数理解析研究所講究録
第 837 巻 1993 年 84-91

On the Generalized Divisor Problem

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§1. The Generalized Divisor Problem

Let $d_z(n)$ be a multiplicative function defined by

$$\zeta^z(s) = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^s} \quad (\sigma > 1)$$

where $s = \sigma + it$, z is a complex number, and $\zeta(s)$ is the Riemann zeta function. Here $\zeta^z(s) = \exp(z \log \zeta(s))$ and let $\log \zeta(s)$ take real values for real $s > 1$.

We note that if z is a natural number, $d_z(n)$ coincides with the divisor function appearing in the Dirichlet-Piltz divisor problem, and $d_{-1}(n)$ with the Möbious function.

The generalized divisor problem is concerned with finding an asymptotic formula for $\sum_{n \le x} d_z(n)$, which was observed for real $z > 0$ by A. Kienast [8] and K. Iseki [5] independently. A. Selberg [17] considered for all complex z, his result being

$$D_z(x) \equiv \sum_{n \le x} d_z(n) = \frac{x(\log x)^{z-1}}{\Gamma(z)} + O(x(\log x)^{\Re z - 2})$$

uniformly for $|z| \le A$, $x \ge 2$, where A is any fixed positive number.

This was extended by Rieger [16] to arithmetic progressions such that

$$D_z(x, q, l) \equiv \sum_{\substack{n \le x \\ n \equiv l \ (\mathrm{mod}\ q)}} d_z(n) = \left(\frac{\varphi(q)}{q}\right)^z \frac{x}{\Gamma(z)\varphi(q)} (\log x)^{z-1}$$

$$+ O\left(\left(\frac{\varphi(q)}{q}\right)^z \frac{x}{\varphi(q)} (\log x)^{\Re z - 2} \log\log 4q\right)$$

uniformly for $|z| \leq A$, $q \leq (\log x)^{\tau}, (q, l) = 1$, where A and τ are any fixed positive numbers.

On the other hand R.D. Dixon [4] derived the following formula which is a sharpening of Selberg's result;

$$D_z(x) \equiv \sum_{n \leq x} d_z(n)$$

$$= x(\log x)^{z-1} \sum_{m=0}^{N-1} \frac{B_m(z)}{(\log x)^m \Gamma(z-m)} + O(x(\log x)^{\Re z - N - 1})$$

uniformly for $|z| \leq A$, where $B_m(z)$ are regular functions of z, especially $B_0(z) = 1$.

We shall consider the connections between the asymptotic formula of $D_z(x, q, l)$ and the location of zeros of the Dirichlet L-function (Riemann zeta function). The main terms of above formulas are, however, inconvenient for our aim so that we introduce the following integral as the main term of $D_z(x, q, l)$:

$$\Phi_z(x, q) = \frac{1}{2\pi i} \int_L (L(s, \chi_0))^z \frac{x^s}{s} \, ds$$

where L is, for any r $(0 < r < 1/2)$, the path which begins at $1/2$, moves to $1 - r$ along the real axis, encircle the point 1 with radius r in the counterclockwise direction, and returns to $1/2$ along the real axis. Here we denote by χ_0 the principal character $mod\, q$.

We note that if $z = k \in N$,

$$\Phi_k(x, 1) = x P_{k-1}(\log x), \quad \Phi_{-k}(x, q) = 0,$$

where $P_{k-1}(x)$ is the same polynomial of degree $k-1$ as in the Dirichlet divisor problem.

The error term is defined by

$$\Delta_z(x, q, l) = D_z(x, q, l) - \frac{1}{\varphi(q)} \Phi_z(x, q),$$

and let

$$\Theta(\chi) = \sup\{\sigma : L(\sigma + it, \chi) = 0\}, \quad \Theta_q = \max_{\chi (mod q)} \Theta(\chi).$$

THEOREM 1. *There exists some constant c such that*

$$\Delta_z(x, q, l) \ll x e^{-c\sqrt{\log x}}$$

uniformly for $|z| \leq A$, $q \leq (\log x)^\tau$, $(q, l) = 1$ *where* A *and* τ *are any fixed positive numbers.*

Further we have

$$\Delta_z(x, q, l) \ll x^{\Theta_q + \epsilon}$$

uniformly for $|z| \leq A$, $q \leq x$, $(q, l) = 1$.

Conversely if $\Delta_z(x, q, l) \ll x^{\Xi + \epsilon}$ *for any* l $((q, l) = 1)$ *and for some* $z \in C - Q^+$, *where* Q^+ *denotes the set of all non negative rational numbers, then any* $L(s, \chi)$ $(mod\, q)$ *has no zeros for* $\sigma > \Xi$.

The main term $\Phi_z(x, q)$ *has an asymptotic expansion*

$$\Phi_z(x, q) = x(\log x)^{z-1} \sum_{m=0}^{N-1} \frac{B_m(z, q)}{(\log x)^m \Gamma(z - m)} + O(x(\log x)^{\Re z - N - 1})$$

uniformly for $|z| \leq A$. *Here* N *is any fixed positive integer and* $B_m(z, q)$ $(0 \leq m \leq N - 1)$ *are regular functions of* z, *especially* $B_0(z, q) = (\varphi(q)/q)^z$.

Remark

1. For $q = 1$, we define α_z by

$$\alpha_z = \inf\{\alpha : \Delta_z(x, 1, 1) \ll x^\alpha\}$$

in a similar way as in the Dirichlet divisor problem. Then it is well known that

$$L.\,H. \iff \alpha_k \le 1/2 \qquad for\ ^{\forall}k \in N.$$

Theorem 1 shows that

$$R.\,H. \implies \alpha_z \le 1/2 \qquad for\ ^{\forall}z \in C,$$

Conversely,

$$\alpha_z \le 1/2 \qquad for\ ^{\exists}z \in C - Q^+ \implies R.\,H.$$

If we suppose that all the zeros of $\zeta(s)$ are simple, the last statement holds for $^{\exists}z \in C - N$.

2. R. Balasubramanian and K. Ramachandra [1] have observed the following asymptotic formura by a method similar to that of Selberg [18] or H. Delange [2][3].

$$\sum_{nd(n)\le x} 1 = M(x) + O(x\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5}))$$

$$M(x) = \frac{1}{2\pi i}\int_C f(s)\frac{x^s}{s}\,ds \quad \asymp \frac{x}{\sqrt{\log x}}.$$

where C is a suitable path which is similer to L.

3. A. Ivić [7] evaluated the integral $\int_1^T |\zeta(1+it)|^{2z}dt$ by using Theorem 1, where $z > 0$ is an arbitrary fixed real number.

4. Similar results are hold for the following sums :

$$\sum_{\substack{n\le x \\ n\equiv l\ (mod\, q)}} z^{\omega(n)}, \qquad \sum_{\substack{n\le x \\ n\equiv l\ (mod\, q)}} z^{\Omega(n)},$$

where $\omega(n)$ means the number of distinct prime factors of n, and $\Omega(n)$ means the number of total prime factors allowing multiplicity.

§2. The Asymptotic Formula for $\pi_k(x,q,l)$

Let $\pi_k(x)$ be the number of integers $\leq x$ which are products of k distinct primes. For $k=1$, $\pi_k(x)$ reduces to $\pi(x)$, the number of primes not exeeding x. C.F. Gauss stated empirically that $\pi_2(x) \sim x(\log\log x)/\log x$, and, by using the prime number theorem, E. Landau proved that $\pi_k(x) \sim x(\log\log x)^{k-1}/(k-1)!\log x$. Selberg considered $D_z(x)$ not only for its own sake but also with an intension to derive

$$\pi_k(x) = \frac{xQ(\log\log x)}{\log x} + O(\frac{x(\log\log x)^k}{k!(\log x)^2})$$

uniformly for $1 \leq k \leq A\log\log x$, where $Q(x)$ is polynomial of degree $k-1$.

H. Delange [3] obtained as a sharpening of Selberg's result,

$$\pi_k(x) = \frac{x}{\log x}\sum_{m=0}^{N-1}\frac{Q_m(\log\log x)}{(\log x)^m} + O(\frac{x(\log\log x)^{k-1}}{(\log x)^{N+1}})$$

for every $k \geq 1$, where N is any fixed integer ≥ 1 and $Q_m(x)$ are polynomials of degree not exeeding $k-1$.

We define $\pi_k(x,q,l)$ as a generalization of $\pi(x,q,l)$ by

$$\pi_k(x,q,l) \equiv \sum_{\substack{n\leq x \\ n\equiv l \,(\mathrm{mod}\, q) \\ n=p_1\cdots p_k\ (p_i\neq p_j)}} 1.$$

In order to concider the connections between $\pi_k(x,q,l)$ and zeros of $L(s,\chi)$ $(\zeta(s))$ we define the following integral as the main term of $\pi_k(x,q,l)$:

$$F_{k,\delta}(x,q) = \frac{1}{(2\pi i)^2}\int_{|z|=1}\int_{L_\delta}(L(s,\chi_0))^z \times \{\prod_p(1+\frac{z\chi_0(p)}{p^s})(1-\frac{\chi_0(p)}{p^s})^z\}\frac{1}{z^{k+1}}\frac{x^s}{s}\,ds\,dz$$

where L_δ is, for every δ and any r ($\delta > 0$, $r > 0$, $\delta + r < 1/2$), the path which begins at $1/2+\delta$, moves to $1-r$ along the real axis, encircle the point 1 with radius r in the counterclockwise direction, and returns to $1/2+\delta$ along the real axis.

For $k = 1$, we can express this in terms of the logalithmic integral. Namely,

$$F_{1,\delta}(x,q) = \int_2^x \frac{du}{\log u} + O(x^{1/2+\delta}).$$

The generating function of $\pi_k(x)$ is

$$\sum_{\ell=0}^{k} \frac{1}{\ell!(k-\ell)!}(\log\zeta(s))^\ell f^{(k-\ell)}(s,0)$$

where

$$f(s,z) = \prod_p (1+\frac{z}{p^s})(1-\frac{1}{p^s})^z,$$

$f^{(n)}(s,z)$ means the n-th derivative of $f(s,z)$ with respect to z.

The error term is defined by

$$R_{k,\delta}(x,q,l) = \pi_k(x,q,l) - \frac{1}{\varphi(q)}F_{k,\delta}(x,q).$$

THEOREM 2. *There is some constant c such that*

$$R_{k,\delta}(x,q,l) \ll xe^{-c\sqrt{\log x}}$$

uniformly for $k \geq 1$, $q \leq (\log x)^\tau$, $(q,\ l) = 1$.

Further we have

$$R_{k,\delta}(x,q,l) \ll x^{\Theta_q+\epsilon}$$

uniformly for $k \geq 1$, $q \leq x$, $(q,\ l) = 1$.

Conversely if $R_{k,\delta}(x,q,l) \ll x^{\Xi+\epsilon}$ *for any* l $((q,\ l) = 1)$ *and for some* $k \geq 1$, *then any* $L(s,\chi)$ $(mod\, q)$ *has no zeros for* $\sigma > \Xi$.

The main term $F_{k,\delta}(x,q)$ has an asymptotic expansion

$$F_{k,\delta}(x,q) = \frac{x}{\log x}\sum_{m=0}^{N-1}\frac{Q_{m,q}(\log\log x)}{(\log x)^m} + O(\frac{x(\log\log x)^{k-1}}{(\log x)^{N+1}})$$

for every k and q. Here N is any fixed positive integer and $Q_{m,q}(x)$ are polynomials of degree not exceeding $k-1$, especially the coefficient of x^{k-1} of $Q_{0,q}(x)$ is 1.

Remark

1. If we define $r_{k,q,l}$ by

$$r_{k,q,l} = \inf_{\delta}\ \inf\{r : R_{k,\delta}(x,q,l) \ll x^r\}$$

Theorem 2 shows that

$$\max_{l}\ r_{k,q,l} = \Theta_q.$$

The statement $\Theta_q = 1/2$ for every q is equivalent to the truth of the Riemann hypothesis for the Dirichlet L-function.

2. Similar results are hold for the following sums :

$$\omega_k(x,q,l) \equiv \sum_{\substack{n\le x\\ n\equiv l\ (\mathrm{mod}\, q)\\ \omega(n)=k}} 1,\quad \Omega_k(x,q,l) \equiv \sum_{\substack{n\le x\\ n\equiv l\ (\mathrm{mod}\, q)\\ \Omega(n)=k}} 1.$$

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