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GENERALIZED GENERALIZED SPIN MODELS

EIICHI BANNAI AND ETSUKO BANNAI

Abstract: The concept of spin model was introduced by V. F. R. Jones. Munemasa and Watatani generalized it by dropping the symmetric condition, and defined a generalized spin model. In this paper, by further generalizing the concept using four functions , we define a generalized generalized spin model. Namely, $(X, w_1; w_2, w_3, w_4)$ is a generalized generalized spin model, if X is a finite set and w_i $(i = 1, 2, 3, 4)$ are complex valued functions on $X \times X$ satisfying the following conditions: (1) $w_1(\alpha, \beta)w_3(\beta, \alpha) = 1$, $w_2(\alpha, \beta)w_4(\beta, \alpha) = 1$ for any α, β in X, (2) $\sum_{x\in X} w_1(\alpha, x)w_3(x, \beta) = n\delta_{\alpha,\beta}$, $\sum_{x\in X} w_2(\alpha, x)w_4(x, \beta) = n\delta_{\alpha,\beta}$ for any α and β in X, (3a) $\sum_{x\in X} w_1(\alpha, x)w_1(x, \beta)w_4(\gamma, x) = Dw_1(\alpha, \beta)w_4(\gamma, \alpha)w_4(\gamma, \beta)$ and (3b) $\sum_{x\in X} w_1(x, \alpha)w_1(\beta, x)w_4(x, \gamma) = Dw_1(\beta, \alpha)w_4(\alpha, \gamma)w_4(\beta, \gamma)$ for any α, β, and γ in X, where $D^2 = n = |X|$.

We call as generalized spin models, the special cases of generalized generalized spin models, where there are only two functions w_+ and w_- from $X \times X$ to $\mathbf{C}$ with two of w_1, w_2, w_3, w_4 being in $\{w_+, {}^t w_+\}$ and the remaining two of w_1, w_2, w_3, w_4 being in $\{w_-, {}^t w_-\}$. We see that we have three types of generalized spin models, namely Jones type, pseudo-Jones type, and Hadamard type. We also see that Munemasa-Watatani's generalized spin model is one special case of Jones type, and Jones' original spin model is a further special case of it. Here we emphasize that there are actually interesting spin models which are considerably different from the original concept of spin model defined by Jones.

§ 1. Introduction

The concept of spin model was defined by Jones [6] (see Definition 7 below). Munemasa and Watatani [7] generalized it by dropping the symmetric condition, and defined a generalized spin model (i.e., the generalized spin model of Jones type in Definition 8). In § 1 of the present paper, we further generalize the concept by using four functions w_i $(i = 1, 2, 3, 4)$, and define generalized generalized spin models (see Definition 3). The purpose of § 1 is to discuss the background of this new definition. In the subsequent sections, we study the special cases where there are only two functions w_+ and w_- from $X \times X$ to $\mathbf{C}$ with two of w_1, w_2, w_3, w_4 being in $\{w_+, {}^t w_+\}$ and the remaining two of w_1, w_2, w_3, w_4 being in $\{w_-, {}^t w_-\}$. We call these models generalized spin models, and they are divided into three types (though these types are not exclusive of each other): Jones type, pseudo-Jones type and Hadamard type. They are discussed in § 2, § 3, and § 4 respectively. We also see that Munemasa-Watatani's generalized spin model is the generalized spin model of Jones type (in Definition 8) and that Jones' original spin model is a further special case of it. Here we

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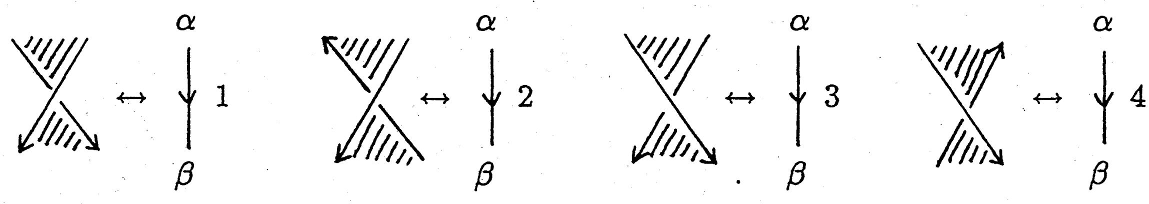
emphasize that there are actually interesting spin models which are considerably different from the original concept of spin model defined by Jones [6].

§ 1. For any diagram L of an oriented link, we color the regions in black and white so that the unbounded region is white and adjacent regions have different colors as in a chess board. Then we get exactly four kinds of crossings. We construct a numbered oriented graph whose vertices are the black regions and edges are the crossings. For each edge (crossing) assign a number and an orientation in the following manner.



For any edge $\alpha \to \beta, n(\alpha \to \beta)$ denotes the number attached to the edge according to the definition given above.

For a diagram L of a link, $v(L)$ denotes the number of the black regions (number of the vertices of the corresponding graph).

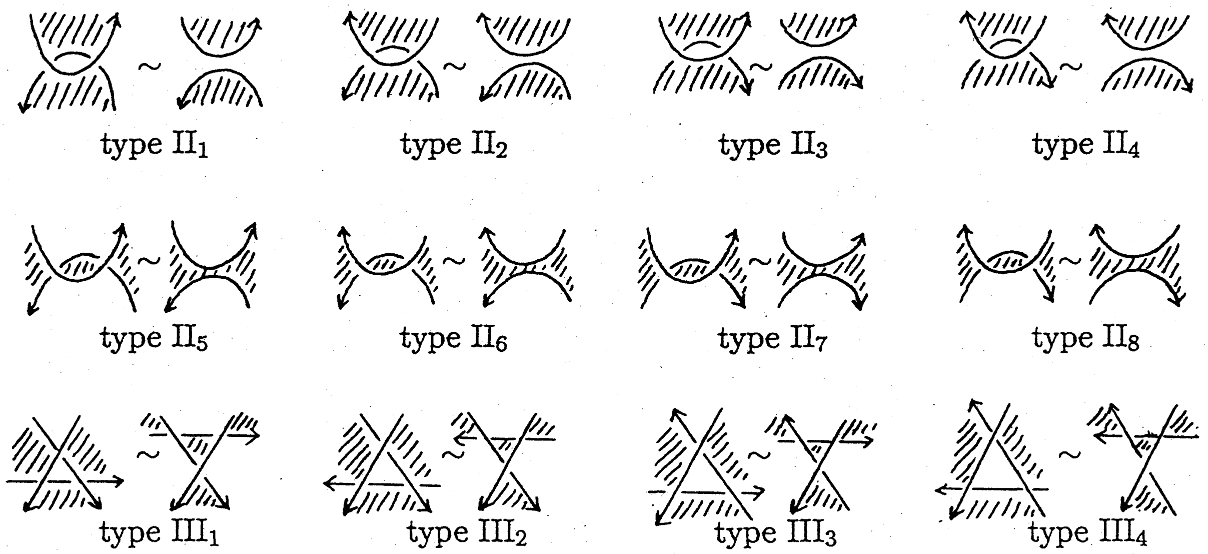
Let X be a finite set with $|X| = n$ and $D^2 = n$. Let w_1, w_2, w_3, and w_4 be complex valued functions defined on $X \times X$.

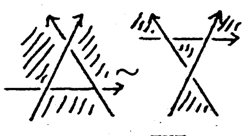
Now we define a partition function Z_L of L by

$$Z_L = D^{-v(L)} \sum_{\substack{\sigma \\ \text{states}}} \prod_{\substack{\alpha\to\beta \\ \text{edges}}} w_{n(\alpha\to\beta)}\big(\sigma(\alpha), \sigma(\beta)\big),$$

where a state σ is a map from the set of vertices of the graph of L to X.

It is easy to see that there exist the following eight kinds of Reidemeister move of type II and sixteen kinds of type III.

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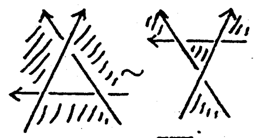
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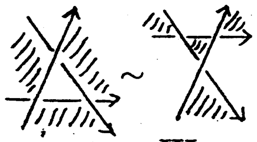
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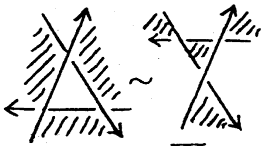
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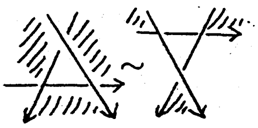
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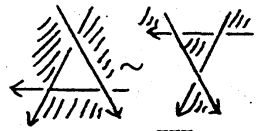
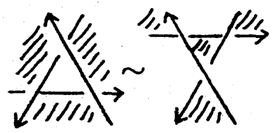
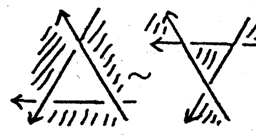
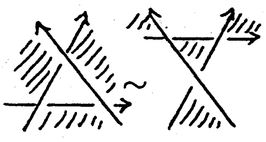

type III_{10}


type III_{11}


type III_{12}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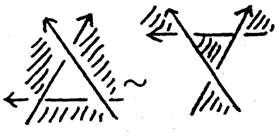
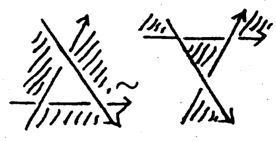
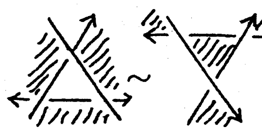

type III_{14}


type III_{15}


type III_{16}

The partition function is invariant under Reidemeister moves of type $\mathrm{II}_1, \cdots$, type II_8 and type $\mathrm{III}_1, \cdots$, type III_{16} if the following conditions $\mathrm{II}_1, \cdots, \mathrm{II}_8$ and $\mathrm{III}_1, \cdots, \mathrm{III}_{16}$ hold respectively.

II_1. $w_1(\alpha,\beta)w_3(\beta,\alpha) = 1$ for any $\alpha, \beta \in X$,

II_2. $w_2(\beta,\alpha)w_4(\alpha,\beta) = 1$ for any $\alpha, \beta \in X$,

II_3. $w_2(\alpha,\beta)w_4(\beta,\alpha) = 1$ for any $\alpha, \beta \in X$,

II_4. $w_1(\beta,\alpha)w_3(\alpha,\beta) = 1$ for any $\alpha, \beta \in X$,

II_5. $\sum_x w_2(\alpha,x)w_4(x,\beta) = n\delta_{\alpha,\beta}$ for any $\alpha, \beta \in X$,

II_6. $\sum_x w_3(\beta,x)w_1(x,\alpha) = n\delta_{\alpha,\beta}$ for any $\alpha, \beta \in X$,

II_7. $\sum_x w_1(\alpha,x)w_3(x,\beta) = n\delta_{\alpha,\beta}$ for any $\alpha, \beta \in X$,

II_8. $\sum_x w_4(\beta,x)w_2(x,\alpha) = n\delta_{\alpha,\beta}$ for any $\alpha, \beta \in X$,

III_1. $\sum_x w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta)$ for any $\alpha,\beta,\gamma \in X$,

III_2. $\sum_x w_1(x,\alpha)w_2(\beta,x)w_3(\gamma,x) = Dw_2(\beta,\alpha)w_3(\gamma,\alpha)w_4(\gamma,\beta)$ for any $\alpha,\beta,\gamma \in X$,

III_3. $\sum_x w_2(x,\alpha)w_2(x,\beta)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_3(\gamma,\alpha)w_3(\beta,\gamma)$ for any $\alpha,\beta,\gamma \in X$,

III_4. $\sum_x w_1(x,\alpha)w_2(x,\beta)w_3(\gamma,x) = Dw_2(\alpha,\beta)w_3(\gamma,\alpha)w_4(\beta,\gamma)$ for any $\alpha,\beta,\gamma \in X$,

III_5. $\sum_x w_1(\alpha,x)w_2(x,\beta)w_3(x,\gamma) = Dw_2(\alpha,\beta)w_3(\alpha,\gamma)w_4(\beta,\gamma)$ for any $\alpha,\beta,\gamma \in X$,

III_6. $\sum_x w_1(x,\alpha)w_1(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_4(\alpha,\gamma)w_4(\beta,\gamma)$ for any $\alpha,\beta,\gamma \in X$,

III_7. $\sum_x w_1(\alpha,x)w_2(\beta,x)w_3(x,\gamma) = Dw_2(\beta,\alpha)w_3(\alpha,\gamma)w_4(\gamma,\beta)$ for any $\alpha,\beta,\gamma \in X$,

III_8. $\sum_x w_2(\alpha,x)w_2(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_3(\alpha,\gamma)w_3(\gamma,\beta)$ for any $\alpha,\beta,\gamma \in X$,

III_9. $\sum_x w_1(\alpha,x)w_3(x,\beta)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_2(\alpha,\gamma)w_4(\gamma,\beta)$ for any $\alpha,\beta,\gamma \in X$,

III_{10}. $\sum_x w_2(\alpha,x)w_3(x,\beta)w_3(\gamma,x) = Dw_2(\alpha,\beta)w_2(\alpha,\gamma)w_3(\gamma,\beta)$ for any $\alpha,\beta,\gamma \in X$,

III_{11}. $\sum_x w_2(x,\alpha)w_4(\beta,x)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_1(\gamma,\alpha)w_3(\beta,\gamma)$ for any $\alpha,\beta,\gamma \in X$,

III_{12}. $\sum_x w_1(x,\alpha)w_3(\beta,x)w_4(\gamma,x) = Dw_1(\beta,\alpha)w_2(\alpha,\gamma)w_4(\gamma,\beta)$ for any $\alpha,\beta,\gamma \in X$,

III_{13}. $\sum_x w_2(x,\alpha)w_3(x,\beta)w_3(\gamma,x) = Dw_2(\beta,\alpha)w_2(\gamma,\alpha)w_3(\gamma,\beta)$ for any $\alpha,\beta,\gamma \in X$,

III_{14}. $\sum_x w_1(x,\alpha)w_3(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_2(\gamma,\alpha)w_4(\beta,\gamma)$ for any $\alpha,\beta,\gamma \in X$,

III_{15}. $\sum_x w_1(\alpha,x)w_3(x,\beta)w_4(x,\gamma) = Dw_1(\alpha,\beta)w_2(\gamma,\alpha)w_4(\beta,\gamma)$ for any $\alpha,\beta,\gamma \in X$,

III_{16}. $\sum_x w_2(\alpha,x)w_4(x,\beta)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_1(\alpha,\gamma)w_3(\gamma,\beta)$ for any $\alpha,\beta,\gamma \in X$.

Let $W_i = \big(w_i(\alpha,\beta)\big)_{\alpha\in X,\beta\in X}$ for $i = 1,2,3,4$. Let I be the identity matrix and J be the matrix whose entries are all 1. Let $Y^{ij}_{\alpha\beta}$ be an n-dimensional column vector whose x-entry is given by $Y^{ij}_{\alpha\beta}(x) = w_i(\alpha,x)w_j(x,\beta)$ for any $i,j \in \{1,2,3,4\}$ and $\alpha,\beta \in X$.

The matrix expressions of II_1 and $\mathrm{II}_4, \mathrm{II}_2$ and $\mathrm{II}_3, \mathrm{II}_6$ and $\mathrm{II}_7, \mathrm{II}_5$ and II_8 are ${}^tW_1 \circ W_3 = J$, ${}^tW_2 \circ W_4 = J$, $W_1W_3 = nI$ and $W_2W_4 = nI$ respectively.

The following $\mathrm{III}_1', \mathrm{III}_2', \cdots, \mathrm{III}_{16}'$ are the matrix expressions of $\mathrm{III}_1, \mathrm{III}_2, \cdots, \mathrm{III}_{16}$ respectively.

III_1'. $W_1Y^{41}_{\alpha\beta} = Dw_4(\alpha,\beta)Y^{41}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_2'. $W_2Y^{31}_{\alpha\beta} = Dw_3(\alpha,\beta)Y^{42}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_3'. ${}^tW_2Y^{42}_{\alpha\beta} = Dw_3(\beta,\alpha)Y^{31}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_4'. ${}^tW_1Y^{32}_{\alpha\beta} = Dw_4(\beta,\alpha)Y^{32}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_5'. ${}^tW_2Y^{13}_{\alpha\beta} = Dw_3(\alpha,\beta)Y^{24}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_6'. ${}^tW_1Y^{14}_{\alpha\beta} = Dw_4(\alpha,\beta)Y^{14}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_7'. $W_1Y^{23}_{\alpha\beta} = Dw_4(\beta,\alpha)Y^{23}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_8'. $W_2Y^{24}_{\alpha\beta} = Dw_3(\beta,\alpha)Y^{13}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_9'. $W_4Y^{13}_{\alpha\beta} = Dw_1(\alpha,\beta)Y^{24}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_{10}'. $W_3Y^{23}_{\alpha\beta} = Dw_2(\alpha,\beta)Y^{23}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_{11}'. $W_4Y^{42}_{\alpha\beta} = Dw_1(\beta,\alpha)Y^{31}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_{12}'. $W_3Y^{41}_{\alpha\beta} = Dw_2(\beta,\alpha)Y^{41}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_{13}'. ${}^tW_3Y^{32}_{\alpha\beta} = Dw_2(\alpha,\beta)Y^{32}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_{14}'. ${}^tW_4Y^{31}_{\alpha\beta} = Dw_1(\alpha,\beta)Y^{42}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_{15}'. ${}^tW_3Y^{14}_{\alpha\beta} = Dw_2(\beta,\alpha)Y^{14}_{\alpha\beta}$ for any $\alpha,\beta \in X$,

III_{16}'. ${}^tW_4Y^{24}_{\alpha\beta} = Dw_1(\beta,\alpha)Y^{13}_{\alpha\beta}$ for any $\alpha,\beta \in X$.

We have the following proposition.

Proposition 1. *Let $W_i, (i = 1,2,3,4)$, satisfy ${}^tW_1 \circ W_3 = {}^tW_2 \circ W_4 = J$ and $W_1W_3 = W_2W_4 = nI$. Then the conditions $III_{12}, III_{11}, III_{14}, III_{13}, III_{16}, III_{15}, III_{10}, III_9$ are equivalent to $III_1, III_2, III_3, III_4, III_5, III_6, III_7, III_8$ respectively.*

Proof. This is obvious from $\mathrm{III}_1', \cdots, \mathrm{III}_{16}'$.

Theorem 2. *Let $W_i, (i = 1,2,3,4)$, satisfy ${}^tW_1 \circ W_3 = {}^tW_2 \circ W_4 = J$ and $W_1W_3 = W_2W_4 = nI$. Then the conditions $III_1, III_4, III_5, III_8$ are equivalent to each other and $III_2, III_3, III_6, III_7$ are equivalent to each other.*

Proof. By III_1 we have

$$\sum_\gamma \{\sum_x w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x)\}w_2(y,\gamma)w_3(\beta,\alpha)$$

$$= \sum_\gamma (Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta))w_2(y,\gamma)w_3(\beta,\alpha)$$

for any $\alpha, \beta, y \in X$. Since $W_2W_4 = nI$ and ${}^tW_1 \circ W_3 = J$, we have III_{16}. Similarly from $\mathrm{III}_{16}, \mathrm{III}_4, \mathrm{III}_9$, by summing over β, γ, and β respectively, we have $\mathrm{III}_4, \mathrm{III}_9$, and III_1 respectively. Therefore $\mathrm{III}_1, \mathrm{III}_{16}, \mathrm{III}_4, \mathrm{III}_9$ are equivalent to each other. Hence, by Proposition 1, $\mathrm{III}_1, \mathrm{III}_5, \mathrm{III}_4, \mathrm{III}_9$ are equivalent to each other. A similar method on $\mathrm{III}_2, \mathrm{III}_{16}, \mathrm{III}_3, \mathrm{III}_{15}$, summing over α, α, β, and α respectively, gives $\mathrm{III}_{10}, \mathrm{III}_3, \mathrm{III}_{15}, \mathrm{III}_2$. Therefore $\mathrm{III}_2, \mathrm{III}_{10}, \mathrm{III}_3, \mathrm{III}_{15}$ are equivalent to each other. Hence, by Proposition 1, $\mathrm{III}_2, \mathrm{III}_7, \mathrm{III}_3, \mathrm{III}_6$ are equivalent to each other.

Theorem 2 tells us that the following definition of generalized generalized spin model is meaningful.

Definition 3. *Let X be a finite set, and let w_i $(i = 1,2,3,4)$ be functions on $X \times X$ to $\mathbf{C}$. Then (X, w_1, w_2, w_3, w_4) is a generalized generalized spin model of loop variable D if the following conditions are satisfied:*

(1) $w_1(\alpha,\beta)w_3(\beta,\alpha) = 1,\ w_2(\alpha,\beta)w_4(\beta,\alpha) = 1$ *for any α and β in X,*

(2) $\sum_{x\in X} w_1(\alpha,x)w_3(x,\beta) = n\delta_{\alpha,\beta},\ \sum_{x\in X} w_2(\alpha,x)w_4(x,\beta) = n\delta_{\alpha,\beta}$ *for any α and β in X,*

(3a) $\sum_{x\in X} w_1(\alpha,x)w_1(x,\beta)w_4(\gamma,x) = Dw_1(\alpha,\beta)w_4(\gamma,\alpha)w_4(\gamma,\beta)$ *for any α, β and γ in X,*

(3b) $\sum_{x\in X} w_1(x,\alpha)w_1(\beta,x)w_4(x,\gamma) = Dw_1(\beta,\alpha)w_4(\alpha,\gamma)w_4(\beta,\gamma)$ *for any α, β and γ in X.*

Note. (3a) and (3b) are III_1 and III_6 respectively.

Note. If (X, w_1, w_2, w_3, w_4) is a generalized generalized spin model, then the partition function Z_L of an oriented link diagram L is invariant under the Reidemeister moves of type II and III.

We have the following matrix expressions of (1), (2), (3a) and (3b).

(1)′ ${}^tW_1 \circ W_3 = J,\ {}^tW_2 \circ W_4 = J,$

(2)′ $W_1W_3 = nI,\ W_2W_4 = nI,$

(3a)′ $W_1Y^{41}_{\gamma\beta} = Dw_4(\gamma,\beta)Y^{41}_{\gamma\beta}$ for any $\gamma, \beta \in X$,

(3b)′ ${}^tW_1Y^{14}_{\beta\gamma} = Dw_4(\beta,\gamma)Y^{14}_{\beta\gamma}$ for any $\gamma, \beta \in X$.

Proposition 4. *Let (X, w_1, w_2, w_3, w_4) be a generalized generalized spin model. Then we have*

(4) $\sum_{x\in X} w_2(\alpha, x) = \sum_{x\in X} w_2(x, \alpha) = Dw_3(\alpha, \alpha) = a^{-1},$

(5) $\sum_{x\in X} w_4(\alpha, x) = \sum_{x\in X} w_4(x, \alpha) = Dw_1(\alpha, \alpha) = a$

for any $\alpha \in X$ with some $a \in \mathbf{C}$. We call this number a the modulus of (X, w_1, w_2, w_3, w_4).

Proof. In $\mathrm{III}_2, \mathrm{III}_3, \mathrm{III}_9$ and III_{14}, put $\alpha = \gamma, \beta = \gamma, \alpha = \beta$, and $\alpha = \beta$ respectively. Then, by (1) and (2), we have the proposition.

The following Proposition 5 is the matrix expression of Proposition 4.

Proposition 5. *Let (X, w_1, w_2, w_3, w_4) be a generalized generalized spin model of modulus a. Then we have the following relations.*

(4)′ $W_2J = {}^tW_2J = a^{-1}J,\ W_3 \circ I = a^{-1}I,$

(5)′ $W_4J = {}^tW_4J = aJ,\ W_1 \circ I = aI.$

Proposition 6. *Let (X, w_1, w_2, w_3, w_4) be a generalized generalized spin model of loop variable D. Then $Dw_4(\alpha, \beta)$ and $Dw_2(\alpha, \beta)$ are eigenvalues of W_1 and W_3 respectively.*

Proof. Obvious from $\mathrm{III}_1{}'$ and $\mathrm{III}_{12}{}'$.
(Note that $Dw_1(\alpha, \beta)$ is not necessarily an eigenvalue of W_4, and that $Dw_3(\alpha, \beta)$ is not necessarily an eigenvalue of W_2.)

§ 2. Generalized spin models of Jones type

In this section we consider the special case of generalized generalized spin models, where there are only two functions w_+ and w_- on $X \times X$ to $\mathbf{C}$ with $w_1, w_2 \in \{w_\epsilon, {}^tw_\epsilon\}$ and $w_3, w_4 \in \{w_{\epsilon'}, {}^tw_{\epsilon'}\}$, where $\{\epsilon, \epsilon'\} = \{+, -\}$.

Definiton 7. *(The original spin model due to Jones [6].)* (X, w_+, w_-) *is a symmetric spin model of Jones type if the following conditions are satisfied.*

(0) $w_+(\alpha,\beta) = w_+(\beta,\alpha),\ w_-(\alpha,\beta) = w_-(\beta,\alpha)$ *for any* α *and* β *in* X,

(1J) $w_+(\alpha,\beta)w_-(\beta,\alpha) = 1$ *for any* α *and* β *in* X,

(2J) $\sum_{x\in X} w_+(\alpha,x)w_-(x,\beta) = n\delta_{\alpha,\beta}$ *for any* α *and* β *in* X,

(3J) $\sum_{x\in X} w_+(\alpha,x)w_+(x,\beta)w_-(x,\gamma) = Dw_+(\alpha,\beta)w_-(\alpha,\gamma)w_-(\beta,\gamma)$ *for any* α,β *and* γ *in* X,

where $|X| = n$ *and* $D^2 = n$.

Definition 8. (X, w_+, w_-) *is a generalized spin model of Jones type if the following conditions are satisfied.*

(1J) $w_+(\alpha,\beta)w_-(\beta,\alpha) = 1$ *for any* α,β *in* X,

(2J) $\sum_{x\in X} w_+(\alpha,x)w_-(x,\beta) = n\delta_{\alpha,\beta}$ *for any* α *and* β *in* X,

(3J) $\sum_{x\in X} w_+(\alpha,x)w_+(x,\beta)w_-(x,\gamma) = Dw_+(\alpha,\beta)w_-(\alpha,\gamma)w_-(\beta,\gamma)$ *for any* α,β *and* γ *in* X, *where* $|X| = n$ *and* $D^2 = n$.

Definition 9. (X, w_+, w_-) *is a generalized spin model of transposed Jones type if the following conditions are satisfied.*

(1JT) $w_+(\alpha,\beta)w_-(\alpha,\beta) = 1$ *for any* α *and* β *in* X,

(2JT) $\sum_{x\in X} w_+(\alpha,x)w_-(\beta,x) = n\delta_{\alpha,\beta}$ *for any* α *and* β *in* X,

(3J) $\sum_{x\in X} w_+(\alpha,x)w_+(x,\beta)w_-(x,\gamma) = Dw_+(\alpha,\beta)w_-(\alpha,\gamma)w_-(\beta,\gamma)$ *for any* α,β *and* γ *in* X, *where* $|X| = n$ *and* $D^2 = n$.

Note. The spin models of symmetric Jones type are special cases of Definitions 8 and 9.

Theorem 10. *Let* $w_+'(\alpha,\beta) = w_+(\beta,\alpha)$ *and* $w_-'(\alpha,\beta) = w_-(\beta,\alpha)$. *Then the following assertions hold.*

(i) (X, w_+, w_-) *is a generalized spin model of Jones type if and only if* (X, w_+, w_-') *is that of transposed Jones type.*

(ii) (X, w_+, w_-) *is a generalized spin model of Jones type if and only if* (X, w_+', w_-) *is that of transposed Jones type.*

Proof. First assume that (X, w_+, w_-) satisfies the conditions (1J) and (2J). Consider (X, w_1, w_2, w_3, w_4) defined by $w_1 = w_+, w_2 = w_+', w_3 = w_-$ and $w_4 = w_-'$. Then

(X, w_1, w_2, w_3, w_4) satisfies the conditions (1) and (2) of Definition 3. Therefore Proposition 1 and Theorem 2 show that III_1 and III_4 are equivalent. In our case III_1 is exactly (3J) and III_4 is exactly

$$\text{(3JT)} \qquad \sum_{x \in X} w_+(\alpha, x) w_+(x, \beta) w_-(\gamma, x) = D w_+(\alpha, \beta) w_-(\gamma, \alpha) w_-(\gamma, \beta)$$

for any α, β, and γ in X. This shows that under (1J) and (2J), (3J) and (3JT) are equivalent. Since the condition (3JT) for (X, w_+, w_-) is the condition (3J) for (X, w_+, w_-'), and conditions (1J) and (2J) for (X, w_+, w_-) are the conditions (1JT) and (2JT) for (X, w_+, w_-'), we have (i).

(ii) Since $\sum_{x \in X} w_+'(\alpha, x) w_+'(x, \beta) w_-(x, \gamma) = \sum_{x \in X} w_+(\beta, x) w_+(x, \beta) w_-(x, \gamma)$ and $D w_+'(\alpha, \beta) w_-(\alpha, \gamma) w_-(\beta, \gamma) = D w_+(\beta, \alpha) w_-(\beta, \gamma) w_-(\alpha, \gamma)$, the condition (3J) for (X, w_+', w_-) is exactly the condition (3J) for (X, w_+, w_-). Therefore we have (ii).

Theorem 11. *Let $w_+' = w_-$ and $w_-' = w_+$. Then (X, w_+, w_-) is a generalized spin model of Jones type (transposed Jones type) if and only if (X, w_+', w_-') is that of Jones type (transposed Jones type respectively).*

Proof. First assume that (X, w_+, w_-) satisfies the conditions (1J) and (2J). Consider (X, w_1, w_2, w_3, w_4) with $w_1(\alpha, \beta) = w_+(\alpha, \beta), w_2(\alpha, \beta) = w_+(\beta, \alpha), w_3(\alpha, \beta) = w_-(\alpha, \beta)$, and $w_4(\alpha, \beta) = w_-(\beta, \alpha)$. Then (X, w_1, w_2, w_3, w_4) satisfies the conditions (1) and (2) of Definition 3. Therefore Proposition 1 and Theorem 2 show that III_7 and III_{10} are equivalent. In our case, III_7 is exactly (3J) and III_{10} is exactly the following condition $(3\mathrm{J}_+)$.

$$(3\mathrm{J}_+) \qquad \sum_{x \in X} w_-(\alpha, x) w_-(x, \beta) w_+(x, \gamma) = D w_-(\alpha, \beta) w_+(\alpha, \gamma) w_+(\beta, \gamma).$$

This shows that under the conditions (1J) and (2J), (3J) and $(3\mathrm{J}_+)$ are equivalent. Next assume (X, w_+, w_-) satisfies the condition (1JT) and (3JT). Consider (X, w_1, w_2, w_3, w_4) with $w_1(\alpha, \beta) = w_+(\alpha, \beta), w_2(\alpha, \beta) = w_+(\beta, \alpha), w_3(\alpha, \beta) = w_-(\beta, \alpha)$ and $w_4(\alpha, \beta) = w_-(\alpha, \beta)$. Then (X, w_1, w_2, w_3, w_4) satisfies the conditions (1) and (2) of Definition 3. Therefore by Proposition 1 and Theorem 2, III_4 and III_{12} are equivalent. In our case III_4 is exactly (3J) and III_{12} is exactly $(3\mathrm{J}_+)$. This shows that under the conditions (1JT) and (2JT), (3J) and $(3\mathrm{J}_+)$ are equivalent. Since the condition $(3\mathrm{J}_+)$ for w_+ and w_- is the condition (3J) for w_+' and w_-', we have the proof for Theorem 11.

Theorem 12. *Let (X, w_1, w_2, w_3, w_4) be a generalized generalized spin model. If $W_1, W_2 \in \{W_\epsilon, {}^tW_\epsilon\}$ and $W_3, W_4 \in \{W_{\epsilon'}, {}^tW_{\epsilon'}\}$ where $\{\epsilon, \epsilon'\} = \{+, -\}$, then the conditions (3a) and (3b) in Definition 3 are equivalent and (X, w_+, w_-) is either a generalized spin model of Jones type or that of transposed Jones type.*

Proof. case (i). $W_1 = W_+, W_2 = W_+, W_3 = W_-, W_4 = W_-$.

The conditions (1) and (2) in Definition 3 show that conditions (1J) and (2J) are satisfied. Both conditions III_1 and III_2 in § 1 give (3JT). Since III_1 and III_2 are equivalent to (3a) and (3b) respectively, the conditions (3a) and (3b) are equivalent. Since (3JT) is equivalent to (3J) under the conditions (1J) and (2J), (X, w_+, w_-) is a generalized spin model of Jones type.

case (ii). $W_1 = W_+, W_2 = W_+, W_3 = W_-, W_4 = {}^tW_-$.

By (2) of Definition 3, we have $W_+W_- = W_+{}^tW_- = nI$. Therefore W_- is symmetric. By (1) of Definition 3, W_+ is also symmetric. Hence the conditions (3a) and (3b) both give condition (3) in Definition 7, and (X, w_+, w_-) is a symmetric spin model of Jones type.

case (iii). $W_1 = W_+, W_2 = {}^tW_+, W_3 = W_-, W_4 = W_-$.

A similar argument as in (ii) proves that (X, w_+, w_-) is a symmetric spin model of Jones type.

case (iv). $W_1 = W_+, W_2 = {}^tW_+, W_3 = W_-, W_4 = {}^tW_-$.

The conditions III_1 and III_7 in § 1 both give (3J). Since III_1 and III_7 are equivalent to (3a) and (3b) respectively, (3a) and (3b) are equivalent. Therefore (X, w_+, w_-) is a generalized spin model of Jones type.

case (v). $W_1 = W_+, W_3 = {}^tW_-$.

Let $W_-' = {}^tW_-$. Then cases (i) and (iv) for W_+ and W_-' prove that (X, w_+, w_-') is a generalized spin model of Jones type and cases (ii) and (iii) for W_+, W_-' show that (X, w_+, w_-') is a symmetric spin model of Jones type. Therefore (X, w_+, w_-) is a generalized spin model of transposed Jones type or a symmetric spin model of Jones type.

case(vi). $W_1 = {}^tW_+$.

Let $W_+' = {}^tW_+$. Then cases (i), (ii), (iii), (iv), and (v) for W_+', W_- and Theorem 10 show that (X, w_+, w_-) is a generalized spin model of Jones type, transposed Jones type, or a symmetric spin model of Jones type.

case (vii). $W_1 \in \{W_-, {}^tW_-\}$.

Let $W_+' = W_-$ and $W_-' = W_+$. Then cases (i), (ii), $\cdots$, (vi) show that (X, W_+', W_-') is a generalized spin model of Jones type, transposed Jones type or a symmetric spin model of Jones type. Therefore by Theorem 11, the proof is completed.

Remark. Combining Theorem 11 and Theorem 10, we can conclude that in order to study generalized spin models of transposed Jones type, we essentially have to consider the generalized spin models of Jones type.

Note. For a given generalized spin model of Jones type, the proof of Theorem 12 shows which signed oriented graph should be taken to construct partition functions of oriented link diagrams which are invariant under the Reidemeister moves of type II and III. Each type of generalized spin model of Jones type has several choices of signed oriented graph which give possibly different partition functions.

§ 3. Generalized spin models of pseudo-Jones type.

In this section we consider the generalized generalized spin model with $W_1, W_4 \in \{W_\epsilon, {}^tW_\epsilon\}$ and $W_2, W_3 \in \{W_{\epsilon'}, {}^tW_{\epsilon'}\}$ where $\{\epsilon, \epsilon'\} = \{+, -\}$.

Definition 13. *(X, w_+, w_-) is a generalized spin model of pseudo-Jones type if the following conditions are satisfied for any α, β, and γ in X.*

(0) $w_+(\alpha,\beta) = w_+(\beta,\alpha),\ w_-(\alpha,\beta) = w_-(\beta,\alpha),$
(1J) $w_+(\alpha,\beta)w_-(\alpha,\beta) = 1,$
(2J) $\sum_{x\in X} w_+(\alpha,x)w_-(x,\beta) = n\delta_{\alpha,\beta},$
(3P) $\sum_{x\in X} w_+(\alpha,x)w_+(x,\beta)w_+(x,\gamma) = Dw_+(\alpha,\beta)w_+(\alpha,\gamma)w_+(\beta,\gamma).$

We have the following theorem.

Theorem 14. *Let (X, w_1, w_2, w_3, w_4) be a generalized generalized spin model. If $W_1, W_4 \in \{W_+, {}^tW_+\}$ (or $W_1, W_4 \in \{W_-, {}^tW_-\}$) and $W_2, W_3 \in \{W_-, {}^tW_-\}$ (or $W_2, W_3 \in \{W_+, {}^tW_+\}$, resp.) for some matrices $W_+ = (w_+(\alpha,\beta))_{\alpha\in X,\beta\in X}$ and $W_- = (w_-(\alpha,\beta))_{\alpha\in X,\beta\in X}$, then the conditions (3a) and (3b) in the Definition 3 coincide and (X, w_+, w_-) is a generalized spin model of pseudo-Jones type.*

Proof. First we will show that W_+ and W_- are symmetric.

case (i). $W_1 = W_+, W_4 = W_+, W_2 = W_-, W_3 = W_-$.

By the assumptions we have ${}^tW_+ \circ W_- = J$ and $W_+W_- = nI$. Also we have $Y^{31}_{\alpha\beta} = Y^{24}_{\alpha\beta}$ and $Y^{13}_{\alpha\beta} = Y^{42}_{\alpha\beta}$. Then by III_3' and III_5' we have $w_-(\beta,\alpha) = w_-(\alpha,\beta)$. Therefore W_+ and W_- are symmetric.

case (ii). $W_1 = W_+, W_4 = W_+, W_2 = W_-, W_3 = {}^tW_-$.

By the assumptions we have $W_+{}^tW_- = W_-W_+ = nI$. Therefore W_- is symmetric and so is W_+.

case (iii). $W_1, W_4 \in \{W_-, {}^tW_-\}$ and $W_2, W_3 \in \{W_+, {}^tW_+\}$.

Let $W_+{}' = W_-$ and $W_-{}' = W_+$. Then case (i) and case (ii) show that $W_+{}'$ and $W_-{}'$ are symmetric. Therefore W_+ and W_- are symmetric.

Thus we see that W_+ and W_- are symmetric. Therefore W_1, W_2, W_3, and W_4 are symmetric and the conditions (3a) and (3b) in Definition 3 are equivalent. For the case $W_1 = W_4 = W_+$, the condition (3a) gives (3P) of Definition 13. For the case $W_1 = W_4 = W_-$, we have that $W_2 = W_3 = W_+$, and that III_{10} gives the condition (3P). Therefore, in both cases, (X, W_+, W_-) is a generalized spin model of pseudo-Jones type.

Note. For a given generalized spin model of psudo-Jones type, the proof of Theorem 14 tells which signed oriented graph should be taken to construct a partition function of oriented link diagram which is invariant under the Reidemeister moves of type II and III.

§ 4. Generalized spin models of Hadamard type.

In this section we consider the cases where $W_1, W_3 \in \{W_\epsilon, {}^tW_\epsilon\}$ and $W_2, W_4 \in \{W_{\epsilon'}, {}^tW_{\epsilon'}\}$, where $\{\epsilon, \epsilon'\} = \{+, -\}$. In these cases, W_+ or W_- is an Hadamard matrix. We call these spin models Hadamard type.

Definition 15. *(X, w_+, w_-) is a generalized spin model of type (H_ϵ) if the following conditions are satisfied.*

(0H$_\epsilon$) $w_\epsilon(\alpha,\beta) = w_\epsilon(\beta,\alpha)$ for any α and β in X,

(1H) $w_+(\alpha,\beta)w_+(\beta,\alpha) = 1,\ w_-(\alpha,\beta)w_-(\beta,\alpha) = 1$ for any α and β in X,

(2H) $\sum_{x\in X} w_+(\alpha,x)w_+(x,\beta) = n\delta_{\alpha,\beta},\ \sum_{x\in X} w_-(\alpha,x)w_-(x,\beta) = n\delta_{\alpha,\beta}$ for any α and β in X,

(3a$_\epsilon$) $\sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(x,\beta)w_\epsilon(x,\gamma) = Dw_{\epsilon'}(\alpha,\beta)w_\epsilon(\alpha,\gamma)w_\epsilon(\beta,\gamma)$ for any α,β, and γ in X,

where $|X| = n = D^2$.

Definition 16. *(X, w_+, w_-) is a generalized spin model of type (HA_ϵ) if the following conditions are satisfied.*

(1H$_\epsilon$) $w_\epsilon(\alpha,\beta)w_\epsilon(\alpha,\beta) = 1,\ w_{\epsilon'}(\alpha,\beta)w_{\epsilon'}(\beta,\alpha) = 1$ for any α and β in X,

(2H$_\epsilon$) $\sum_{x\in X} w_\epsilon(\alpha,x)w_\epsilon(\beta,x) = n\delta_{\alpha,\beta},\ \sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(x,\beta) = n\delta_{\alpha,\beta}$ for any α and β in X,

(3a$_\epsilon$) $\sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(x,\beta)w_\epsilon(x,\gamma) = Dw_{\epsilon'}(\alpha,\beta)w_\epsilon(\alpha,\gamma)w_\epsilon(\beta,\gamma)$ for any α,β, and γ in X,

(3b$_\epsilon$) $\sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(x,\beta)w_\epsilon(\gamma,x) = Dw_{\epsilon'}(\alpha,\beta)w_\epsilon(\gamma,\alpha)w_\epsilon(\gamma,\beta)$ for any α,β, and γ in X,

where $|X| = n = D^2$.

Definition 17. *(X, w_+, w_-) is a generalized spin model of type (HB_ϵ) if the following conditions are satisfied.*

(1H$_\epsilon$) $w_\epsilon(\alpha,\beta)w_\epsilon(\alpha,\beta) = 1,\ w_{\epsilon'}(\alpha,\beta)w_{\epsilon'}(\beta,\alpha) = 1$ for any α and β in X,

(2H$_\epsilon$) $\sum_{x\in X} w_\epsilon(\alpha,x)w_\epsilon(\beta,x) = n\delta_{\alpha,\beta},\ \sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(x,\beta) = n\delta_{\alpha,\beta}$ for any α and β in X,

(3a$_{\epsilon'}$) $\sum_{x\in X} w_\epsilon(\alpha,x)w_\epsilon(x,\beta)w_{\epsilon'}(x,\gamma) = Dw_\epsilon(\alpha,\beta)w_{\epsilon'}(\alpha,\gamma)w_{\epsilon'}(\beta,\gamma)$ for any α,β, and γ in X,

(3b$_{\epsilon'}$) $\sum_{x\in X} w_\epsilon(\alpha,x)w_\epsilon(x,\beta)w_{\epsilon'}(\gamma,x) = Dw_\epsilon(\alpha,\beta)w_{\epsilon'}(\gamma,\alpha)w_{\epsilon'}(\gamma,\beta)$ for any α,β, and γ in X,

where $|X| = n = D^2$.

Definition 18. *(X, w_+, w_-) is a generalized spin model of type (HC_ϵ) if the following conditions are satisfied.*

$(0H_\epsilon)$ $w_\epsilon(\alpha,\beta) = w_\epsilon(\beta,\alpha)$ *for any* α *and* β *in* X,

$(1H_{\epsilon'})$ $w_{\epsilon'}(\alpha,\beta)w_{\epsilon'}(\alpha,\beta) = 1,\ w_\epsilon(\alpha,\beta)w_{\epsilon'}(\beta,\alpha) = 1$ *for any* α *and* β *in* X,

$(2H_{\epsilon'})$ $\sum_{x\in X} w_\epsilon(\alpha,x)w_\epsilon(x,\beta) = n\delta_{\alpha,\beta},\ \sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(\beta,x) = n\delta_{\alpha,\beta}$ *for any* α *and* β *in* X,

$(3a_{\epsilon'})$ $\sum_{x\in X} w_\epsilon(\alpha,x)w_\epsilon(x,\beta)w_{\epsilon'}(x,\gamma) = Dw_\epsilon(\alpha,\beta)w_{\epsilon'}(\alpha,\gamma)w_{\epsilon'}(\beta,\gamma)$ *for any* α, β, *and* γ *in* X,

$(3b_{\epsilon'})$ $\sum_{x\in X} w_\epsilon(\alpha,x)w_\epsilon(x,\beta)w_{\epsilon'}(\gamma,x) = Dw_\epsilon(\alpha,\beta)w_{\epsilon'}(\gamma,\alpha)w_{\epsilon'}(\gamma,\beta)$ *for any* α, β, *and* γ *in* X,

where $|X| = n = D^2$.

Note. Let $w_+{}'(\alpha,\beta) = w_+(\beta,\alpha)$, $w_-{}'(\beta,\alpha) = w_-(\beta,\alpha)$. If (X, w_+, w_-) is a generalized spin model of type (H_ϵ), (HA_ϵ), (HB_ϵ) and (HC_ϵ), then $(X, w_+{}', w_-)$, $(X, w_+, w_-{}')$, $(X, w_+{}', w_-{}')$ are also generalized spin models of type (H_ϵ), (HA_ϵ), (HB_ϵ), and (HC_ϵ) respectively.

Note. Let $w_+{}'(\alpha,\beta) = w_-(\alpha,\beta)$ and $w_-{}'(\alpha,\beta) = w_+(\alpha,\beta)$. If (X, w_+, w_-) is a generalized spin model of type (H_ϵ), (HA_ϵ), (HB_ϵ), and (HC_ϵ), then $(X, w_+{}', w_-{}')$ is a generalized spin model of type $(H_{\epsilon'})$, $(HA_{\epsilon'})$, $(HB_{\epsilon'})$, and $(HC_{\epsilon'})$.

Definition 19. *(X, w_+, w_-) is a symmetric Hadamard type spin model if the following conditions are satisfied. We denote it by type (SH_ϵ)*

(0) $w_+(\alpha,\beta) = w_+(\beta,\alpha),\ w_-(\alpha,\beta) = w_-(\beta,\alpha)$,

(1H) $W_+ \circ W_+ = J,\ W_- \circ W_- = J$,

(2H) $W_+{}^2 = nI,\ W_-{}^2 = nI$,

$(3a_\epsilon)$ $\sum_{x\in X} w_{\epsilon'}(\alpha,x)w_{\epsilon'}(x,\beta)w_\epsilon(x,\gamma) = Dw_{\epsilon'}(\alpha,\beta)w_\epsilon(\alpha,\gamma)w_\epsilon(\beta,\gamma)$.

Theorem 20. *Let (X, w_1, w_2, w_3, w_4) be a generalized generalized spin model. Assume that $W_1, W_3 \in \{W_\epsilon, {}^tW_\epsilon\}$ and $W_2, W_4 \in \{W_{\epsilon'}, {}^tW_{\epsilon'}\}$ for some matrices $W_\epsilon = (w_\epsilon(\alpha,\beta))_{\alpha\in X,\beta\in X}$ and $W_{\epsilon'} = (w_{\epsilon'}(\alpha,\beta))_{\alpha\in X,\beta\in X}$. Then (X, w_+, w_-) is one of the generalized spin models of type (H_ϵ), (HA_ϵ), (HB_ϵ), and (HC_ϵ), where $\epsilon \in \{+,-\}$.*

Proof. case (i). $W_1 = W_+, W_2 = W_-, W_3 = W_+, W_4 = W_-$.

By (1) and (2) of Definition 3, we have (1H) and (2H). By $\mathrm{III}_1{}'$ of § 2, $Dw_4(\alpha,\beta) = Dw_-(\alpha,\beta)$ is an eigenvalue of $W_1 = W_+$ for any $\alpha, \beta \in X$. By (2H) we have $W_+{}^2 = W_-{}^2 = nI$. Therefore $w_-(\alpha,\beta)w_-(\alpha,\beta) = 1$. Therefore by $(1H_-)$ we have $w_-(\alpha,\beta) = w_-(\beta,\alpha)$ for any α and β in X. Hence $(0H_-)$ is satisfied. Since $W_4 = W_-$ is symmetric, (3a) and (3b) of Definition 3 are both equivalent to $(3a_-)$. Therefore (X, w_+, w_-) is a generalized spin model of type (H_-).

case (ii). $W_1 = W_+, W_2 = W_-, W_3 = W_+, W_4 = {}^tW_-$.

By (1) and (2) of Definition 3, we have (1H$_-$) and (2H$_-$). (3a) and (3b) of Definition 3 give (3a$_-$) and (3b$_-$) respectively. Therefore (X, w_+, w_-) is a generalized spin model of type (HA$_-$).

case (iii). $W_1 = W_+, W_2 = W_-, W_3 = {}^tW_+, W_4 = W_-$.

By (1) and (2) of Definition 3, we have (1H$_+$) and (2H$_+$). (3a) and (3b) of Definiton 3 give (3a$_-$) and (3b$_-$) respectively. Therefore (X, w_+, w_-) is a generalized spin model of type (HB$_+$).

case (iv). $W_1 = W_+, W_2 = W_-, W_3 = {}^tW_+, W_4 = {}^t W_-$.

By (1) and (2) of Definition 3, we have $W_+ \circ W_+ = J$, $W_- \circ W_- = J$, $W_+ {}^tW_+ = nI$ and $W_- {}^tW_- = nI$. By III$_1{}'$ of § 2, $W_+{}^2 Y^{41}_{\alpha\beta} = D^2 w_-(\beta, \alpha)^2 Y^{41}_{\alpha\beta} = nY^{41}_{\alpha\beta}$ for any α and β in X. Since $\{Y^{41}_{\alpha\beta}\}_{\alpha,\beta\in X}$ is a spanning set, $W_+{}^2 = nI$. Hence W_+ is symmetric. Therefore we have (0H$_+$), (1H$_-$), (2H$_-$) of Definition 18. (3a) and (3b) of Definition 3 give (3a$_-$) and (3b$_-$) respectively. Therefore (X, w_+, w_-) is of type (HC$_+$).

case (v). $W_1 = W_+, W_2 = {}^tW_-$.

Let $W_-{}' = {}^tW_-$. Then $W_1 = W_+$ and $W_2 = W_-{}'$, and $(X, W_+, W_-{}')$ satisfies the conditions of case (i), (ii), (iii), or (iv). Therefore $(X, W_+, W_-{}')$ is of type (H$_-$), type (HA$_-$), type (HB$_+$), and type (HC$_+$) respectively.

case (vi). $W_1 = {}^tW_+$.

Let $W_+{}' = {}^t W_+$. Then cases (i), (ii), (iii), (iv), (v) show that $(X, w_+{}', w_-)$ is a generalized spin model of type (H$_-$), (HA$_-$), (HB$_+$), and (HC$_+$). Therefore (X, w_+, w_-) is also of those type.

caes (vii). $W_1 \in \{W_-, {}^t W_-\}$.

Let $W_+{}' = W_-$ and $W_-{}' = W_+$. Then $W_1, W_3 \in \{W_+{}', {}^t W_+{}'\}$ and $W_2, W_4 \in \{W_-{}', {}^t W_-{}'\}$. Therefore (X, W'_+, W'_-) is a generalized spin model of type (H$_-$), (HA$_-$), (HB$_+$), (HC$_+$). Therefore (X, w_+, w_-) is those of type (H$_+$), (HA$_+$), (HB$_-$) and (HC$_-$).

Note. As for the partition function Z_L of an oriented link diagram L attached to the generalized spin models of Hadamard type, the proof of Theorem 20 shows what kinds of signed oriented graphs are suitable for each type of generalized spin model of Hadamard type. The choice is not unique.

§ 5. Concluding Remarks.

Generalized generalized spin models (X, w_1, w_2, w_3, w_4) seem to exist considerably in abundance when compared with the original (symmetric) spin models due to Jones. The generalized spin models considered in § 2, § 3, § 4 are special cases of generalized generalized spin models, but they exist also considerably in abundance.

As we have discussed in § 2, § 3, and § 4, we have three types of generalized spin models: Jones type, pseudo-Jones type and Hadamard type.

1) In order to consider (non-symmetric) Jones type, essentially we only have to consider Definition 8 (because of Theorem 10). Such generalized spin models were first considered by Munemasa and Watatani [7]. They gave two explicit examples with $n = 3$ and $n = 5$. A family of such examples were constructed on the group association schemes of finite cyclic

groups by Bannai and Bannai [1]. For symmetric Jones type, there are many examples attached to symmetric association schemes, in particular to strongly regular graphs (*cf.* [4], [5]). Nomura [8] systematically gives examples of symmetric spin models (in the original sense of Jones) attached to an Hadamard graph (i.e., the distance-regular graph of intersection array

$$\left\{\begin{matrix} * & 1 & m & 2m-1 & 2m \\ 0 & 0 & 0 & 0 & 0 \\ 2m & 2m-1 & m & 1 & * \end{matrix}\right\}$$

which is canonically constructed from each Hadamard matrix (see [3, p.19]).

2) In pseudo-Jones type, the matrices W_+ and W_- are always symmetric. The following is an explicit example of pseudo-Jones type which is not of Jones type nor of Hadamard type.

$$W_+ = \begin{pmatrix} 1 & i & 1 & -i \\ i & 1 & -i & 1 \\ 1 & -i & 1 & i \\ -i & 1 & i & 1 \end{pmatrix}, \quad W_- = \begin{pmatrix} 1 & -i & 1 & i \\ -i & 1 & i & 1 \\ 1 & i & 1 & -i \\ i & 1 & -i & 1 \end{pmatrix}$$

with $i = \sqrt{-1}$. (We can check all the conditions in Definition 13 easily.) It is expected that there are many other generalized spin models of pseudo-Jones type.

3) In Hadamard type, we only have to consider the following ones: symmetric Hadamard type (H_+), (HA_+), (HB_+), and (HC_+) (because of Notes in § 4).

a) The following is an example of symmetric Hadamard type, which is not of Jones type, nor of pseudo-Jones type.

$$W_+ = \begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}, \quad W_- = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

(We can easily check the conditions in Definition 19.)

b) The following is an example of non-symmetric Hadamard type (H_-), which is not of Jones type, nor of pseudo-Jones type.

$$W_+ = \begin{pmatrix} 1 & -i & -1 & -i \\ i & 1 & i & -1 \\ -1 & -i & 1 & -i \\ i & -1 & i & 1 \end{pmatrix}, \quad W_- = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

(We can easily check the conditions in Definition 15.) It is expected that there exist many other generalized spin models of Hadamard type.

Remark. Let $(X_1, w_1^{(1)}, w_2^{(1)}, w_3^{(1)}, w_4^{(1)})$ and $(X_2, w_1^{(2)}, w_2^{(2)}, w_3^{(2)}, w_4^{(2)})$ be generalized generalized spin models. Let us set $X = X_1 \times X_2$, $w_i = w_i^{(1)} \otimes w_i^{(2)}$ $(i = 1, 2, 3, 4)$, namely, $W_i =$

$W_i^{(1)} \otimes W_i^{(2)}$, where $W_i^{(j)}$ is the matrix representation of $w_i^{(j)}$. Then (X, w_1, w_2, w_3, w_4) is a generalized generalized spin model. (We can immediately prove this claim by checking Definition 3.) Also, we can easily see that if $(X_i, w_+^{(i)}, w_-^{(i)})$ $(i = 1, 2)$ are two generalized spin models of a same type, i.e., symmetric Jones type, Jones type, transposed Jones type, pseudo-Jones type, symmetric Hadamard type, or Hadamard type (H_ϵ), (HA_ϵ), (HB_ϵ), (HC_ϵ), then (X, w_+, w_-) with $X = X_1 \times X_2$, $w_+ = w_+^{(1)} \otimes w_+^{(2)}$, $w_- = w_-^{(1)} \otimes w_-^{(2)}$ is a generalized spin model of the same type. Therefore, by this tensor product construction, we get many more examples of various spin models. Note that if we take two generalized spin models of different types, then their tensor product is generally not a generalized spin model, but a generalized generalized spin model.

Anyway, it seems interesting to notice that in many instances, the existence of spin models is closely connected with the existence of interesting combinatorial objects such as Hadamard matrices, association schemes, etc,. (See [2] and [3] for general information on such combinatorial objects.)

We want to discuss further examples of (various kinds of) spin models and the link invariants attached to them in subsequent papers by looking at more combinatorial objects, and by considering (generalized) generalized spin models, we hope to be able to find missing mechanisms of systematically constructing spin models which Jones [6, p.325] wanted to discover.

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Department of Mathematics
Faculty of Science, Kyushu University
Hakozaki 6-10-1, Higashi-ku
Fukuoka, 812, Japan

and

Yakuin 4-1-18-126
chuo-ku, Fukuoka, 810, Japan