数理解析研究所講究録
第 838 巻 1993 年 40-43


# An Uncountable Group of Fibre Homotopy Equivalences

K. Tsukiyama

Department of Mathematics
Shimane University
Matsue, Shimane 690, Japan

Abstract. Let $G$ be a compact connected Lie group which is not a torus and let $T$ be a maximal torus of $G$. For the fibre bundle with structure group $G$

$$G/T \to BT \to BG,$$

one can consider the group of fibre homotopy equivalences of this fibration, which we denote by $\mathcal{L}(BT)$. In this note we show that $\mathcal{L}(BT)$ is a semi-direct product of a rational vector space and the Weyl group $W(G)$ with the operator induced by the natural action $W(G) \times BT \to BT$.

## 1. Introduction

Let $G$ be a compact connected Lie group which is not a torus and let $T$ be a maximal torus of $G$. We have the following fibre bundle with structure group $G$

$$G/T \to BT \to BG, \tag{1.1}$$

where $BG$ and $BT$ are classifying spaces of $G$ and $T$ respectively.

Then the set of all fibre homotopy classes of fibre homotopy equivalences of the total space $BT$ of (1.1) forms a group under the multiplication defined by the composition of maps. This group is called the group of fibre homotopy equivalences, and we denote it by $\mathcal{L}(BT)$ (cf. [4], [7]).

It is shown in [5, Theorem, p.422] that for the fibre space $(E, p, B, F)$ with fibre $F$, $\mathcal{L}(E)$ is a finitely presented group, when $E$, $B$ and $F$ have the homotopy type of finite $CW$-complexes.

In this note we show that for the fibration (1.1), $\mathcal{L}(BT)$ is an extension of a non-trivial rational vector space by the Weyl group $W(G)$.

In a similar way, we can define the group of self homotopy equivalences of a topological space $X$, which we denote by $\mathcal{E}(X)$.

By [3, Theorem 3.6], the Weyl group of a given connected compact Lie group $G$ is given as follows.

Let $i : T \subset G$ be the inclusion of a maximal torus, then the Weyl group $W(G)$ is isomorphic to the group of homotopy classes of homotopy equivalences $\alpha : BT \to BT$ over $Bi$ such that the following diagram

$$\begin{array}{ccc} BT & \xrightarrow{\quad\alpha\quad} & BT \\ & {\scriptstyle Bi}\searrow \qquad \swarrow{\scriptstyle Bi} & \\ & BG & \end{array} \tag{1.2}$$

is homotopy commutative.

Then, one can easily see that the forgetful homomorphism $N$

$$\mathcal{L}(BT) \xrightarrow{N} W(G) \subset \mathcal{E}(BT) = GL(n; Z) \tag{1.3}$$

is surjective to the Weyl group $W(G)$, by using the covering homotopy property of the fibration (1.1).

Thus we have an example where the group of fibre homotopy equivalences is an infinite group whose image in the group of homotopy equivalences is finite, since $W(G)$ is finite for any compact connected Lie group $G$.

Let

$$W(G) \times T \to T \tag{1.4}$$

be the natural action of the Weyl group on the torus.

$W(G)$ acts on $BT$ and hence on

$$\pi_1(map(BT, BG), Bi). \tag{1.5}$$

We have the following

THEOREM A. *For the fibre bundle (1.1), the group of fibre homotopy equivalences $\mathcal{L}(BT)$ is a semi-direct product of a rational vector space (1.5) and the Weyl group $W(G)$ with the operator as mentioned above. $\mathcal{L}(BT)$ is an uncountable group if $G$ is simply connected.*

## 2. Proof of Theorem A

Let $(E, p, B, F)$ be a fibration with fibre $F$. Then

$$p^E : map(E, E) \to map(E, B), \ p^E(f) = pf, \tag{2.1}$$

is also a fibration with fibre $(p^E)^{-1}(p)$, which is the space of all fibre preserving maps of $E$ to itself.

We apply this to the case that $(E, p, B, F) = (BT, Bi, BG, G/T)$.

Since $\pi_1(map(BT, BG), Bi)$ acts on $\mathcal{L}(BT)$, the long exact sequence of the homotopy group restricts to the following exact sequence(cf. [7, Remark 7.9], [4, p.448])

$$(2.2) \qquad \pi_1(map(BT, BT), 1) \to \pi_1(map(BT, BG), Bi) \xrightarrow{\partial} \mathcal{L}(BT) \xrightarrow{N} \mathcal{E}(BT).$$

The image of $\mathcal{L}(BT)$ by $N$ is $W(G)$ as stated in §1. And $\pi_1(map(BT, BT), 1) = \pi_1(map(K(\pi, 2), K(\pi, 2), 1)) = 0$ by R. Thom [6, Theorem 2].

We consider the homotopy group $\pi_1(map(BT, BG), Bi)$, where $i$ is an inclusion of a maximal torus $T$ to $G$.

Let $i : T \to G$ be an inclusion of a maximal torus to the compact connected Lie group. Denote by $C(i)$ the centralizer of the image $i$. The obvious homomorphism

$$(2.3) \qquad C(i) \times T \to G$$

passes to a map of a classifying space, which has an adjoint

$$(2.4) \qquad ad(i) : BC(i) \to map(BT, BG)_\rho (\rho = Bi)$$

Let us consider the following fibration (see [2, p.164])

$$(2.5) \qquad X_i \to BC(i) \xrightarrow{ad(i)} map(BT, BG)_\rho \ (\rho = Bi),$$

where $X_i$ is a homotopy fibre of $ad(i)$.

We have $BC(i) = BZ(T) = BT$, since $T$ is a maximal torus ($Z(T)$ denotes the centralizer of $T$). Therefore by the homotopy exact sequence of (2.5), we have

$$(2.6) \qquad \pi_1(map(BT, BG), Bi) \cong \pi_0(X_i).$$

By ([3, Lemma 2.3 and Proposition 6.1]), $\pi_0(X_i)$ is a non-trivial rational vector space. Therefore (2.6) is a rational vector space. We show that (2.6) is a rational vector space of uncountable dimension if $G$ is simply connected at least.

$$(2.7) \qquad \pi_1(map(BT, BG), Bi) \cong \prod_n Ext(H_n(BT, Q), \pi_{n+2}(BG)/Torsion)$$

by [8, Theorem D] and the homotopy exact sequence of the evaluation fibration

$$(2.8) \qquad \omega : map(BT, BG)_\rho \to BG.$$

$\pi_{n+2}(BG) \cong \pi_{n+1}(G)$ ($n \geq 1$) contains a torsion free subgroup for some even $n$ and $\pi_{n+1}(G)$ is a torsion group for n big enough. Since

$$(2.9) \qquad Ext(Q, Z) \cong R(= real\ numbers),$$

(2.7) is a direct sum of some copies of real numbers $R$. Hence (2.6) is a non-trivial rational vector space of uncountable dimension if $G$ is simply connected at least.

Now we have the following exact sequence

$$1 \to \pi_1(map(BT, BG), Bi) \to \mathcal{L}(BT) \to W(G) \to 1. \tag{2.10}$$

$W(G)$ acts on $T$ as in (1.4), hence on $BT$ and hence on $\pi_1(map(BT, BG), Bi)$. Let

$$\varphi : W(G) \to Aut\ \pi_1(map(BT, BG), Bi) \tag{2.11}$$

be the operator induced by this action.

Let *Opext* denote the set of all congruence classes of extensions of the Abelian group (2.6) by $W(G)$ with operator $\varphi$ as in (2.11). Since

$$H^2_\varphi(W(G), \pi_1(map(BT, BG), Bi)) = 0 \tag{2.12}$$

by (2.6) and [1, Corollary 5.2], *Opext* consists of a single element by [1, Theorem 4.1].

Therefore $\mathcal{L}(BT)$ is a semi-direct product of the vector space (2.6) and the Weyl group $W(G)$ with operator (2.11).

The author is grateful to the referee for helpful comments and suggestions.

## Bibliography


1. S. MacLane, *Homology*, Springer-Verlag, 1975.
2. D. Notbohm, *Maps between classifying spaces*, Math. Z. **207** (1991), 153–168.
3. D. Rector, *Subgroups of finite dimensional topological groups*, J. Pure Appl.Algebra (1971), 253–273.
4. S. Sasao, *Fibre homotopy self-equivalences*, Kodai Math. J. **5** (1982), 446–453.
5. H. Scheerer, *Arithmeticity of group of fibre homotopy equivalence classes*, Manuscripta Math. **31** (1980), 413–424.
6. R. Thom, *L'homologie des espaces fonctionels*, Colloque de topologie algébrique, Louvain (1956), 29–39.
7. K. Tsukiyama, *On the group of fibre homotopy equivalences*, Hiroshima Math. J. **12** (1982), 349–376.
8. A. Zabrodsky, *On phantom maps and a theorem of H. R. Miller*, Israel J. Math. **58** (1987), 129–143.