n\le x}\phi^{\beta}(n)=\zeta^{-\beta}(2)g_{\beta}(2)\times\begin{Bmatrix} \dfrac{x^{\beta+1}}{\beta+1} & {\scriptstyle(-1<\beta<0)} \\ \log x & {\scriptstyle(\beta=-1)} \end{Bmatrix}+B+x^{\beta}\sum_{r=0}^{[-\beta]}b'_r(\log x)^{-\beta-r}+E_{g_\beta}(x)+o(x^{\beta})\ , \tag{18}$$

where $b'_r=b'_r(\beta)$ *and* $B=B(\beta)$ *are constants and*

$$E_{g_\beta}(x)=\begin{cases} O(x^{\beta}(\log x)^{2|\beta|/3}(\log\log x)^{4|\beta|/3}) \\ \Omega_{\pm}(x^{\beta}(\log\log x)^{|\beta|})\ . \end{cases} \tag{19}$$

Corollary 3 . *If* $\beta<-1$ *We have*

$$\sum_{n>x}\sigma^{\beta}(n)=-\frac{\zeta^{\beta}(2)f_{\beta}(2)}{\beta+1}x^{\beta+1}+E_{f_\beta}(x)+o(x^{\beta})\ , \tag{20}$$

where $E_{f_\beta}(x)$ *satisfies (17), and*

$$\sum_{n>x}\phi^{\beta}(n)=-\frac{\zeta^{-\beta}(2)g_{\beta}(2)}{\beta+1}x^{\beta+1}+x^{\beta}\sum_{r=0}^{[-\beta]}b'_r(\log x)^{-\beta-r}+E_{g_\beta}(x)+o(x^{\beta})\ , \tag{21}$$

where $E_{g_\beta}(x)$ *satisfies (19).*

3 The methods

3.1 Theorem 1

The proof of Theorem 1 develops further Balakrishnan's technique in [1]. It relies on two main ideas. The first consists in making use of the inverse transform of

$$F(s) = \sum_{n\geq 1} g(n)n^{-s} = \int_{0-}^{\infty} e^{-ts} d(A(e^t)), \tag{22}$$

known as Perron's formula

$$A(x^-) + \frac{g(x)}{2} = \frac{1}{2\pi i}\int_{\kappa-i\infty}^{\kappa+i\infty} F(s)\frac{x^s}{s} ds \tag{23}$$

($g(x) = 0$ if $x \notin N$; $\kappa > \max(0, \sigma_a(F))$), in order to estimate the sum of coefficients

$$A(x) := \sum_{n\leq x} g(n). \tag{24}$$

For an adequate choice of T and κ the contribution of the two infinite vertical segments from $\kappa \pm it$ to $\kappa \pm i\infty$ on the right side of (23) is shown to be small (thus yielding an "effective" Perron's formula). Then the singularities of the integrand in (23) are exploited, the poles with the theorem of residues, and the other singularities s_0 by expanding $F(s)$ in (complex) powers of $s - s_0$ and using Hankel's formula (see [17, Théorème II.5.2]). In our cases the generating function F of the arithmetical function g has an expression as, or similar to, that in the title of this paper, and some classical estimates on the size of ζ inside the path of integration can be used.

This technique is sometimes referred to as the "Selberg-Delange" method. Directly applied to $g(n) = a(n)$ however, it doesn't yield satisfactory results: we obtain Theorem 1 with a O-estimate on the error term so weak we cannot even ensure that the term $B_0(\log x)^\alpha$ is significant.

The second idea consists in exploiting the fact that

$$a = 1 * v. \tag{25}$$

This easily yields

$$\sum_{n\leq x} a(n) = x\sum_{n\leq x}\frac{v(n)}{n} - \frac{1}{2}\sum_{n\leq x} v(n) - \sum_{n\leq x} v(n)\psi(\frac{x}{n}), \tag{26}$$

where we put $\psi(y) := \{y\} - 1/2$. The "Selberg-Delange" method is then efficient in dealing with $g(n) = nv(n)$, and partial summation takes care of the two first sums on the right of (26). As for the third one it is truncated by elementary (i.e. real analysis) means using an idea due to P. Codecà [4].

3.2 Theorem 2

The proof of Theorem 2 is based on Walfisz' treatment of the case $a(n) = \phi(n)/n$ in Chapter IV of [18]. Very briefly:

(i) we replace (with a resulting small error)

$$\sum_{Q}^{Q'} \psi(\frac{x}{q}) \quad \text{by} \quad \sum_{Q}^{Q'} \int_0^{\frac{1}{x}} \psi(\frac{x}{q} + \vartheta) d\vartheta \qquad (Q \leq Q' \leq 2Q, Q' \leq y) ;$$

(ii) we expand ψ in its Fourier series

$$\psi(x) = \frac{-1}{2\pi i} \lim_{N \to \infty} \sum_{-N}^{N} \frac{e(nx)}{n} ;$$

(iii) we exchange the summation order of the left side of (3), and then use methods due to Weyl for rather large M and to Vinogradov and Korobov for rather small M to bound sums of the type

$$\sum_{M}^{M'} e(\frac{t}{m}) \qquad (M \leq M' \leq 2M) ;$$

(iv) we thus obtain the estimate

$$\sum_{w \leq n \leq y} v_n \psi(\frac{x}{n}) = O(1) \qquad (w := \exp(t^{2/3} u^{4/3})) ;$$

(v) and we conclude with the trivial remark that

$$\sum_{n \leq w} v_n \psi(\frac{x}{n}) = O(t^{2\alpha/3} u^{4\alpha/3}) .$$

3.3 Theorem 3

The proof of Theorem 3 is based on a method that to our knowledge originated in a work by Erdös ans Shapiro [6]. It was developed further by Codecà [4] and Pétermann [12]. It consists in averaging the error term

$$E(x) := \sum_{n \leq y} v_n \psi(\frac{x}{n}) \tag{27}$$

over arithmetical progressions $An + B$ $(n \leq x)$ of very large moduli $A = A(x)$ (in our case A is as large as $\exp((\log x)^a))$ for some a with $0 < a < 1$). We generalise to our functions v the formula

$$\frac{1}{x} \sum_{n \leq x} E(An + B) = \sum_{k \leq u(x)} \frac{v(k)}{k} (A, k) \psi(\frac{B}{(A, k)}) + O(1) , \tag{28}$$

(where u is a certain function with $u(x) = o(x \log^{1-\alpha} x)$ proved in [12] for bounded v's. With its help the oscillations estimates of Theorem 3 are obtained by making adequate choices of A and B, ensuring that k divides A "often", and that the quantity

$$\frac{v(k)}{|v(k)|}\psi\Big(\frac{B}{(A,k)}\Big)$$

stays "often" away from 0 with the same sign.

References

[1] U. Balakrishnan. *On the sum of divisors function,* preprint.

[2] U. Balakrishnan and Y.-F.S. Pétermann. *The Dirichlet series of* $\zeta(s)\zeta^{\alpha}(s+1)f(s+1)$*: On an error term associated to its coefficients,* preprint.

[3] S.D. Chowla. *An order result involving Euler's* ϕ*- function,* J. Indian Math. Soc. 18 (1929/30), 138-141.

[4] P. Codecà. *On the properties of oscillations and almost periodicity of certain convolutions,* Rend. Sem. Math. Univ. Padova 71 (1984) 373-387.

[5] P.J.G.L. Dirichlet. *Ueber die Bestimmung der mittleren Werthe in der Zahlentheorie,* Abh. Kön. Preuss. Akad. (1849), 69-83. (Also in: Werke II, Berlin 1897, 51-66).

[6] P. Erdös and H.N. Shapiro. *On the changes of sign of a certain error function,* Canad. J. Math. 3 (1951), 375-385.

[7] **И.И. Ильясов. Оценка остаточного члена суммы $\sum_{n\leq x}(\phi(n)/n)^{\alpha}$, Изв. Акад. Наук. Казак. ССР сер. Физ. Мат. 3 (1963), 77-79.**

[8] A. Ivić. *The Riemann zeta-function.* John Wiley & Sons 1985.

[9] E. Landau. *Ueber die zahlentheoretische Funktion* $\phi(n)$ *und ihre Beziehung zur Goldbachschen Satz,* Gött. Nachr. (1900), 177-186.

[10] H.L.Montgomery, *Fluctuations in the mean of Euler's phi function*, Proc. Indian Acad. Sci. (Math. Sci.) 97 (1987) 239-245.

[11] Y.-F.S. Pétermann. *An* Ω*-theorem for an error term related to the sum-of-divisors function,* Mh. Math. 103 (1987), 145-157; Addendum ibid. 104 (1988), 193-194.

[12] Y.-F.S. Pétermann. *About a theorem of Paolo Codecà's and* Ω*-estimates for arithmetical convolutions,* J. Number Theory 30 (1988), 71-85; Addendum ibid. 36 (1990), 322-327.

[13] S. Ramanujan. *Some formulae in the theory of numbers,* Mess. Math. XLV (1916), 81-84. (Also in: Collected papers, Cambridge Univ. press 1927.)

[14] A. Sivaramasarma. *Some problems in the theory of Farey series and the Euler totient function.* Doctoral thesis (Chapter 8), Waltair 1979.

[15] R. Sitaramachandrarao. *On an error term of Landau,* Indian J. pure appl. Math. 13 (1982), 882-885.

[16] R.A. Smith. *An error term of Ramanujan,* J. Number Theory 2 (1970), 91-96.

[17] G. Tenenbaum. *Introduction à la théorie analytique et probabiliste des nombres.* Institut Elie Cartan 13, 1990.

[18] A. Walfisz. *Weylsche Exponentialsummen in der neueren Zahlentheorie.* VEB Deutscher Verlag der Wissenschaften, Berlin 1963.

[19] S. Wigert. *Sur quelques fonctions arithmétiques,* Acta Math. 37 (1914), 113-140.

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