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# DOUBLE TRANSFERS AT THE PRIME 2

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The $S^1$–transfer map $\tau : CP_0^n \to S^{-1}$ is a stable map defined using the transfer construction for the principal $S^1$–bundle over $CP^n$, where $CP_0^n = CP^n \cup *$ is the disjoint union of the complex projective space $CP^n$ and a base point. Then, $\tau_2 = \tau \wedge \tau : CP_0^\infty \wedge CP_0^\infty \to S^{-2}$ is called the double transfer map, and gives information of filtration 2 in the stable homotopy groups of spheres. Knapp [Kn] has given the first great step toward understanding the double transfer map, and the authors in [BC] have extended his results.

In [Hi], Hilditch has found a factorization $\bar{u}_2 : CP_0^\infty \wedge CP_0^\infty \to \Sigma^{-4}N_2$ of the double transfer map $\tau_2$ through $N_2$, the realization of the second stage of the chromatic filtration given by Ravenel [Ra], under the condition that spectra are localized at an odd prime $p$. His result extends that of Miller in [Mi], and enables the transfer images to be compared with the chromatic filtration.

In the case that spectra are localized at 2, just the same factorization of $\tau_2$ as above does not exist (cf. [Hi;Remark 3.20]). In this note, I show that the restriction of $\tau_2$ on $CP^\infty \wedge CP^\infty$ is factored through a map $\bar{u}_2 : CP^\infty \wedge CP^\infty \to \Sigma^{-4}N_2$ even if it is localized at $p = 2$. Although we limit the content to this point, we notice that such factorization enables us to calculate the image of $(\tau_2)_* : \pi_*^s(CP^\infty \wedge CP^\infty)_{(2)} \to \pi_*^s(S^0)_{(2)}$ to some extent, using the result of Shimomura [Sh] concerning a calculation of the Novikov-Adams spectral sequence at prime 2.

## §1 Preliminaries.

First, we prepare some properties about the $K$ and $KO$–cohomology of relevant spaces.

Let $\xi$ be the canonical complex line bundle over $CP^m$ for $0 \leq m \leq \infty$, and $(CP^m)^{k\xi}$ the Thom space of $k\xi$ for an integer $k$. We denote the Thom space by $CP_k^{m+k} = (CP^m)^{k\xi}$, and $CP_k = (CP^\infty)^{k\xi}$. Also, we put $CP = CP_1$ as in the introduction. $k\xi$ is $K$–orientable for any integer $k$, and thus we have a $K$–Thom class $U_k^K \in K^{2k}(CP_k)$ ([AB]). Here and hereafter, we only discuss the cases in which $\mathrm{Lim}^1$ of $K$ and $KO$–cohomology groups are 0, and thus we may regard $K^i(CP_k)$ as $K^i(CP_k^{2N+k})$ for some large $N$.

We remark that $2k\xi$ is $KO$–orientable but $(2k+1)\xi$ is not $KO$–orientable. Since we need the information of $KO^*(CP)$ and $KO^*(CP_{-1})$ later, we recall the structure of $KO^*(CP_{2k+1})$ first. Let $r : K \to KO$ and $c : KO \to K$ be the realification and complexification respectively, and $t \in K_2$ the generator. We put $X = [\xi - 1] \in K^0(CP)$ and $Y = r(X) \in KO^0(CP)$. Then, the following is shown by Fujii [Fu] in the case of $n \geq 0$, and we can prove it even in the case of $n < 0$.

**Lemma 1.1.** *Let $n = 2k + 1$ and $m \geq 1$. Then there is an element $\bar{u}_n \in KO^{2n}(CP_n)$ which satisfies the following:*

(1) $KO^{2n}(CP_n^{2m+n-1}) = Z\{\bar{v}_n Y^i \mid 0 \leq i \leq m-1\}$.

(2) $c(\bar{v}_n) = U_n^K (2+X)/(1+X)^{k+1}$.

**Corollary 1.2.**

(1) *For any odd integer $n$, $i^*(\bar{v}_n) = 2\iota$.*

(2) $c(\bar{v}_{-1}) = U_{-1}^K(2+X)$.

Let $\sinh^{-1}(T)$ be the inverse of the formal power expansion on $T$ of the function $\sinh(T)$, and put $S(T) = (\sqrt{T}/2)/(\sinh^{-1}(\sqrt{T}/2))$. Then, we define an element $G_n(Y) \in KO^{2n}(CP_n; Q)$ for odd $n$ as follows:

$$G_n(Y) = \frac{1}{2} S(Y)^{-n} \left(1 + \frac{Y}{4}\right)^{-1/2} \bar{v}_n. \tag{1.3}$$

Let $ch : K^*(-;Q) \to H^{2*}(-;Q)$ be the Chern character, and $ph = ch \circ c : KO^*(-;Q) \to H^{2*}(-;Q)$ the Pontrjagin character. We denote by $U_n^H \in H^{2n}(CP_n; Z)$ the Thom class of $n\xi$ in the ordinary cohomology group. Then we have the following lemma, which is implicit in [CK].

**Lemma 1.4.** *For any odd $n$, we have $ph(G_n(Y)) = U_n^H$.*

Consider the following commutative diagram:

(1.5)

$$\begin{array}{ccccc}
\pi_s^{2n}(CP_n; Q) & \xrightarrow{h^{KO}} & KO^{2n}(CP_n; Q) & \xrightarrow{c} & K^{2n}(CP_n; Q) \\
\downarrow{\scriptstyle h^H} & & \downarrow{\scriptstyle ph} & & \downarrow{\scriptstyle ch} \\
H^{2n}(CP_n; Q) & \longrightarrow & \bigoplus_{i\geq 0} H^{2n+4i}(CP_n; Q) & \longrightarrow & \bigoplus_{j\geq 0} H^{2n+2j}(CP_n; Q),
\end{array}$$

where $\pi_s^{2n}(-)$ denotes the stable cohomotopy group and $h^{KO}$ is the $KO$–Hurewicz map. In this diagram, the vertical homomorphisms are all isomorphisms and the horizontal two maps are inclusions. We put

$$u_1 = (h^H)^{-1}(U_n^H) \in \pi_s^{2n}(CP_n; Q). \tag{1.6}$$

Then we can show the following, in which $h^K = c \circ h^{KO}$ is the $K$–Hurewicz map.

**Lemma 1.7.**

(1) $h^{KO}(u_1) = G_n(Y)$ *for odd $n$.*

(2) $h^K(u_1) = U_n^K(log(1 + X)/X)^n$ *for any $n$.*

**Corollary 1.8.** *We have $c(G_n(Y)) = U_n^K(\log(1 + X)/X)^n$. In particular, $c(G_1(Y)) = t^{-1}\log(1 + X) \in K^2(CP; Q)$.*

We note that Corollary 1.8 can be also proved directly. We need in §3 the following corollary of Lemma 1.7 and Corollary 1.8.

**Corollary 1.9.** *For odd $n = 2k-1$, there is an element $V \in KO^{2n}(CP_{n+1}; Q)$ which is uniquely defined by the equation $j^*(V) = h^{KO}(u_1) - (1/2)\bar{v}_n$ and satisfies*

$$c(V) = U^K_{n+1} t \left( \frac{(\log(1+X))^n}{X^{n+1}} - \frac{1+\frac{X}{2}}{X(1+X)^k} \right).$$

## §2 THE COFIBER OF THE TRANSFER MAP

Stably, we can consider $CP$ as a subspace of $CP_0$, and we denote by $\tilde{\tau}$ : $CP \to S^{-1}$ the restriction of the $S^1$-transfer map $\tau : CP_0 \to S^{-1}$. Let $W = S^{-2} \bigcup_{\tilde{\tau}} C(\Sigma^{-1}CP)$ be the cofiber of $\tilde{\tau} : \Sigma^{-1}CP \to S^{-2}$. Since the cofiber of $\tau : \Sigma^{-1}CP_0 \to S^{-2}$ is stably homotopy equivalent to $CP_{-1}$ (cf. [Mi],[Kn]), we have inclusion maps $i' : CP \to CP_0$ and $i' : W \to CP_{-1}$, and the following homotopy commutative diagram up to sign:

$$(2.1)\qquad \begin{array}{ccccccc} S^{-2} & \xrightarrow{i} & W & \xrightarrow{j} & CP & \xrightarrow{\tilde{\tau}} & S^{-1} \\ \| & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i'} & & \| \\ S^{-2} & \xrightarrow{i} & CP_{-1} & \xrightarrow{j} & CP_0 & \xrightarrow{\tau} & S^{-1}. \end{array}$$

Then the following is obvious.

**Lemma 2.2.**

(1) $0 \to H^k(CP; Z) \xrightarrow{j^*} H^k(W; Z) \xrightarrow{i^*} H^k(S^{-2}; Z) \to 0$ *is a split exact sequence for any $k$, and $(i')^* : H^k(CP_{-1}; Z) \to H^k(W; Z)$ is an epimorphism with the kernel $H^0(CP_{-1}; Z) = Z\{U^H_{-1}x\}$.*

(2) $0 \to K^k(CP) \xrightarrow{j^*} K^k(W) \xrightarrow{i^*} K^k(S^{-2}) \to 0$ *is a split exact sequence for any $k$, and $(i')^* : K^*(CP_{-1}) \to K^*(W)$ is an epimorphism with the kernel $K_*\{U^K_{-1}X\}$.*

Concerning $KO^{-2}(W)$, we have the following:

**Proposition 2.3.** *There is an element $w \in KO^{-2}(W)$ which satisfies the following:*

*(i)* $KO^{-2}(W) = Z\{w\} \oplus j^*(KO^{-2}(CP))$ *and* $j^* : KO^{-2}(CP) \to KO^{-2}(W)$ *is a monomorphism.*
*(ii)* $(i')^*(\bar{v}_{-1}) = 2w$ *for* $(i')^* : KO^{-2}(CP_{-1}) \to KO^{-2}(W)$.
*(iii)* $i^*(w)$ *is a generator of* $KO^{-2}(S^{-2})$ *for the inclusion* $i : S^{-2} \to W$.
*(iV)* $c(w) = (i')^*(U^K_{-1})$ *in* $K^{-2}(W)$.

## §3 Factorization

In this section, we show a factorization of $\tilde{\tau}\wedge\tilde{\tau}$, which is one of our our main results. We will use the following notations: $S(G)$ denotes the Moore spectrum of a group $G$, and $S^iG = \Sigma^i S(G)$; $\psi = \psi^3 - 1 : KO_{(2)} \to KO_{(2)}$ is the stable Adams operation, and $Ad$ the fiber spectrum of $\psi$. Thus we have a cofibering $Ad \xrightarrow{j} KO_{(2)} \xrightarrow{\psi} KO_{(2)}$, and we put $Ad^iG = Ad \wedge S^iG$.

Let $N_i \xrightarrow{l_i} M_i \xrightarrow{j_i} N_{i+1} \xrightarrow{\delta_{i+1}} \Sigma N_i$ be the cofiber sequence such that

$$(3.1) \qquad \cdots \to \Sigma^{-2}N_2 \xrightarrow{\delta_2} \Sigma^{-1}N_1 \xrightarrow{\delta_1} S^0$$

is the geometrical realization of the chromatic filtration by Ravenel [Ra], where $l_i$ is the Bousfield localization [Bo] with respect to the $v_i^{-1}BP_*$–homology. Then $N_0 \xrightarrow{l_0} M_0 \xrightarrow{j_0} N_1$ is identified with $S^0 \xrightarrow{i} S^0Q \xrightarrow{\rho_Z} S^0Q/Z$ by definition. where $\rho_Z$ is the mod $Z$ reduction. By [Bo], the second cofiber sequence $N_1 \xrightarrow{l_1} M_1 \xrightarrow{j_1} N_2$ is canonically identified with

$$(3.2) \qquad S^0Q/Z \xrightarrow{h^{Ad}} Ad^0Q/Z \xrightarrow{\bar{j}} \overline{Ad}^0Q/Z,$$

where $\overline{Ad} = Ad/S^0$. Hereafter, we denote $KO_{(2)}$ simply by $KO$, and put $KO^iG = KO \wedge S^iG$.

Let $u_1 \in \pi_s^{-2}(CP_{-1}; Q)$ be the element in (1.6) for $n = -1$. Then we have an element $\bar{u}_1 \in \pi_s^{-2}(CP; Q/Z)$ which makes the following diagram homotopy commutative up to sign:

$$\begin{array}{ccccccc} S^{-2} & \xrightarrow{i} & W & \xrightarrow{j} & CP & \xrightarrow{\tilde{\tau}} & S^{-1} \\ \| & & \downarrow{\scriptstyle u_1 \circ i'} & & \downarrow{\scriptstyle \bar{u}_1} & & \| \\ S^{-2} & \xrightarrow{i} & S^{-2}Q & \xrightarrow{\rho_Z} & S^{-2}Q/Z & \xrightarrow{\delta_1} & S^{-1}, \end{array} \tag{3.3}$$

where the upper cofiber sequence is that of (2.1).

For $V \in KO^{-2}(CP_0; Q)$ in Lemma 1.9 for $n = -1$, we put $\tilde{V} = (i')^*(V) \in KO^{-2}(CP; Q)$. Then from Lemma 1.9, the following is clear.

**Lemma 3.4.** *$j^*(\tilde{V}) = h^{KO}(u_1 \circ i') - w$, $\rho_Z \tilde{V} = h^{KO}\bar{u}_1$ and $c(\tilde{V}) = t(1/\log(1 + X) - 1/X)$, where $w$ is the element in Proposition 2.3.*

Let $g_i \in KO_{4i}$ be the generator and $a(i) = 1$ (resp.2) if $i$ is even (resp. odd). For the Berunoulli number $B_i \in Q$ defined by the equation $z/(e^z - 1) = \sum_{i \geq 0}(B_i/i!)z^i$, we consider the following $K$-theoretical Bernoulli numbers:

(3.5)

$$\tilde{B}_i^{KO} = (B_{2i}/(2i)!)(g_i/a(i)) \in KO_{4i} \otimes Q \quad \text{and} \quad \tilde{B}_i^{K} = (B_i/i!)t^i \in K_{2i} \otimes Q.$$

For $CP_0 \wedge CP_0$, we will denote its $K$-cohomology groups by $KO^*(CP_0 \wedge CP_0) = KO_*[[Y_1, Y_2]]$ and $K^*(CP_0 \wedge CP_0) = K_*[[X_1, X_2]]$, where $Y_i$ and $X_i$ denote the respective Euler classes of $\xi$. We can consider as $KO^{-4}(CP \wedge CP; Q) \subset KO^{-4}(CP \wedge W; Q) \subset KO^{-4}(CP_0 \wedge CP_0; Q)$, and define an element $h(Y_1, Y_2) \in KO^{-4}(CP \wedge CP; Q) \subset KO^{-4}(CP \wedge W; Q)$ by

$$h(Y_1, Y_2) = \sum_{i,j \geq 0} \frac{9^j - 1}{9^{i+j} - 1} \tilde{B}_i^{KO} \tilde{B}_j^{KO} G_1(Y_1)^{2i-1} \otimes G_1(Y_2)^{2j-1},$$

where $G_1(Y)$ is the element of (1.3) for $n = 1$. Using this element, we define

$$\tilde{u} = \tilde{V} \otimes w + h(Y_1, Y_2) \in KO^{-4}(CP \wedge W; Q). \tag{3.6}$$

Similarly as $h(Y_1, Y_2)$, we can define $h_C(X_1, X_2) = \sum_{i,j>0}(3^j - 1)/(3^{i+j} - 1)\tilde{B}_i^K \tilde{B}_j^K (t^{-1}\log(1 + X_1))^{i-1} \otimes (t^{-1}\log(1 + X_2))^{j-1}$. Then, by using Corollary 1.9 and Proposition 2.3, we have the following lemma, in which we denote by $1/X$ the element $t^{-1}U_{-1}^K \in K^0(CP_{-1})$.

**Lemma 3.7.**

(1) $c(h(Y_1, Y_2)) = h_C(X_1, X_2) \in K^{-4}(CP \wedge CP; Q)$.

(2) $c(\tilde{V} \otimes w) = (i' \wedge i')^*(t^2(1/\log(1+X_1) - 1/X_1) \otimes (1/X_2))$, *as an element of* $K^{-4}(CP \wedge W; Q)$.

Let $U(X_1, X_2) = t^2(1/\log(1+X_1) - 1/X_1) \otimes (1/X_2) \in K^{-4}(CP_0 \wedge CP_{-1}; Q)$. Then we have the following corollary.

**Corollary 3.8.** $c(\tilde{u}) = (i' \wedge i')^*(U(X_1, X_2) + h_C(X_1, X_2)) \in K^{-4}(CP \wedge W; Q)$.

This corollary shows that, through $(i' \wedge i')^*$, $c(\tilde{u})$ has just the same formula with that in the case of an odd prime $p$ in [BC]. The following is crucial.

**Proposition 3.9.** *The element* $\tilde{u} \in KO^{-4}(CP \wedge W; Q)$ *satisfies the following:*

(1) $(1 \wedge i)^*(\tilde{u}) = \tilde{V} \in KO^{-4}(CP \wedge S^{-2}; Q)$ *and*

(2) $\psi(\tilde{u}) \in \mathrm{Im}[I : KO^{-4}(CP \wedge W) \to KO^{-4}(CP \wedge W; Q)]$, *where* $\psi = \psi^3 - 1 : KO^0Q \to KO^0Q$ *is the stable Adams operation.*

*Proof.* (1) follows immediately from the definition of $\tilde{u}$, because $(i)^*(w) = 1$ by Proposition 2.3 (iii) and $(1 \wedge i)^* h(Y_1, Y_2) = h(Y_1, 0) = 0$. Also, we have (2) by a direct calculation in $KO$–theory, but it is better to apply the complexification $c$ once and consider it in the $K$–theory. Then, by Corollary 3.8 the calculation is just the same as that done in [Hi] or [BC] for the case of an odd prime, and we have $c\psi(\tilde{u}) = c((\psi^3(w) - w) \otimes \psi^3(w)) \in K^{-4}(CP \wedge W) \subset K^{-4}(W \wedge W)$. Since $c : KO^{-4}(CP \wedge W) \to K^{-4}(CP \wedge W)$ is a monomorphism, we have $\psi(\tilde{u}) = (\psi^3(w) - w) \otimes \psi^3(w) \in KO^{-4}(CP \wedge W)$, and thus we have (2).

Let $\bar{u}_1$ be the element in (3.3), $\bar{j} : \overline{Ad} \to \overline{KO}$ the map induced from $j : Ad \to KO$ and $\bar{\rho} : KO \to \overline{KO}$ the reduction.

**Theorem 3.10.** *We have an element* $\bar{u}_2 \in \overline{Ad}^{-4}(CP \wedge CP; Q/Z)$ *which satisfies*

$$\delta_1(\bar{u}_2) = [\bar{u}_1 \circ (1 \wedge \bar{\tau})] \quad \textit{and} \quad (1 \wedge j)^* \bar{j}_*(\bar{u}_2) = \bar{\rho}\rho_Z(\tilde{u}).$$

*Proof.* Proposition 3.9 (2) means $\psi \circ \rho_Z \circ \tilde{u} \simeq 0$, and thus there is an element $u_2 \in Ad^{-4}(CP \wedge W; Q/Z)$ with $j_*(u_2) = \rho_Z(\tilde{u})$. Proposition 3.9 (1) and Corollary 1.9 yield $(1 \wedge i)^* \rho_Z(\tilde{u}) = \rho_Z(\tilde{V}) = h^{KO}(\bar{u}) = j_* h^{Ad}(\bar{u})$. Then These two equations give $[\bar{u}_1 \circ h^{Ad}] = [u_2 \circ (1 \wedge i)]$, since $j_* : Ad^{-2}(CP; Q/Z) \to KO^{-2}(CP; Q/Z)$ is a monomorphism. Then it derives a map from the cofiber sequence $CP \wedge W \xrightarrow{1 \wedge j} CP \wedge CP \xrightarrow{1 \wedge \tilde{\tau}} CP \wedge S^{-1}$ to the cofiber sequence $Ad^{-4} Q/Z \xrightarrow{\bar{\rho}} \overline{Ad}^{-4} Q/Z \xrightarrow{\delta_2} S^{-3} Q/Z$. Thus, we have $\bar{u}_2 : CP \wedge CP \to \overline{Ad}^{-4} Q/Z$ with the required properties, and it completes the proof.

Since the chromatic filtration $\Sigma^{-2} N_2 \xrightarrow{\delta_2} \Sigma^{-1} N_1 \xrightarrow{\delta_1} S^0$ is equal to $\overline{Ad}^{-2} Q/Z \xrightarrow{\delta_2} S^{-1} Q/Z \xrightarrow{\delta_1} S^0$, we have the desired factorization of $\tilde{\tau} \wedge \tilde{\tau}$ as follows:

**Corollary 3.11.** $\tilde{\tau} \wedge \tilde{\tau} \simeq \delta_2 \delta_1 \bar{u}_2$.

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