数理解析研究所講究録
第 837 巻 1993 年 92-100


# On the representation of numbers as the sum of a prime and a k-th power.

by Koichi KAWADA (University of Tsukuba).
川田 浩一 （筑波大学大学院）

## 1. Introduction.

Let $k \geq 2$ be a fixed integer, and, for a natural number $n$, let $r_k(n)$ be the number of representations of $n$ as the sum of a prime and a $k$-th power.

In the case $k=2$, Hardy and Littlewood [3] conjectured that, unless $n$ is a square,

$$(1) \qquad r_2(n) \sim \frac{\sqrt{n}}{\log n} \prod_{p>2} \left(1 - \frac{\left(\frac{n}{p}\right)}{p-1}\right),$$

where $p$ denotes prime numbers, and $\left(\frac{n}{p}\right)$ denotes the Legendre symbol. In 1968, Miech [5] showed that the above asymptotic formula (1) is valid for almost all $n$. More precisely, he proved that

$$r_2(n) = \frac{\sqrt{n}}{\log n} \prod_{p>2} \left(1 - \frac{\left(\frac{n}{p}\right)}{p-1}\right) \cdot \left\{ 1 + O\left(\frac{\log\log n}{\log n}\right) \right\},$$

for all but $O(N(\log N)^{-A})$ natural numbers $n \leq N$ with any $A>0$. It seems impossible, for the present, to improve Miech's result because of the possible existence of the Siegel zeros.

On the other hand, to show that $n$ is representable as the sum of a prime and a square, we need only a positive lower bound for $r_2(n)$, and which was obtained with less exceptional $n$'s. A.I.Vinogradov [8] and Brünner, Perelli and Pintz [1] proved that there exist a positive constant $\delta$ such that $r_2(n)>0$ for $n\le N$ with at most $O(N^{1-\delta})$ exceptions.

Proofs of these result for the case $k=2$ are based on the circle method, and most part of the proofs in [5] and [1] still work for the case $k>2$. Essential difference between the cases $k=2$ and $k>2$ occurs in the treatment of the sum called "singular series". So we investigate the "singular series" for the general case $k\ge 2$.

We denote by $\rho_n(d)$ the number of solutions of the congruence

$$x^k - n \equiv 0 \pmod{d}.$$

Then the singular series for our problem is the sum of the form;

$$\mathfrak{S}(n,M) = \sum_{m\le M} \frac{\mu(m)}{\varphi(m)} \prod_{p|m} (\rho_n(p)-1),$$

where $\mu$ and $\varphi$ denote the Möbius function and Euler's totient function, respectively. It is proved that, for almost all $n$, the sum $\mathfrak{S}(n,M)$ is approximated by the finite product of the form $\prod_{p\le M'}\left(1-\frac{\rho_n(p)-1}{p-1}\right)$, and then a good positive lower bound for $\mathfrak{S}(n,M)$ is obtained. This is essentially due to Plaksin[6]. This work with the arguement in [1] yields the corresponding result for $k\ge 3$ of [1] and [8], namely, there exist a positive constant $\delta$ depending only on $k$ such that $r_k(n)>0$ for all $n\le N$ with at most $O(N^{1-\delta})$ exceptions.

Next, we consider the corresponding result for $k\ge 3$ of Miech's result [5]. We define the set

$$E_k = \{n\in \mathbb{N}\ ;\ \text{the polynomial } x^k-n \text{ is irredusible in } \mathbb{Q}[x]\}.$$

Then, instead of (1), we can expect that, for $n\in E_k$,

$$r_k(n) \sim \frac{n^{1/k}}{\log n} \prod_p \left(1 - \frac{\rho_n(p)-1}{p-1}\right) .$$

And our result is

THEOREM. For $k \geq 3$ and for any $A > 0$, we have

$$r_k(n) = \frac{n^{1/k}}{\log n} \prod_p \left(1 - \frac{\rho_n(p)-1}{p-1}\right) \cdot \left\{1 + O\left(\frac{\log\log n}{\log n}\right)\right\} ,$$

for all $n \leq N$ with at most $O(N(\log N)^{-A})$ exceptions.

In order to prove this, we need more precise treatment for the singular series $\mathfrak{S}(n, M)$ than Plaksin's way [6]. The rest of this article outlines the main features of our proof of the result. As for the details, refer to [4].

## 2. Treatment of the singular series.

Let $N$ be a sufficiently large real number. By a standard application of the circle method, it follows that

$$(2) \quad r_k(n) = \mathfrak{S}(n, (\log N)^B) \cdot \left\{\frac{n^{1/k}}{\log n} + O\left(\frac{n^{1/k} \log\log n}{(\log n)^2}\right)\right\} + O(N^{1/k} (\log N)^{-A'}) ,$$

for all but $O(N(\log N)^{-A})$ natural numbers $n \leq N$, where $A$ and $A'$ are arbitrary positive constants, and $B$ is a positive constant depending on $A$, $A'$ and $k$. On the proof of this fact, there is no essential difference between the cases $k=2$ and $k>2$.

Making use of the inequality (8) below, we see easily that

$$(3) \quad \mathfrak{S}(n, (\log N)^B) = \mathfrak{S}(n, \sqrt{N}) + O((\log N)^{-A'}) ,$$

for all $n \leq N$ with at most $O(N(\log N)^{-A})$ exceptions. And we proceed to investigate $\mathfrak{S}(n, \sqrt{N})$. We start with applying Perron's formula. As usual, let $s=\sigma+it$ be a complex variable. We introduce the function

$$Z_n(s)=\prod_p\left(1-\frac{\rho_n(p)-1}{p^{s-1}(p-1)}\right),$$

for $\sigma>1$. And we put $b=\frac{1}{\log N}$ and $T=\exp(\sqrt{\log N})$. Then we have routinely

$$(5)\ \mathfrak{S}(n,\sqrt{N})=\frac{1}{2\pi i}\int_{b-iT}^{b+iT}Z_n(s)\frac{\sqrt{N}^{\,s-1}}{s-1}\,ds+\text{(Admissible remainder)},$$

for $n\le N$. So we need some information about $Z_n(s)$ near the line $\sigma=1$.

On the other hand, let $\zeta(s)$ and $\zeta_n(s)$ be the Riemann zeta function and the Dedekind zeta function of the field $\mathbb{Q}(n^{1/k})$, respectively. The Euler product for $\zeta_n(s)$ is written as

$$\zeta_n(s)=\prod_p\prod_{1\le f\le k}(1-p^{-fs})^{-a_n(f,p)},$$

where $a_n(f,p)$ is the number of prime ideals $\mathfrak{p}$ in $\mathbb{Q}(n^{1/k})$ such that the norm of $\mathfrak{p}$ is $p^f$. By a known fact about $a_n(f,p)$'s, we see

$$a_n(1,p)=\rho_n(p),$$

providing $n\in E_k$ and $p\nmid kn$. Then the Euler product for $\zeta(s)/\zeta_n(s)$ becomes

$$\frac{\zeta(s)}{\zeta_n(s)}=\prod_p(1-p^{-s})^{-1+\rho_n(p)}\cdot\prod_p\prod_{2\le f\le k}(1-p^{-fs})^{a_n(f,p)}\cdot\prod_{p\mid kn}(1-p^{-s})^{a_n(1,p)-\rho_n(p)}$$

$$(6)\qquad =\prod_p\left(1-\frac{\rho_n(p)-1}{p^s}\right)\cdot\prod_p\left\{(1-p^{-s})^{-1+\rho_n(p)}\cdot\left(1-\frac{\rho_n(p)-1}{p^s}\right)^{-1}\right\}\times$$
$$\times\prod_p\prod_{2\le f\le k}(1-p^{-fs})^{a_n(f,p)}\cdot\prod_{p\mid kn}(1-p^{-s})^{a_n(1,p)-\rho_n(p)}$$

$$=Z_n(s)\,\xi_n(s)\,\Xi_n(s),$$

where

$$\zeta_n(s) = \prod_p \left\{ (1-p^{-s})^{-1+\rho_n(p)} \left(1-\frac{\rho_n(p)-1}{p^{s-1}(p-1)}\right)^{-1} \right\} \cdot \prod_p \prod_{2\le f\le k} (1-p^{-fs})^{a_n(f,p)},$$

and

$$\Xi_n(s) = \prod_{p|kn} (1-p^{-s})^{a_n(1,p)-\rho_n(p)}.$$

We note here that $\Xi_n(s)$ is written as the finite product, and that $\zeta_n(s)$ is treated easily near the line $\sigma=1$. Hence, in view of (6), we regard, essentially, $Z_n(s)$ as $\zeta(s)/\zeta_n(s)$.

In our case, $\zeta_n(s)/\zeta(s)$ is an entire function, which was due to Uchida [7] and van der Waall [9] (independetly). Therefore, if $\zeta_n(s)/\zeta(s)$ has no zero near the line $\sigma=1$, then $\zeta(s)/\zeta_n(s)$ is analytic near $\sigma=1$, and so is $Z_n(s)$. Then, using Hadamard's three circle theorem, we have a good estimate for $Z_n(s)$ near $\sigma=1$, and, by (5), we get, with a suitable constant $\eta>0$,

$$\mathfrak{S}(n,\sqrt{N}) = Z_n(1) + \frac{1}{2\pi i}\left(\int_{b-iT}^{1-\eta-iT} + \int_{1-\eta-iT}^{1-\eta+iT} + \int_{1-\eta+iT}^{b+iT}\right) Z_n(s)\frac{\sqrt{N}^{\,s-1}}{s-1}\,ds +$$

$$+ \text{(Admissible remainder)}$$

$$= Z_n(1) + \text{(Admissible remainder)}.$$

In fact, we obtain the following Lemma 1.

LEMMA 1. Let $\mathcal{N}(n;\alpha,T)$ be the number of zeros of $\zeta_n(s)/\zeta(s)$ in the region $\sigma\ge\alpha$ and $|t|\le T$. Assume that $n\le N$, $n\in E_k$ and $\mathcal{N}(n;1-\delta,\exp(\sqrt{\log N}))=0$ with some positive constant $\delta$. Then we have

$$\mathfrak{S}(n,\sqrt{N}) = \prod_p \left(1-\frac{\rho_n(p)-1}{p-1}\right) + O\left(\exp\left(-\tfrac{1}{2}\sqrt{\log N}\right)\right).$$

3. Zero density estimate.

We see plainly that the number of $n$'s such that $n \le N$ and $n \notin E_k$ is $O(\sqrt{N})$. So, in view of (2), (3) and Lemma 1, in order to prove our theorem, it suffices to show that there are positive constants $\delta$ and $\delta'$ such that

$$\sum_{\substack{n \le N \\ n \in E_k}} \mathcal{N}(n; 1-\delta, \exp(\sqrt{\log N})) \ll N^{1-\delta'} . \tag{7}$$

In other words, we need a zero density estimate for $\zeta_n(s)/\zeta(s)$'s. We note that, for the case $k=2$, the function $\zeta_n(s)/\zeta(s)$ is the Dirichlet L-function for a certain real primitive character, if $n$ is not a square. And, in [5], Miech used Bombieri's zero density theorem for L-functions proved in 1965. We see here the most important difference between $k=2$ and $k>2$.

The inequality (7) follows at once from the following Lemma 2. Therefore, our proof of the theorem is completed by justifying Lemma 2.

LEMMA 2. For a natural number $r$, we put $\sigma_1 = 1 - \frac{1}{r(r-1)}$. We suppose $\sigma_1 \ge \frac{\log(k-1)}{\log(k+1)}$, $T \ge 1$ and

$$(NT)^{(r+1)(k-1)(3-2\sigma_1)} \le N^{r(r-1)} .$$

Then we have, for $\frac{1}{2} \le \sigma < 1$,

$$\sum_{\substack{n \le N \\ n \in E_k}} \mathcal{N}(n; \sigma, T) \ll (NT)^{1 - \frac{\sigma - \sigma_1}{3 - \sigma - \sigma_1} + \varepsilon} ,$$

with any $\varepsilon > 0$.

Remark. In application of Lemma 2 to show (7), we take $T = \exp(\sqrt{\log N})$ and $r = k+1$.

It is well known that zero density estimates for Dirichlet

L-functions is obtained from the large sieve inequality, namely,

$$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi\,(\mathrm{mod}.\,q)} \left|\sum_{m=M_0+1}^{M_0+M} a_m\chi(m)\right|^2 \ll (Q^2+M)\sum_{m=M_0+1}^{M_0+M}|a_m|^2 ,$$

where $\sideset{}{^*}\sum_{\chi\,(\mathrm{mod}.\,q)}$ indicates the summation over all primitive characters (mod $q$).

We should prepare the inequalities which work in our zero density estimate instead of the large sieve inequality. Now we put $\beta_n(m)=\mu(m)^2\prod_{p|m}(\rho_n(p)-1)$ . As we see in the preceding section, we can regard, essentially, $\zeta_n(s)/\zeta(s)$ as

$$\prod_p\left(1+\frac{\rho_n(p)-1}{p^s}\right)=\sum_{m=1}^{\infty}\beta_n(m)\,m^{-s} ,$$

for $\sigma>1$ , because of (6). So we consider how to estimate the sum of the form;

$$\sum_{n\le N}\left|\sum_{m\le M}a_m\beta_n(m)\right|^2 .$$

For a square-free natural number $m$, we define the sets $C_m$ and $C_m^*$ of Dirichlet characters (mod $m$) as follows;

$$C_m=\{\chi\ (\mathrm{mod}\ m)\ ;\ \chi^k=\chi_{0,m}\ \text{and}\ \chi\ne\chi_{0,m}\} ,$$
$$C_m^*=\{\ \chi\in C_m\ ;\ \chi\ \text{is primitive.}\} ,$$

where $\chi_{0,m}$ denotes the principal character (mod $m$). As is mentioned in [6], we find easily the relation

$$\beta_n(m)=\sum_{\chi\in C_m^*}\chi(n)$$

for any square-free $m$. Making use of this fact, we get

$$\sum_{n\le N}\left|\sum_{m\le M}a_m\beta_n(m)\right|^2=\sum_{m_1\le M}\sum_{m_2\le M}a_{m_1}\overline{a_{m_2}}\sum_{\chi_1\in C_{m_1}^*}\sum_{\chi_2\in C_{m_2}^*}\sum_{n\le N}\chi_1\overline{\chi_2}(n) ,$$

and, by the Pólya-Vinogradov inequality, we have

$$(8) \qquad \sum_{n\le N}\Big|\sum_{m\le M} a_m\beta_n(m)\Big|^2 \ll (N+M^2\log M)\sum_{m\le M}\mu(m)^2\tau_k(m)^2|a_m|^2,$$

where $\tau_k(m)$ is the number of the factorizations of $m$ into $k$ positive numbers.

We see that the inequality (8) gives only a trivial bound when $M>N$. In this case, we need, instead of the Pólya-Vinogradov inequality, a non-trivial bound for the sum

$$\sum_{\substack{m\le M\\ \mu(m)^2=1}}\ \sum_{\chi\in C_m}\Big|\sum_{n\le N}\chi(n)\Big| .$$

We estimate this sum by the method indicated in [2], and obtain that the quantity of the sum is

$$\ll N^{1-\frac{1}{r+1}+\varepsilon}M ,$$

where $r$ is a natural number satisfying $M^{r+1}\le N^{r(r-1)}$. Applying this estimate, we have

$$(9) \qquad \sum_{n\le N}\Big|\sum_{m\le M} a_m\beta_n(m)\Big|^2 \ll N\sum_{m\le M}\tau_k(m)|a_m|^2 + N^{1-\frac{1}{r+1}+\varepsilon}\max_{M_1\le M}\Big(M_1\max_{M_1<m\le 2M_1}|a_m|\Big)^2,$$

where $r$ is a natural number satisfying $M^{2(r+1)}\le N^{r(r-1)}$.

Then our Lemma 2 is derived by the standard method in the study of zero density for L-functions, using the inequalities (8) and (9) instead of the large sieve inequality.

## References

[1] Brünner,R., Perelli,A. and Pintz,J.; The exceptional set for the sum of a prime and a square., Acta Math. Hungar. 53 (1989), 347-365.

[2] Elliott,P.D.T.A.; The distribution of primitive roots., Canad. J. Math. 21(2) (1969), 822-841.

[3] Hardy,G.H. and Littlewood,J.E.; Some problem of 'partitio numerorum' (III)., Acta Math. 44 (1923), 1-70.

[4] Kawada,K.; On the sum of a prime and a k-th power., preprint.

[5] Miech,R.J.; On the equation $n=p+x^2$., Trans. Amer. Math. Soc. 130 (1968), 494-512.

[6] Plaksin,V.A.; On a question of Hua Loo-Keng. (Russian), Mat. Zametki 47(3) (1990), 78-90. Engl. transl.: Math. Notes 47(3/4) (1990), 278-286.

[7] Uchida,K.; On Artin L-functions., Tôhoku Math. J. (2) 27 (1975), 75-81.

[8] Vinogradov,A.I.; On a binary additive problem of Hardy-Littlewood. (Russian), Acta Arith. XLVI (1985), 33-56.

[9] van der Waall,R.W.; On a conjecture of Dedekind on zeta functions., Indag. Math. 37 (1975), 83-86.

Institute of Mathematics, University of Tsukuba,
Tsukuba-city, Ibaraki Pref., 305 Japan.
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