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K_*-LOCALIZATIONS OF SPECTRA X WITH $KU_*X \cong Z\oplus Z$

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0. Introduction

Let E be an associative ring spectrum with unit. For any CW-spectra X , Y we say that X is quasi E_*-equivalent to Y if there exists an equivalence $h : E\wedge Y \to E\wedge X$ of E-module spectra (see [10] or [5]). Let KO and KU be the real and the complex K-spectrum respectively. We denote by S_K the K_*-localization of the sphere spectrum $S = \Sigma^0$. Since the K_*-localization of an arbitrary CW-spectrum X is obtained as the smash product $S_K\wedge X$ ([3] or [9]), we observe that two CW-spectra X and Y have the same K_*-local type if and only if X is quasi S_{K*}-equivalent to Y.

Let $\bar{\eta} : \Sigma^1 SZ/2 \to \Sigma^0$ and $\tilde{\eta} : \Sigma^2 \to SZ/2$ be an extension and a coextension of the Hopf map $\eta : \Sigma^1 \to \Sigma^0$. Their cofibers $C(\bar{\eta})$ and $C(\tilde{\eta})$ are quasi KO_*-equivalent to Σ^4 and Σ^7 respectively. In [12] we first dealt with CW-spectra X such that $KU_0X \cong Z$ and $KU_1X = 0$ in order to study their K_*-local types. Such a CW-spectrum X with $X\wedge SQ = SQ$ is quasi KO_*-equivalent to either of Σ^0 and Σ^4 , and more precisely it has the same K_*-local type as either of Σ^0 and $C(\bar{\eta})$ (see [12, Theorem 1.2 i)] or [4, Proposition 10.6]). In particular, the cofibers $C(\bar{\eta})$ and $\Sigma^{-3}C(\tilde{\eta})$ have the same K_*-local type.

The purpose of this paper is to determine completely the K_*-local types of CW-spectra X with $KU_0X\oplus KU_1X \cong Z\oplus Z$ (see [13] for details). Such a CW-spectrum X with $X\wedge SQ = (\Sigma^t\vee\Sigma^0)\wedge SQ$

is quasi KO_*-equivalent to the wedge sum $\Sigma^t \vee \Sigma^0$, $\Sigma^{t+4} \vee \Sigma^0$, $\Sigma^t \vee \Sigma^4$, $\Sigma^{t+4} \vee \Sigma^4$ or the following cofiber $Y = \Sigma^{-1} C(\eta^2)$, $C(\eta)$ or $C(\eta^2)$ according as $t \equiv 1$, 2 or 3 mod 4 . A CW-spectrum X is said to be a Wood spectrum if it is quasi KO_*-equivalent to $C(\eta)$, and an Anderson spectrum if it is quasi KO_*-equivalent to $C(\eta^2)$. In the previous paper [12, Theorems 1 and 2] we have determined the K_*-local types of Wood and Anderson spectra except when $t = -1$, applying the following powerful tool due to Bousfield [4] (see [12, Theorem 4]):

Theorem 1. Let Y be a certain CW-spectrum such that i) KU_*Y is either free or divisible and $[\Sigma^1 Y, Y \wedge SQ] = 0$, or ii) $KU_1 Y = 0$ (or $KU_0 Y = 0$). Assume that a CW-spectrum X is quasi KO_*-equivalent to Y and the stable real Adams operations ψ_R^k on $KO_*X \otimes Z[1/k]$ and $KO_*Y \otimes Z[1/k]$ behave as the same action. Then X has the same K_*-local type as Y .

1. Spectra with $KU_*X \cong Z \oplus Z$ and $X \wedge SQ = (\Sigma^t \vee \Sigma^0) \wedge SQ$, $t \neq \pm 1$

1.1. Recall that the homotopy groups $\pi_i S_K$ are given as follows ([9] or [3]): $Z \oplus Z/2$ for $i = 0$; $Z/2$ for $i \equiv 0 \bmod 8$ with $i \neq 0$; $Z/2 \oplus Z/2$ for $i \equiv 1 \bmod 8$; $Z/2$ for $i \equiv 2 \bmod 8$; $Z/m(2s)$ for $i = 4s-1$ with $s \neq 0$; Q/Z for $i = -2$; and 0 otherwise where $m(2s)$ is the numerical function defined explicitly in [1] or [2]. We first realize generators of the homotopy groups $\pi_i S_K$ except $i = -2$ by constructing maps between small spectra. By the solution of the Adams conjecture we get an Adams' K_*- equivalence

(1.1) $\quad A_r : \Sigma^{8r} SZ/m(4r) \to SZ/m(4r)$

such that the composite map $jA_r i : \Sigma^{8r-1} \to \Sigma^0$ is the generator

ρ'_r of the J-image, where SZ/m denotes the Moore Spectrum of type Z/m . Set

$\bar{\rho}_r = jA_r : \Sigma^{8r-1}SZ/m(4r) \to \Sigma^0$ and $\tilde{\rho}_r = A_r i : \Sigma^{8r} \to SZ/m(4r)$

and consider their cofibers $C(\bar{\rho}_r)$ and $C(\tilde{\rho}_r)$ for which there exist K_*-equivalences

(1.2) $e_r : C(\bar{\rho}_r) \to \Sigma^0$ and $e'_r : \Sigma^{8r+1} \to C(\tilde{\rho}_r)$.

For $r \geq 1$ we denote by

(1.3) $\rho_{-r} : C(\bar{\rho}_r) \to \Sigma^{8r+1}$ and $\rho'_{-r} : \Sigma^0 \to C(\tilde{\rho}_r)$

the top cell projection and the bottom cell inclusion.

Let $\bar{\eta} : \Sigma^1 SZ/2m \to \Sigma^0$ and $\tilde{\eta} : \Sigma^2 \to SZ/2m$ be an extension and a coextension of the Hopf map $\eta : \Sigma^1 \to \Sigma^0$. We construct the following maps of order 2 (cf. [12, (1.13) and (2.5)]):

(1.4) $\mu_r = \bar{\eta}A_r i : \Sigma^{8r+1} \to \Sigma^0$, $\mu'_r = jA_r\tilde{\eta} : \Sigma^{8r+1} \to \Sigma^0$,

$\mu_{-r} = \bar{\eta}i_r : \Sigma^{-8r+1}C(\bar{\rho}_r) \to \Sigma^0$, $\mu'_{-r} = j_r\tilde{\eta} : \Sigma^{-8r+1} \to \Sigma^{-8r-1}C(\tilde{\rho}_r)$

where $i_r : C(\bar{\rho}_r) \to \Sigma^{8r}SZ/m(4r)$ and $j_r : SZ/m(4r) \to C(\tilde{\rho}_r)$ denote the canonical projection and inclusion respectively.

Choose an extension and a coextension

$\bar{\xi}_r : \Sigma^{8r+3}SZ/m(8r+4) \to \Sigma^0$ and $\tilde{\xi}_r : \Sigma^{8r+4} \to SZ/m(8r+4)$

of the generator ξ_r of the J-image, and then construct the following maps

(1.5) $\xi_{-r-1} = \bar{\xi}_r i_{2r+1} : \Sigma^{-8r-5}C(\bar{\rho}_{2r+1}) \to \Sigma^0$ and

$\xi'_{-r-1} = j_{2r+1}\tilde{\xi}_r : \Sigma^{-8r-5} \to \Sigma^{-16r-9}C(\tilde{\rho}_{2r+1})$.

Consider the cofibers $C(\bar{\eta})$ and $C(\tilde{\eta})$ of the maps $\bar{\eta} : \Sigma^1 SZ/2 \to \Sigma^0$ and $\tilde{\eta} : \Sigma^2 \to SZ/2$. The homotopy groups $\pi_i S_K \wedge C(\bar{\eta}) \cong \pi_{i+3} S_K \wedge C(\tilde{\eta})$ are given as follows: Z for $i = 0$; $Z/2$ for $i \equiv 4$ or $6 \bmod 8$ with $i \neq -2$; $Z/2 \oplus Z/2$ for $i \equiv 5 \bmod 8$; $Z/m(2s)$ for $i = 4s-1$ with $s \neq 0$; $Z/2 \oplus Q/Z$ for $i = -2$; and 0 otherwise. We next realize generators of the homotopy groups $\pi_i S_K \wedge C(\bar{\eta})$ except $i = -2$ by constructing maps between small spectra. Choose maps $\bar{\lambda} : C(\bar{\eta}) \to \Sigma^0$ and $\tilde{\lambda} : \Sigma^3 \to C(\tilde{\eta})$ satisfying $\bar{\lambda}\bar{i} = 4$ and $\tilde{j}\tilde{\lambda} = 4$ where $\bar{i} : \Sigma^0 \to C(\bar{\eta})$

and $\tilde{j} : C(\tilde{\eta}) \to \Sigma^3$ denote the bottom cell inclusion and the top cell projection. We now construct the following maps

(1.6)
$$s_r = \tilde{\lambda}\rho_r : \Sigma^{8r-1} \to \Sigma^{-3}C(\tilde{\eta}) \ , \quad s'_r = \rho_r\bar{\lambda} : \Sigma^{8r-1}C(\bar{\eta}) \to \Sigma^0 \ ,$$
$$s_{-r} = \tilde{\lambda}\rho_{-r} : \Sigma^{-8r-1}C(\bar{\rho}_r) \to \Sigma^{-3}C(\tilde{\eta}) \text{ and}$$
$$s'_{-r} = \rho'_{-r}\bar{\lambda} : \Sigma^{-8r-1}C(\bar{\eta}) \to \Sigma^{-8r-1}C(\tilde{\rho}_r) \ .$$

Take the composite maps $\bar{h} = \tilde{i}\pi : SZ/2m \to C(\tilde{\eta})$ and $\check{h} = \pi\bar{j} : C(\bar{\eta}) \to \Sigma^2 SZ/2m$ where $\tilde{i} : SZ/2 \to C(\tilde{\eta})$ and $\bar{j} : C(\bar{\eta}) \to \Sigma^2 SZ/2$ denote the canonical inclusion and the bottom cell collapsing respectively and the maps π are the canonical maps between Moore spectra. Then we can construct the following maps of order 2 (cf. [12, (1.13) and (2.5)]):

(1.7)
$$m_r = \bar{h}A_r i : \Sigma^{8r-3} \to \Sigma^{-3}C(\tilde{\eta}) \ , \quad m'_r = jA_r\check{h} : \Sigma^{8r-3}C(\bar{\eta}) \to \Sigma^0 \ ,$$
$$m_{-r} = \bar{h}i_r : \Sigma^{-8r-3}C(\bar{\rho}_r) \to \Sigma^{-3}C(\tilde{\eta}) \text{ and}$$
$$m'_{-r} = j_r\check{h} : \Sigma^{-8r-3}C(\bar{\eta}) \to \Sigma^{-8r-1}C(\tilde{\rho}_r) \ .$$

Choose maps $\bar{n} : \Sigma^6 SZ/16 \to C(\tilde{\eta})$ and $\tilde{n} : \Sigma^4 C(\bar{\eta}) \to SZ/16$ with $\tilde{j}\bar{n} = \bar{3}\bar{\nu}$ and $\tilde{n}\bar{i} = \tilde{3}\tilde{\nu}$, and moreover choose maps $\bar{u}_r : \Sigma^{8r+6}SZ/m \to C(\tilde{\eta})$ and $\tilde{u}_r : \Sigma^{8r+4}C(\bar{\eta}) \to SZ/m$ with $\tilde{j}\bar{u}_r = \bar{8}\xi_r$ and $\tilde{u}_r\bar{i} = \tilde{8}\xi_r$ where $\nu : \Sigma^3 \to \Sigma^0$ is the Hopf map and $m = m(4r+2)/8$. Using these maps we can construct the following maps

(1.8)
$$n_r : \Sigma^{8r+3} \to \Sigma^{-3}C(\tilde{\eta}) \ , \quad n'_r : \Sigma^{8r+3}C(\bar{\eta}) \to \Sigma^0 \ ,$$
$$n_{-r-1} = \bar{n}_r i_{2r+1} : \Sigma^{-8r-5}C(\bar{\rho}_{2r+1}) \to \Sigma^{-3}C(\tilde{\eta}) \text{ and}$$
$$n'_{-r-1} = j_{2r+1}\tilde{n}_r : \Sigma^{-8r-5}C(\bar{\eta}) \to \Sigma^{-16r-9}C(\tilde{\rho}_{2r+1}) \ .$$

Using the composite maps $\sigma\mu_{-1} = \sigma\bar{\eta}i_1 : C(\bar{\rho}_1) \to \Sigma^0$ and $\mu'_{-1}\sigma = j_1\tilde{\eta}\sigma : \Sigma^9 \to C(\tilde{\rho}_1)$ where $\sigma : \Sigma^7 \to \Sigma^0$ is the Hopf map, we introduce the following two maps

(1.9)
$$\theta_m = 2^m e_1 + \sigma\mu_{-1} : C(\bar{\rho}_1) \to \Sigma^0 \text{ and}$$
$$\theta'_m = 2^m e'_1 + \mu'_{-1}\sigma : \Sigma^0 \to \Sigma^{-9}C(\tilde{\rho}_1)$$

whose cofibers are respectively denoted by SZ_m and SZ'_m for $m \geq 1$. Denote by V_m and V'_m respectively the cofibers of the composite maps $i\bar{\eta} : \Sigma^1 SZ/2 \to SZ/2^{m-1}$ and $\tilde{\eta}j : \Sigma^1 SZ/2^{m-1} \to SZ/2$. The spectrum $\Sigma^{-2}V'_m$ is quasi KO_*-equivalent to $\Sigma^4 V_m$, and more precisely it has the same K_*-local type as $V_m \wedge C(\bar{\eta})$ (use [12, Theorem 1.2 and (1.3)]). Consider the following maps

$$\text{(1.10)}\quad \begin{aligned} \varphi_m &= i\bar{\eta}(e_1\wedge 1)+\tilde{\eta}j(\sigma\mu_{-1}\wedge 1) : \Sigma^1 C(\bar{\rho}_1)\wedge SZ/2 \to SZ/2^{m-1} \\ \varphi'_m &= (e'_1\wedge 1)i\bar{\eta}+(\mu'_{-1}\sigma\wedge 1)\tilde{\eta}j : \Sigma^1 SZ/2 \to \Sigma^{-9}C(\tilde{\rho}_1)\wedge SZ/2^{m-1} \\ \psi_m &= \tilde{\eta}j(e_1\wedge 1)+i\bar{\eta}(\sigma\mu_{-1}\wedge 1) : \Sigma^1 C(\bar{\rho}_1)\wedge SZ/2^{m-1} \to SZ/2 \\ \psi'_m &= (e'_1\wedge 1)\tilde{\eta}j+(\mu'_{-1}\sigma\wedge 1)i\bar{\eta} : \Sigma^1 SZ/2^{m-1} \to \Sigma^{-9}C(\tilde{\rho}_1)\wedge SZ/2 \end{aligned}$$

whose cofibers are respectively denoted by ΔV_m , $\Delta V''_m$, $\Delta V'_m$ and $\Delta V''_m$. For convenience' sake we set $V_1 = \Sigma^2 SZ/2$, $V'_1 = SZ/2$, $\Delta V_1 = \Sigma^2 SZ_1$ and $\Delta V'_1 = SZ'_1$. As is easily seen,

(1.11) i) $\psi^3_R = -1$ on $KO_2SZ_1 \cong KO_2SZ'_1 \cong Z/4$, and $\psi^3_R = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ on $KO_2SZ_m \cong KO_2SZ'_m \cong Z/2\oplus Z/2$ whenever $m \geq 2$.

ii) $\psi^3_R = 1+2^m$ on $KO_4\Delta V_m \cong KO_4\Delta V''_m \cong KO_2\Delta V'_m \cong KO_2\Delta V''_m \cong Z/2^{m+1}$ for $m \geq 1$.

Applying Theorem 1 by means of (1.11) we obtain the following result (cf. [4, Proposition 10.5]).

Theorem 2. Let X be a CW-spectrum such that $KU_0X \cong Z/2^m$ on which $\psi^k_C = 1$ and $KU_1X = 0$. Then X has the same K_*-local type as one of the following small spectra: $SZ/2^m$, $SZ/2^m\wedge C(\bar{\eta})$, SZ_m , $SZ_m\wedge C(\bar{\eta})$, V_m , $\Sigma^{-2}V'_m$, ΔV_m and $\Sigma^{-2}\Delta V'_m$.

1.2. Let X be a CW-spectrum with $KU_0X \cong Z\oplus Z$ and $KU_1X = 0$. For such a CW-spectrum X we may assume that $X\wedge SQ = (\Sigma^{2t}\vee \Sigma^0)\wedge SQ$ for some integer $t \geq 0$. When t is odd, X is quasi KO_*-equivalent to the wedge sum $\Sigma^2\vee\Sigma^0$, $\Sigma^6\vee\Sigma^0$, $\Sigma^2\vee\Sigma^4$, $\Sigma^6\vee\Sigma^4$ or the cofiber $C(\eta)$. The Adams operation ψ^k_C on $KU_0X\otimes Z[1/k]$ is represented by either of the following two matrices as left action (cf. [2, Theorem 7.18]):

$$A_{k,t,0} = \begin{pmatrix}1/k^t & 0\\ 0 & 1\end{pmatrix} \quad\text{and}\quad A_{k,t,1} = \begin{pmatrix}1/k^t & 0\\ 1-k^t/2k^t & 1\end{pmatrix} .$$

When t is even, X is quasi KO_*-equivalent to the wedge sum

$\Sigma^0 \vee \Sigma^0$, $\Sigma^4 \vee \Sigma^0$, $\Sigma^0 \vee \Sigma^4$ or $\Sigma^4 \vee \Sigma^4$. If X is quasi KO_*-equivalent to either $\Sigma^0 \vee \Sigma^0$ or $\Sigma^4 \vee \Sigma^4$, then ψ_C^k on $KU_0 X \otimes Z[1/k]$ behaves as the following matrix

$$A_{k,t,m} = \begin{pmatrix} 1/k^t & 0 \\ m(1-k^t)/m(t)k^t & 1 \end{pmatrix}$$

for some m , $0 \leq m < m(t)$. If X is quasi KO_*-equivalent to either $\Sigma^4 \vee \Sigma^0$ or $\Sigma^0 \vee \Sigma^4$ then ψ_C^k on $KU_0 X \otimes Z[1/k]$ behaves as the matrix $A_{k,t,2m}$ for some m , $0 \leq m < m(t)$. Applying Theorem 1 we easily obtain

Theorem 3. Let X be a CW-spectrum such that $KU_0 X \cong Z \oplus Z$ $KU_1 X = 0$ and $X \wedge SQ = (\Sigma^{2t} \vee \Sigma^0) \wedge SQ$ for some even integer t . Then X has the same K_*-local type as the following small spectrum Y :

i) $Y = C(m\rho_r)$, $C(m\rho_r) \wedge C(\bar{\eta})$, $C(ms_r)$ or $C(ms'_r)$ for some m , $0 \leq m < m(4r)$ when $t = 4r$;

ii) $Y = C(m\xi_r)$, $C(m\xi_r) \wedge C(\bar{\eta})$, $C(mn_r)$ or $C(mn'_r)$ for some m , $0 \leq m < m(4r+2)$ when $t = 4r+2$.

Theorem 4. Let X be a CW-spectrum such that $KU_0 X \cong Z \oplus Z$, $KU_1 X = 0$ and $X \wedge SQ = (\Sigma^{2t} \vee \Sigma^0) \wedge SQ$ for some odd integer t .

i) If X is quasi KO_*-equivalent to the wedge sum $\Sigma^2 \vee \Sigma^0$, then X has the same K_*-local type as the following small spectrum Y : $Y = \Sigma^{8r+2} \vee \Sigma^0$ or $C(\mu'_{r-1}\eta\sigma)$ if $t = 4r+1$; and $Y = \Sigma^{8r-2} C(\bar{\eta}) \vee \Sigma^0$ or $C(m'_{r-1}\eta\sigma)$ if $t = 4r-1$.

ii) If X is a Wood spectrum, then X has the same K_*-local type as the cofiber $C(\mu'_r)$ or $C(m'_r)$ according as $t = 4r+1$ or $4r-1$.

Let X be a CW-spectrum with $KU_0 X \cong KU_1 X \cong Z$. For such a CW-spectrum X we may assume that $X \wedge SQ = (\Sigma^{2t+1} \vee \Sigma^0) \wedge SQ$

for some odd integer t . Thus $\psi_C^k = 1$ on $KU_0X\otimes Z[1/k]$ and $\psi_C^k = 1/k^t$ on $KU_1X\otimes Z[1/k]$. Then X is quasi KO_*-equivalent to the wedge sum $\Sigma^3\vee\Sigma^0$, $\Sigma^7\vee\Sigma^0$, $\Sigma^3\vee\Sigma^4$, $\Sigma^7\vee\Sigma^4$ or the cofiber $C(\eta^2)$. Similarly to Theorems 1.2 and 1.3 we obtain

Theorem 5. Let X be a CW-spectrum such that $KU_0X \cong KU_1X \cong Z$ and $X\wedge SQ = (\Sigma^{2t+1}\vee\Sigma^0)\wedge SQ$ for some odd integer $t \neq -1$.

i) If X is quasi KO_*-equivalent to the wedge sum $\Sigma^3\vee\Sigma^0$, then X has the same K_*-local type as the wedge sum $\Sigma^{8r+3}\vee\Sigma^0$ or $\Sigma^{8r-1}C(\bar{\eta})\vee\Sigma^0$ according as $t = 4r+1$ or $4r-1$.

ii) If X is quasi KO_*-equivalent to the wedge sum $\Sigma^{-1}\vee\Sigma^0$ then X has the same K_*-local type as the following small spectrum Y : $Y = \Sigma^{8r+3}C(\bar{\eta})\vee\Sigma^0$ or $\Sigma^{8r+3}C(m_{-r-1}(\sigma\wedge 1))$ if $t = 4r+1$; and $Y = \Sigma^{8r-1}\vee\Sigma^0$ or $\Sigma^{8r-1}C(\mu_{-r-1}\sigma)$ if $t = 4r-1$.

iii) If X is an Anderson spectrum, then X has the same K_*-local type as the cofiber $C(\mu_r'\eta)$ or $C(m_r'(\eta\wedge 1))$ according as $t = 4r+1$ or $4r-1$.

2. Spectra with $KU_0X \cong KU_1X \cong Z$ and $X\wedge SQ = (\Sigma^{-1}\vee\Sigma^0)\wedge SQ$

2.1. we here realize elements of the homotopy groups $\pi_{-2}S_K \cong Q/Z$ and $\pi_{-2}S_K\wedge C(\bar{\eta}) \cong Z/2\oplus Q/Z$ by constructing maps between small spectra. Given an integer $n \geq 1$ we take the minimum $r = r(n) \geq 1$ such that $m(4r)$ is divisible by n , and then set

$$
\begin{aligned}
&\tau_{1/n} = m\rho_r\rho_{-r} : \Sigma^{-2}C(\bar{\rho}_r) \to \Sigma^0 \\
(2.1)\quad &\tau'_{1/n} = m\rho'_{-r}\rho_r : \Sigma^{-2} \to \Sigma^{-8r-1}C(\tilde{\rho}_r) \\
&t_{1/n} = \tilde{\lambda}\tau_{1/n} : \Sigma^{-2}C(\bar{\rho}_r) \to \Sigma^{-3}C(\tilde{\eta}) \\
&t'_{1/n} = \tau'_{1/n}\bar{\lambda} : \Sigma^{-2}C(\bar{\eta}) \to \Sigma^{-8r-1}C(\tilde{\rho}_r)
\end{aligned}
$$

in which $r = r(n)$ and $m = m(4r)/n$. Whenever $f_{1/n}$ is one of the maps given in (2.1) we simply write $f_\alpha = \ell f_{1/n}$ for each rational number $\alpha = \ell/n$. Notice that the cofibers $C(\tau_\alpha)$ and $C(\tau'_\alpha)$ have the same K_*-local type, and $C(t_\alpha)$ and $C(t'_\alpha)\wedge C(\bar{\eta})$ have also the same K_*-local type. For simplicity we hereafter set $D_\alpha = C(\bar{\rho}_r)$ and $D'_\alpha = \Sigma^{-8r-1}C(\tilde{\rho}_r)$ with $\alpha = \ell/n \in Q$ and $r = r(n)$ both of which have the same K_*-local type as Σ^0. The exact sequences

$$0 \to S_{K-1}C(\tau'_\alpha) \to S_{K0}\Sigma^0 \to S_{K-2}D'_\alpha \to S_{K-2}C(\tau'_\alpha) \to 0$$

$$0 \to S_{K-1}C(t'_\alpha) \to S_{K0}C(\bar{\eta}) \to S_{K-2}D'_\alpha \to S_{K-2}C(t'_\alpha) \to 0$$

are respectively given in the following forms:

$$(2.2)\qquad \begin{array}{l} 0 \to Z\oplus Z/2 \xrightarrow{\begin{pmatrix} n & 0 \\ 0 & 1\end{pmatrix}} Z\oplus Z/2 \xrightarrow{(\alpha\ 0)} Q/Z \xrightarrow{n} Q/Z \to 0 \\ 0 \to Z \xrightarrow{n} Z \xrightarrow{\alpha} Q/Z \xrightarrow{n} Q/Z \to 0 \end{array}$$

where n is the denominator of $\alpha \in Q$. Then we can show

Proposition 6. Let α and β be rational numbers.

i) If the cofibers $C(\tau'_\alpha)$ and $C(\tau'_\beta)$ have the same K_*-local type, then $\alpha \equiv \beta \bmod Z$.

ii) If the cofibers $C(t'_\alpha)$ and $C(t'_\beta)$ have the same K_*-local type, then $\alpha \equiv \beta \bmod Z$.

For any rational number $\alpha = \ell/n$ we now construct the following two maps

$$(2.3)\qquad \begin{array}{l} w_\alpha = \tilde{i}i\eta e_r + \ell t_{1/4n} : \Sigma^{-2}C(\bar{\rho}_r) \to \Sigma^{-3}C(\tilde{\eta}) \quad \text{and} \\ w'_\alpha = e'_r\eta j\bar{j} + \ell t'_{1/4n} : \Sigma^{-2}C(\bar{\eta}) \to \Sigma^{-8r-1}C(\tilde{\rho}_r) \end{array}$$

where $r = r(n)$ is the integer used in (2.1) and e_r and e'_r are the K_*-equivalences given in (1.2). Notice that the cofibers $C(w_\alpha)$ and $C(w'_\alpha)\wedge C(\bar{\eta})$ have the same K_*-local type. As is easily seen, the exact sequence

$$0 \to S_{K-1}C(w_\alpha) \to S_{K0}D_\alpha \to S_{K1}C(\tilde{\eta}) \to S_{K-2}(w_\alpha) \to 0$$

is given in the following form:

$$(2.4)\quad 0 \to Z \xrightarrow{\binom{2n}{\ell}} Z\oplus Z/2 \xrightarrow{\begin{pmatrix}1 & 0\\ \ell/4n & 1/2\end{pmatrix}} Z/2\oplus Q/Z \xrightarrow{(\ell/2\ \ 2n)} Q/Z \to 0$$

where $\alpha = \ell/n \in Q$. By virtue of (2.4) we can show

Proposition 7. Let α and β be rational numbers. If the cofibers $C(w_\alpha)$ and $C(w_\beta)$ have the same K_*-local type, then $\alpha \equiv \beta \mod 2Z$.

Let X be a CW-spectrum such that $KU_0X \cong KU_1X \cong Z$ and $X\wedge SQ \cong (\Sigma^{-1}\vee\Sigma^0)\wedge SQ$. If X is quasi KO_*-equivalent to either of $\Sigma^3\vee\Sigma^0$ and $C(\eta^2)$, then $S_{K-1}X \cong Z$ and $S_{K-2}X \cong Q/Z$. For such a CW-spectrum X the exact sequence

$$0 \to S_{K-1}X \to KO_{-1}X \to F_{-2}X \to S_{K-2}X \to 0$$

is written into the following form:

$$(2.5)\qquad 0 \to Z \xrightarrow{n} Z \xrightarrow{\alpha} Q/Z \xrightarrow{n} Q/Z \to 0$$

with some $\alpha = \ell/n \in Q$ where F denotes the fiber of the K_*-localized unit $\iota_R : S_K \to KO$. Then the cofiber sequence $S_{K(p)} \to Ad_{(p)} \to \Sigma^{-1}SQ \to \Sigma^1 S_{K(p)}$ induces an exact sequence

$$0 \to S_{K-1}X\otimes Z_{(p)} \to Ad_{(p)-1}X \to SQ_0X \to S_{K-2}X\otimes Z_{(p)} \to 0$$

given in the following form for each prime p :

$$(2.6)\quad 0 \to Z_{(p)} \xrightarrow{\binom{n}{-\ell_p}} Z_{(p)}\oplus Z_{(p)} \xrightarrow{(\ell_p/n\ \ 1)} Q \xrightarrow{n} Z/p^\infty \to 0$$

with some $\ell_p \in Z_{(p)}$ satisfying $\ell_p \equiv \ell \mod nZ_{(p)}$.

For the rational number $\alpha = \ell/n$ obtained in (2.5) we set $\beta = \alpha$ or $\alpha+1$ according as $(\ell_2-\ell)/n \in Z_{(2)}$ is even or odd, where ℓ_2 is given in (2.6). Then we obtain the following result as one of our main results.

Theorem 8. Let X be a CW-spectrum such that $KU_0X \cong KU_1X \cong Z$ and $X\wedge SQ = (\Sigma^{-1}\vee\Sigma^0)\wedge SQ$.

i) If X is quasi KO_*-equivalent to the wedge sum $\Sigma^3\vee\Sigma^0$, then X has the same K_*-local type as the cofiber $C(t'_\alpha)$ for some $\alpha \in Q$ with $0 \le \alpha < 1$.

ii) If X is an Anderson spectrum, then X has the same K_*-local type as the cofiber $C(w_\beta)$ for some $\beta \in Q$ with $0 \le \beta < 2$.

2.2. Let $\eta_{4m,2} : \Sigma^2 SZ/4m \to SZ/2$ be a map satisfying $\eta_{4m,2}i = \tilde{\eta}$ and $j\eta_{4m,2} = \bar{\eta}$. Using the composite map $\tilde{\mu}_{-1} = \eta_{240,2}i_1 : C(\bar{\rho}_1) \to \Sigma^6 SZ/2$ we introduce the following map

(2.7) $\quad i\eta e_1 + (\sigma\wedge 1)\tilde{\mu}_{-1} : \Sigma^1 C(\bar{\rho}_1) \to SZ/2$

whose cofiber is denoted by ΔM_1 . The map $2(\rho_r\wedge 1) : \Sigma^{8r-1}C(\bar{\rho}_1) \to C(\bar{\rho}_1)$ is factorized through ΔM_1 as $2(\rho_r\wedge 1) = j_M v_r$ for a suitable map $v_r : \Sigma^{8r+1}C(\bar{\rho}_1) \to \Delta M_1$ where $j_M : \Delta M_1 \to \Sigma^2 C(\bar{\rho}_1)$ denotes the canonical projection, because $\sigma^2\eta = 0$ and $\sigma\rho_r = 0$ for any $r \ge 2$. For any integer $n \ge 1$ we set

(2.8) $\quad \upsilon_{1/n} = mv_r(\rho_{-r}\wedge 1) : C(\bar{\rho}_r)\wedge C(\bar{\rho}_1) \to \Delta M_1$

where $r = r(n)$ and $m = m(4r)/n$. For any rational number $\alpha = \ell/n$ we now introduce the following map

(2.9) $\quad \omega_\alpha = i_M i(e_r\wedge e_1) + \ell\upsilon_{1/4n} : C(\bar{\rho}_r)\wedge C(\bar{\rho}_1) \to \Delta M_1$

where $i_M : SZ/2 \to \Delta M_1$ denotes the canonical inclusion. Note that the cofiber $C(\omega_\alpha)$ is quasi KO_*-equivalent to the wedge sum $\Sigma^1 \vee \Sigma^2$. For simplicity we hereafter set $D_{\alpha,1} = C(\bar{\rho}_r)\wedge C(\bar{\rho}_1)$, which has the same K_*-local type as Σ^0 via the K_*-equivalence $e_r\wedge e_1$. Since the induced homomorphism $\omega_{\alpha*} : S_{K0}D_{\alpha,1} \to S_{K0}\Delta M_1$ is expressed as the matrix $\begin{pmatrix} 1 & 0 \\ \ell/4n & 1/2 \end{pmatrix} : Z\oplus Z/2 \to Z/2\oplus Q/Z$ and $\omega_{\alpha*} : S_{K1}D_{\alpha,1} \to S_{K1}\Delta M_1$ is an epimorphism, the exact sequence

$$0 \to S_{K1}C(\omega_\alpha) \to S_{K0}D_{\alpha,1} \to S_{K0}\Delta M_1 \to S_{K0}C(\omega_\alpha) \to 0$$

is written into the same form as (2.4).

Similarly to Proposition 7 we can show

Proposition 9. Let α and β be rational numbers. If the cofibers $C(\omega_\alpha)$ and $C(\omega_\beta)$ have the same K_*-local type, then $\alpha \equiv \beta \mod 2Z$.

Let X be a CW-spectrum such that $KU_0X \cong KU_1X \cong Z$ and $X\wedge SQ = (\Sigma^{-1}\vee\Sigma^0)\wedge SQ$. If X is quasi KO_*-equivalent to the wedge sum $\Sigma^{-1}\vee\Sigma^0$, then $S_{K-2}X \cong Q/Z$ and $S_{K-1}X \cong Z\oplus Z/2$ or Z according as $\psi_R^3 = 1$ or $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$ on $KO_1X \cong Z/2\oplus Z/2$. For such a CW-spectrum X the exact sequence

$$S_{K-1}X \to KO_{-1}X \to F_0X \to S_{K-2}X \to 0$$

is written into the following form:

(2.10) $$Z\oplus Z/2 \xrightarrow{(n\ 0)} Z \xrightarrow{\alpha} Q/Z \xrightarrow{n} Q/Z \to 0 \quad \text{if} \quad S_{K-1}X \cong Z\oplus Z/2$$

$$Z \xrightarrow{n} Z \xrightarrow{\alpha} Q/Z \xrightarrow{n} Q/Z \to 0 \quad \text{if} \quad S_{K-1}X \cong Z$$

with some $\alpha = \ell/n \in Q$. Then the exact sequence

$$0 \to S_{K-1}X\otimes Z_{(p)} \to Ad_{(p)-1}X \to SQ_0X \to S_{K-2}X\otimes Z_{(p)} \to 0$$

is given in the following form for each prime p :

(2.11) i) In the case when $S_{K-1}X \cong Z\oplus Z/2$ and $p = 2$,

$$0 \to Z_{(2)}\oplus Z/2 \xrightarrow{\begin{pmatrix}n & 0\\ -\ell_2 & 0\\ 0 & 1\end{pmatrix}} Z_{(2)}\oplus Z_{(2)}\oplus Z/2 \xrightarrow{(\ell_2/n\ 1\ 0)} Q \xrightarrow{n} Z/2^\infty \to 0$$

ii) In the case when $S_{K-1}X \cong Z$ or p is odd,

$$0 \to Z_{(p)} \xrightarrow{\begin{pmatrix}n\\ -\ell_p\end{pmatrix}} Z_{(p)}\oplus Z_{(p)} \xrightarrow{(\ell_p/n\ 1)} Q \xrightarrow{n} Z/p^\infty \to 0 .$$

Here $\ell_p \in Z_{(p)}$ satisfies $\ell_p \equiv \ell \mod nZ_{(p)}$ for each prime p .

Similarly to Theorem 2.3 we obtain the following result as another of our main results.

Theorem 10. Let X be a CW-spectrum such that $KU_0X \cong KU_1X \cong Z$ and $X\wedge SQ = (\Sigma^{-1}\vee\Sigma^0)\wedge SQ$. Assume that X is quasi KO_*-equivalent to the wedge sum $\Sigma^{-1}\vee\Sigma^0$.

i) If $\psi_R^3 = 1$ on $KO_1X \cong Z/2\oplus Z/2$, then X has the same

K_*- local type as the cofiber $C(\tau'_\alpha)$ for some $\alpha \in Q$ with $0 \leq \alpha < 1$.

ii) If $\psi^3_R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ on $KO_1X \cong Z/2 \oplus Z/2$, then X has the same K_*-local type as the cofiber $\Sigma^{-2}C(\omega_\beta)$ for some $\beta \in Q$ with $0 \leq \beta < 2$.

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