

Linear independence results on values related to  
 higher dimensional continued fractions

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SUMMARY: We are intending to describe our results [T4, T6] in a rather self-contained fashion with some remarks. We shall give no proofs of our theorems. Let  $s \geq 1$ ,  $k \geq 1$  be fixed integers,  $A := \{a_0, a_1, \dots, a_s\}$  be a set, and let  $\sigma \in \text{Hom}(A^*, A^*)$  be a monoid homomorphism defined by

$$\sigma(a_0) := a_0^k a_1, \quad \sigma(a_i) := a_{i+1} \quad (1 \leq i \leq s-1), \quad \sigma(a_s) := a_0,$$

where  $A^*$  denotes the free monoid generated by  $A$ . A homomorphism  $\tau \in \text{Hom}(A^*, B^*)$  can be extended to  $A^* \cup A^\infty$  by defining  $\tau(u_1 u_2 \dots u_n \dots) := \tau(u_1) \tau(u_2) \dots \tau(u_n) \dots$  ( $u_n \in A$ ), where  $A^\infty$  denotes the set of all infinite words (to the right) over  $A$ . We denote by  $\omega = \omega_0 \omega_1 \dots \omega_n \dots = \lim_{m \rightarrow \infty} \sigma^m(a_0)$  ( $\omega_n \in A$ ) the fixed point of the  $\sigma$  prefixed by  $a_0$ , where  $\sigma^m$  is the  $m$ -fold iteration of the  $\sigma$  ( $\sigma^0$  is the identity map on  $A^* \cup A^\infty$ ), and  $\lim_{m \rightarrow \infty} \sigma^m(a_0)$  indicates the word  $\omega \in A^\infty$  having  $\sigma^m(a_0)$  as its prefix for all  $m \geq 0$ .

We can show that there exists uniquely a number  $\alpha = \alpha(s, k)$  such that

$$f(\alpha) = 0, \quad \alpha > 1, \quad f(x) := x^{s+1} - kx^s - 1 \in \mathbb{Z}[x].$$

Throughout the paper,  $H(\underline{h})$  denotes the height of  $\underline{h}$ , i. e.,

$$H(\underline{h}) := \max\{|h_0|, |h_1|, \dots, |h_s|\}, \quad \underline{h} := (h_0, h_1, \dots, h_s) \in \mathbb{Z}^{s+1}.$$

Theorem 1. Let  $k \geq s \geq 1$ ,  $2 \leq g \in \mathbb{Z}$ , and let  $\theta_i$  be numbers defined by

$$\theta_i = \theta_i(g; s, k) := \sum_{\omega_n = a_i} g^{-n-1} \quad (0 \leq i \leq s).$$

Then

$$\left| \sum_{i=0}^s h_i \theta_i \right| > \kappa_- / H(\underline{h})^{\alpha(\alpha^s - 1) / (\alpha - 1)}$$

for all  $\underline{h} \in \mathbb{Z}^{s+1}$  with  $\underline{h} \neq \underline{0} := (0, 0, \dots, 0)$ , and

$$\left| \sum_{i=0}^s h_i \theta_i \right| < \kappa_+ / H(\underline{h})^{\alpha(\alpha^s - 1) / (\alpha - 1)}$$



Theorem 3. Let  $(s, k) \in \mathbb{N}^2$ , and let  $g, \theta_i, a$  be as in Theorem 1. Put  $\delta := \max\{1, |\beta|; f(\beta) = 0, (a \neq \beta \in \mathbb{C})\}$ . Then

$$\left| \sum_{i=0}^s h_i \theta_i \right| > \kappa_- / (H(\underline{h}))^{\frac{a(a^s-1)}{(a-1)} \frac{(\log H(\underline{h}))}{\kappa} (\log \delta) / \log a}$$

for all  $\underline{h} \in \mathbb{Z}^{s+1}$  with  $\underline{h} \neq \underline{0}$ , and

$$\left| \sum_{i=0}^s h_i \theta_i \right| < \kappa_+ (\kappa_-)^{\frac{(\log H(\underline{h}))}{\kappa} (\log \delta) / \log a} / (H(\underline{h}))^{\frac{a(a^s-1)}{(a-1)}}$$

for infinitely many  $\underline{h} \in \mathbb{Z}^{s+1}$ , where  $\kappa_+ > 0$ ,  $\kappa_- > 0$ , and  $\kappa > 1$  are constants independent of  $\underline{h}$ .

Remark 1.  $1 \leq \delta < a$  holds. The polynomial  $f(x)$  is irreducible for all  $s \geq 1$ ,  $k \geq 1$ . The  $a$  is not always a Pisot number, for instance,  $\delta > 1$  if  $(s, k) = (5, 1)$ . If the  $a$  is a Pisot number, then  $\delta = 1$ , so that the estimates in Theorem 3 turn out to be exact ones as in Theorem 1.

Corollary 3. The number  $\theta_i(g; s, k)$  is a non-Liouville number for all  $0 \leq i \leq s$ ,  $2 \leq g \leq \mathbb{Z}$ , and  $(s, k) \in \mathbb{N}^2$ .

Let  $s \geq 1$ , and  $k \geq 2$  be integers, and let  $\tilde{\sigma} \in \text{Hom}(\tilde{A}^*, \tilde{A}^*)$  ( $\tilde{A} := \{a_{-1}, a_0, \dots, a_s\}$ ) be a monoid homomorphism defined by

$$\begin{aligned} \tilde{\sigma}(a_{-1}) &:= a_{-1}^k, & \tilde{\sigma}(a_0) &:= a_0 a_1 a_{-1}^{k-2}, \\ \tilde{\sigma}(a_j) &:= a_{-1} a_{j+1} a_{-1}^{k-2} \quad (1 \leq j \leq s-1), & \tilde{\sigma}(a_s) &:= a_0 a_{-1}^{k-1}. \end{aligned}$$

We denote by  $\tilde{\omega} = \tilde{\omega}_0 \tilde{\omega}_1 \cdots \tilde{\omega}_n \cdots$  ( $\tilde{\omega}_n \in \tilde{A}$ ) the fixed point of the  $\tilde{\sigma}$  prefixed by  $a_0$ . Then we can show

Theorem 4. Let  $\varphi_i = \varphi_i(g; s, k)$  be numbers defined by

$$\begin{aligned} \varphi_i &:= \varphi(g^{-k^i}) \quad (0 \leq i \in \mathbb{Z}, 2 \leq g \in \mathbb{Z}), \\ \varphi(z) = \varphi(z; k, s) &:= \sum_{\tilde{\omega}_n \in \{a_0, a_s\}} z^n. \end{aligned}$$

Then

$$\left| \sum_{i=0}^s h_i \varphi_i \right| > \kappa_- / (H(\underline{h}))^{k(k^s-1)/(k-1)}$$

for all  $\underline{h} \in \mathbb{Z}^{s+1}$  with  $\underline{h} \neq \underline{0}$ , and

$$\left| \sum_{i=0}^s h_i \varphi_i \right| < \kappa_+ / H(\underline{h})^{k(k^s-1)/(k-1)}$$

for infinitely many  $\underline{h} \in \mathbb{Z}^{s+1}$ , where  $\kappa_+$ ,  $\kappa_-$  are positive constants independent of  $\underline{h}$ .

Corollary 4. The number  $\varphi(g^{-1})$  is a non-Liouville number for all  $g$  with  $2 \leq g \in \mathbb{Z}$ .

Corollary 5. Let  $i \geq 0$  be an integer. Then, the  $s+1$  numbers  $\varphi_i, \varphi_{i+1}, \dots, \varphi_{i+s}$  are linearly independent over  $\mathbb{Q}$ ; the  $s+2$  numbers  $1$ , and  $\varphi_i, \varphi_{i+1}, \dots, \varphi_{i+s}$  are linearly dependent over  $\mathbb{Q}$ .

Example 2.  $(s,k)=(1,2)$ : Put  $a_{-1}=c, a_0=a, a_1=b$ . Then,  $\tilde{\sigma}(a)=ab, \tilde{\sigma}(b)=ac, \tilde{\sigma}(c)=c^2$ , and the base- $g$  expansion of the numbers  $\varphi_i$  ( $i \geq 0$ ) is given by

$\tilde{\omega} = a \quad b \quad a \quad c \quad a \quad b \quad c \quad c \quad a \quad b \quad a \quad c \quad c \quad c \quad c \quad c \quad a \quad b \quad a \dots$ ,  
 $\varphi_0 = 1. \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1 \dots$ ,  
 $\varphi_1 = 1. \quad 01 \quad 01 \quad 00 \quad 01 \quad 01 \quad 00 \quad 00 \quad 01 \quad 01 \quad 01 \quad 00 \quad 00 \quad 00 \quad 00 \quad 00 \quad 01 \quad 01 \quad 01 \dots$ ,  
 $\varphi_2 = 1.000100010000000100010000000000010001000100000000000000000000000000000100010001 \dots$ ,  
 $\dots$ . The base- $g$  expansion of  $\varphi_i/g^{2^i}$  is written by  ${}_g 0.\tau(\tilde{\omega})$  with  $\tau \in \text{Hom}(\tilde{A}^*, B^*)$  defined by  $\tau(a)=\tau(b):=0^{2^i-1}1, \tau(c):=0^{2^i}$ .

We denote by  $\tilde{\omega} = \tilde{\omega}_0 \tilde{\omega}_1 \dots \tilde{\omega}_n \dots$  ( $\tilde{\omega}_n \in A$ ) the fixed point of  $\tilde{g}$  prefixed by  $a_0$  with  $\tilde{g} \in \text{Hom}(\tilde{A}^*, \tilde{A}^*)$  defined by

$$\begin{aligned} \tilde{g}(a_{-1}) &:= a_{-1}^k, & \tilde{g}(a_0) &:= a_0 a_{-1}^{k-2} a_1, \\ \tilde{g}(a_j) &:= a_{-1}^{k-1} a_{j+1} \quad (1 \leq j \leq s-1), & \tilde{g}(a_s) &:= a_{-1}^{k-1} a_0, \quad k \geq 2. \end{aligned}$$

Then, we can show

Theorem 5. Let  $\eta_i = \eta_i(g; s, k)$  be numbers defined by

$$\begin{aligned} \eta_i &:= \eta(g^{k^i}) \quad (0 \leq i \leq s, 2 \leq g \in \mathbb{Z}), \\ \eta(z) = \eta(z; s, k) &:= \sum_{\tilde{\omega}_n = a_0} z^{-(n+1)/(k-1)}. \end{aligned}$$

Then the same assertion with  $\eta_i$  in place of  $\varphi_i$  in Theorem 4 holds.

Corollary 6. The number  $\eta(g)$  is a non-Liouville number for all  $g$  with  $2 \leq g \in \mathbb{Z}$ .

Corollary 7. Let  $i \geq 0$  be an integer. Then, the  $s+1$  numbers  $\eta_i, \eta_{i+1}, \dots, \eta_{i+s}$  are linearly independent over  $\mathbb{Q}$ ; the  $s+2$  numbers  $1$ , and  $\eta_i, \eta_{i+1}, \dots, \eta_{i+s}$  are linearly dependent over  $\mathbb{Q}$ .

We refer to the fact that our results have some connection with higher dimensional continued fractions, and certain Boolean equations, cf. [T4, T6]. For instance, let  $\chi = \chi(\tilde{\omega}; a_0, a_s)$  with  $\tilde{\omega}$  of Theorem 4, where  $\chi(u; p, q, r, \dots)$  denotes the set defined by

$$\chi(u; p, q, r, \dots) := \{n \geq 0; u_n \in \{p, q, r, \dots\}\}$$

for  $u = u_0 u_1 u_2 \dots \in A^* \cup A^\infty$  ( $u_n \in A$ ),  $p, q, r, \dots \in A$ . Then, considering an automaton, we can show that  $X = \chi$  is a solution of a Boolean equation

$$(1) \quad X = kX \cup (k^{s+1}X + (k^s - 1)/(k-1)), \quad \emptyset \neq X \subset \mathbb{N} \cup \{0\},$$

which implies

$$(2) \quad \varphi(z) = \varphi(z^k) + z^{(k^s - 1)/(k-1)} \varphi(z^{k^{s+1}}),$$

where  $pX + q := \{px + q; x \in X\}$ , and  $\varphi(z)$  is the function given in Theorem 4. Using the functional equation (2), we can construct a formal  $s$ -dimensional continued fraction:

$$(3) \quad \underline{\psi}(z) = \underline{1} + \frac{z}{\underline{1} + \frac{z^k}{\underline{1} + \frac{z^{k^2}}{\underline{1} + \frac{z^{k^n}}{\underline{1} + \dots}}}}$$

where

$$\underline{\psi}(z) := (\psi_1(z), \psi_2(z), \dots, \psi_s(z)),$$

$$\psi_i(z) = z^{(k^i-1)/(k-1)} / (\psi_s(z)^k \psi_s(z)^{k^2} \cdots \psi_s(z)^{k^i}) \quad (1 \leq i \leq s-1, s \geq 2),$$

$$\psi_s(z) := \varphi(z) / \varphi(z^k) \quad (s \geq 1),$$

$$\underline{1} := (0, 0, \dots, 0, 1) \in \mathbb{R}^{s+1},$$

and

$$\frac{c}{(d_1, d_2, \dots, d_s)} := c(1/d_s, d_1/d_s, \dots, d_{s-1}/d_s)$$

for given  $c, d_1, d_2, \dots, d_s \in \mathbb{C}$  ( $d_s \neq 0$ ). We can show that the  $n$ th convergent of the continued fraction (3) converges component-wise to  $\underline{\psi}(z)$  if  $z=1/g$  with  $2 \leq g \in \mathbb{Z}$ . Using (3), we can give the following expansion of  $\underline{\psi}(1/g)$  by the Jacobi-Perron algorithm:

$$(4) \quad \underline{\psi}(1/g) = b_0 \underline{1} + \frac{1}{b_1 \underline{1} + \frac{1}{b_2 \underline{1} + \dots}}$$

$$b_n := g^{k^{n-1}} \quad (n \not\equiv 0 \pmod{s+1}),$$

$$b_n := g^{(k^s-1)(k^n-1)/(k^{s+1}-1)} \quad (n \equiv 0 \pmod{s+1}),$$

cf. [T6]. The continued fraction (4) is a regular one in the sense of Korobov [K], p. 84. We can apply a result due to Korobov ([K], p. 91) on higher dimensional regular continued fractions, and we obtain the exact measure, except for  $O$ -constants, of linear independence for the values  $\varphi_i$  as in Theorem 4.

Theorem 5 has a connection with the following continued fraction:

$$(5) \quad \underline{\zeta}(z) = z \underline{1} + \frac{1}{z^k \underline{1} + \frac{1}{z \underline{1} + \frac{k^2}{z \underline{1} + \dots} + \frac{1}{z \underline{1} + \frac{k^n}{z \underline{1} + \dots}}}}$$

where

$$\begin{aligned}\zeta(z) &= (\zeta_1(z), \zeta_2(z), \dots, \zeta_s(z)), \\ \zeta_i(z) &:= 1/(\zeta_s(z)^k \zeta_s(z^{k^2}) \cdots \zeta_s(z^{k^i})) \quad (1 \leq i \leq s-1, s \geq 2), \\ \zeta_s(z) &:= \eta(z)/\eta(z^k) \quad (s \geq 1).\end{aligned}$$

We can show that (5) follows from the functional equation

$$\eta(z) = z\eta(z^k) + \eta(z^{k^{s+1}}), \quad \eta(z) = \sum_{m \in X} z^{-m},$$

where  $X$  is a solution of the Boolean equation

$$(6) \quad X = (kX-1) \cup k^{s+1}X$$

with countable set  $X \subset \mathbb{R}$  such that  $X$  is bounded from the left, or from the right.

We note that (6) with  $k \geq 3$  has no solutions under  $\phi \neq X \subset \mathbb{N} \cup B$  ( $B$  is a finite subset of  $\mathbb{Z}$ ), but (6) has a unique solution under  $\phi \neq X \subset \mathbb{Z}$ , so that  $X$  is unbounded from both sides, and  $\zeta(z)$  can not be well-defined in the case  $\phi \neq X \subset \mathbb{Z}$ . Nevertheless, we can show that the equation (6) has a unique solution given by

$$X = (k-1)^{-1} \chi(\tilde{\omega}; a_0) + (k-1)^{-1}$$

with  $\tilde{\omega}$  in Theorem 5 under the condition

$$\phi \neq X \subset (k-1)^{-1}\mathbb{N}.$$

The solution  $X = \chi(\tilde{\omega}; a_0, a_s)$  of (1) is also a unique one. In general, it will be difficult to solve such a uniqueness problem. In fact, we can show the following

Remark 2. The Boolean equation

$$3X+1 = (6X+1) \cup (X \cap (6\mathbb{N}-2)) \quad (1 \in X \subset \mathbb{N})$$

has a solution  $X = \mathbb{N}$ . Suppose that  $X = \mathbb{N}$  is the unique solution, then one can prove that Syracuse conjecture holds, and vice versa. (Syracuse conjecture is a well-known conjecture as the so called  $3x+1$  problem, or Collatz problem, or Kakutani's problem, that is still open. The conjecture states that for any given positive integer  $m$ , there exist a positive integer  $n = n(m)$  satisfying  $F^n(m) = 1$ , where  $F(m) := 3m+1$  ( $m$ : odd),  $F(m) := m/2$  ( $m$ : even), and  $F^n$  denotes the  $n$ -fold iteration of the map  $F$ .)

It is convenient to use a locally catinative formula among the words  $\sigma^n(a_0)$  ( $n \geq 0$ ) to construct a higher dimensional continued fraction connected with Theorems 1, 3, cf. [T4]. Let  $\theta_i$  be as in Theorem 1. Then, the following continued fraction is a corresponding one to Theorems 1, and 3:

$$(7) \quad \underline{\gamma} = \underline{\gamma}(g; s, k) = \frac{1}{c_1 \underline{1} + \frac{1}{c_2 \underline{1} + \frac{1}{c_3 \underline{1} + \dots}}} \quad (g \geq 2, s \geq 1, k \geq 1)$$

with

$$c_n = c_n(g; s, k) =: g \sum_{h=0}^{k-1} g^{hf_{n+s}},$$

$$f_n := kf_{n-1} + f_{n-s-1} \quad (n \geq s+2), \quad f_n := (k^n - 1)/(k-1) \quad (1 \leq n \leq s+1),$$

and

$$\underline{\gamma} = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(s)}),$$

$$\gamma^{(i)} := (\theta_{i-1} + \sum_{j=1}^s p_j^{(i)} \theta_j) / \theta_s \quad (1 \leq i \leq s),$$

$$(p_j^{(i)})_{0 \leq i \leq s, 0 \leq j \leq s} := Q^{-1}, \quad Q = (q_j^{(i)})_{0 \leq i \leq s, 0 \leq j \leq s},$$

$$q_j^{(i)} := \sum_{h=0}^{k-1} g^{hf_j} \times \sum_{m \in \chi(\mathbb{R}(\sigma^j(a_0)); a_i)} g^m \quad (0 \leq i < j \leq s),$$

$$q_i^{(i)} := 1 \quad (0 \leq i \leq s), \quad q_j^{(i)} := 0 \quad (0 \leq j < i \leq s),$$

where we denote by  $\sigma$  the morphism defined in the first paragraph, and by  ${}^R u$  the word  $u_1 u_{i-1} \dots u_1 \in A^*$  for a given word  $u_1 u_2 \dots u_t$  ( $u_m \in A$ ). We note that  $p_j^{(i)}$  are integers, since  $Q \in \text{SL}_+(s+1; \mathbb{Z})$ . Thus, using the result of Korobov [K], we can find the exact measure of linear independence for the  $s+1$  numbers 1, and  $\gamma_1, \gamma_2, \dots, \gamma_s$  when  $a$  is a Pisot number, from where we get Theorem 1, see Remark 1. The continued fraction (7) can be regarded as a higher dimensional version of some of the classical results, see, for example [B], [D], [M], [Bu], see also [K-S-T], [T1-T3]. Related to our transcendence result (Theorem 2), we note that functions  $\theta_i(z)$  ( $0 \leq i \leq s$ ),  $\varphi(z)$ , and  $\eta(z)$  appearing in our theorems are transcendental



functions, which follows from a Theorem due to Fatou [F]. We can prove Theorem 2 by Roth's theorem, estimating the irrationality measure of the number  $_{g}0.\tau(\omega)$  from below. The estimate from above is not easy in general. But, in some cases, we can find the exact value of the irrationality measure of the number  $_{g}0.\tau(\omega)$  under a minor condition on  $\tau$ , cf. [T7]. We gave Theorems 1-3, and Theorems 4-5, respectively, in [T4], and in [T6]. We gave a higher dimensional version of the Ramanujan's continued fraction, and the linear independence measure, which is also an exact one except for O-constants, of values of certain q-series, cf. [T5].

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