数理解析研究所講究録
第 840 巻 1993年 54-64

Spin Models Constructed from Hadamard matrices

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A new spin model M is constructed from an arbitrary Hadamard matrix H through a distance-regular graph which is called a Hadamard graph. F. Jaeger gives a formula for the link invariant of the model M, and V. F. R. Jones gives two links which have the same V-polynomial but different polynomials of M.

1 Definition of a Spin Model

The following definition is essentially due to V. F. R. Jones [8].

Definition 1 Let n be a positive integer, D be one of the square roots of n. A *spin model* with loop variable D is a pair (X, w) of a finite non-empty set X of size n, and a complex-valued symmetric function w on $X \times X$ which satisfy the following equations for all $\alpha, \beta, \gamma \in X$:

$$\frac{1}{n}\sum_{x\in X}\frac{w(\alpha,x)}{w(\beta,x)} = \delta_{\alpha,\beta} \tag{1}$$

$$\frac{1}{D}\sum_{x\in X}\frac{w(\alpha,x)w(\beta,x)}{w(\gamma,x)} = \frac{w(\alpha,\beta)}{w(\alpha,\gamma)w(\beta,\gamma)} \tag{2}$$

Each element of X is called a *spin*, and the function w is called *Boltzmann weight*. The $(n\times n)$-matrix $W = (w(\alpha,\beta))$, is called the *weight matrix* of the spin model. The equation (2) is called star-triagle relation.

Example Let X be a finite set of size $n = D^2 > 1$ and let a, b be complex numbers such that

$$b^2 + \frac{1}{b^2} + D = 0, \quad a = -\frac{1}{b^3}.$$

Define a function w by

$$w(\alpha,\beta) = \begin{cases} a & \text{if } \alpha = \beta \\ b & \text{if } \alpha \neq \beta \end{cases}$$

As easily shown, (X, w) becomes a spin model with the weight matrix

$$M = (a-b)I + bJ.$$

This spin model is called *Potts model*.

Remark 1 If (X, w) is a spin model with $D = \sqrt{n}$, then $(X, \sqrt{-1}\,w)$ becomes a spin model with $D = -\sqrt{n}$.

Remark 2 Under (1), the star-triagle relation (2) is equivalent to:

$$\frac{1}{D}\sum_{x\in X}\frac{w(\alpha,x)}{w(\beta,x)w(\gamma,x)}=\frac{w(\alpha,\beta)w(\alpha,\gamma)}{w(\beta,\gamma)}. \tag{3}$$

Remark 3 By putting $\beta=\gamma$ in 2, we get

$$\frac{1}{D}\sum_{x\in X}w(\alpha,x)=\frac{1}{w(\beta,\beta)}.$$

This shows $w(\beta,\beta)$ is independent on the choise of $\beta\in X$:

$$w(\beta,\beta)=a$$

is a constant called *modulus* of the model. Thus we have

$$\frac{1}{D}\sum_{x\in X}w(\alpha,x)=\frac{1}{a}.$$

From 3, we have

$$\frac{1}{D}\sum_{x\in X}\frac{1}{w(\alpha,x)}=a.$$

Remark 4 The equation (1) is equivallent to

$$\sum_{x\in X}\frac{w(\alpha,x)}{w(\beta,x)}=0 \qquad \text{if } \alpha\neq\beta.$$

2 Spin Models on Distance-Regular Graphs

A connected graph Γ is said to be *distance-regular* if there are integers b_i, c_i $(i\geq 0)$ such that for any two vertices u, x at distance $i=\partial(u,x)$, there are precisely c_i neighbours of x in $\Gamma_{i-1}(u)$ and b_i neighbours of x in $\Gamma_{i+1}(u)$. In particular, Γ is regular of valency $k=b_0$. The sequence

$$\iota(\Gamma)=\{b_0,b_1,\ldots,b_{d-1};\, c_1,c_2,\ldots,c_d\},$$

where d is the diameter of Γ, is called the intersection array of G. For two vertices u, v, the size

$$p^{\alpha}_{ij}=|\Gamma_i(u)\cap\Gamma_j(v)|$$

depends only on the distance $\alpha=\partial(u,v)$, rather than the individual vertices u, v with $\partial(u,v)=\alpha$ (see [4] 4.1). In particular $k_i=|\Gamma_i(u)|$, which is called the *i-th valency*, does not depend on the choice of a vertex u. For three vertices u, v, w, put

$$P_{ij\ell}(u,v,w)=|\Gamma_i(u)\cap\Gamma_j(v)\cap\Gamma_\ell(w)|.$$

More presice descriptions about distance-regular graphs will be found in [3], [4].

The following Proposition is obtained directly from the definition and remarks in the previous section.

Proposition 1 *Let Γ be a distance-regular graph of diameter d with the vertex set X. Put $|X| = n$ and let D be one of the square roots of n. Let t_0, t_1, ..., t_d be non-zero complex numbers and let w be the complex valued function on $X \times X$ defined by $w(u,v) = t_i$ where $i = \partial(u,v)$. Then (X, w) becomes a spin model if and only if the following conditions hold:*

(C1) $\sum_{i=0}^{d} k_i t_i = D t_0^{-1}$,

(C2) $\sum_{i=0}^{d} k_i t_i^{-1} = D t_0$,

(C3) $\sum_{i=0}^{d}\sum_{j=0}^{d} p_{ij}^{\alpha} t_i t_j^{-1} = 0 \qquad (\alpha = 1, 2, \ldots, d)$,

(C4) For all vertices u, v, w in X,

$$\sum_{\ell=0}^{d}\sum_{i=0}^{d}\sum_{j=0}^{d} P_{ij\ell}(u,v,w) t_i t_j t_\ell^{-1} = D t_\alpha t_\beta^{-1} t_\gamma^{-1},$$

where $\alpha = \partial(u,v)$, $\beta = \partial(u,w)$, $\gamma = \partial(v,w)$.

Remark 5 Though conditions (C1) and (C2) can be removed in the above, these are useful to find solutions of the equations.

3 Result

A distance-regular graph having the intersection array

$$\{4m,\ 4m-1,\ 2m,\ 1;\ 1,\ 2m,\ 4m-1,\ 4m\}$$

is called a *Hadamard graph* of order $4m$. There is a natural one-to-one correspondence between Hadamard graphs of order $4m$ and Hadamard matrices of order $4m$ (see [4] 1.8). Now our main result follows:

Theorem 2 *Let Γ be a Hadamard graph of order $4m$. Let s, t_0, t_1 be complex numbers such that*

$$s^2 + 2(2m-1)s + 1 = 0, \qquad t_0^2 = \frac{2\sqrt{m}}{(4m-1)s+1}, \qquad t_1^4 = 1.$$

Put $t_2 = s t_0$, $t_3 = -t_1$ and $t_4 = t_0$. Then t_0,...,t_4 satisfy the conditions in Proposition 1 with $D = 4\sqrt{m}$.

Theorem 2 can be described without using distance-regular graphs as follows:

Theorem 3 *Let H be a Hadamard matrix of order n, $n \equiv 0 \pmod 4$, and let M be the weight matrix of the Potts model of size n. Let ω be one of the 4-th roots of 1, $\omega^4 = 1$. Define a $4n \times 4n$-matrix W as:*

$$W = \begin{pmatrix} M & M & \omega H & -\omega H \\ M & M & -\omega H & \omega H \\ \omega H^t & -\omega H^t & M & M \\ -\omega H^t & \omega H^t & M & M \end{pmatrix}$$

Then W becomes the weight matrix of a spin model having $4n$ spins.

4 Proof of Theorem 2

Let H be a Hadamard graph of order $4m$ and let $s, t_0, \ldots, t_4$ be complex numbers such that

$$s^2 + 2(2m-1)s + 1 = 0, \quad t_0^2 = \frac{2\sqrt{m}}{(4m-1)s+1},$$
$$t_1^4 = 1, \quad t_2 = st_0, \quad t_3 = -t_1, \quad t_4 = t_0.$$

By $k_{i-1}b_{i-1} = k_i c_i$, we get

$$k_0 = 1, \quad k_1 = 4m, \quad k_2 = 8m-2, \quad k_3 = 4m, \quad k_4 = 1.$$

So (C1) becomes

$$t_0 + 4mt_1 + (8m-2)t_2 + 4mt_3 + t_4 = 4\sqrt{m}\, t_0^{-1}.$$

By $t_3 = -t_1$, $t_0 = t_4$ and $t_2 = st_0$, this becomes

$$2t_0 + (8m-2)st_0 = 4\sqrt{m}\, t_0^{-1}.$$

Clearly this holds by the assumption $t_0^2 = 2\sqrt{m}((4m-1)s+1)^{-1}$.

Condition (C2) becomes

$$t_0^{-1} + 4mt_1^{-1} + (8m-2)t_2^{-1} + 4mt_3^{-1} + t_4^{-1} = 4\sqrt{m}\, t_0,$$

and it becomes

$$2t_0^{-1} + (8m-2)t_2^{-1} = 4\sqrt{m}\, t_0,$$
$$1 + (4m-1)s^{-1} = 2\sqrt{m}\, t_0^2.$$

By the assumption $t_0^2 = 2\sqrt{m}((4m-1)s+1)^{-1}$, it is equivalent to

$$1 + (4m-1)s^{-1} = 2\sqrt{m} \cdot 2\sqrt{m}\,((4m-1)s+1)^{-1}.$$

This is implied by the assumption $s^2 + 2(2m-1)s + 1 = 0$.

Next consider condition (C3). The values of p_{ij}^{α} are easily computed by the following formula ([4] 4.1.7).

$$p_{j+1,\ell}^{\alpha} = \frac{1}{c_{j+1}}(p_{j,\ell-1}^{\alpha} b_{\ell-1} + p_{j,\ell+1}^{\alpha} c_{\ell+1} - p_{j-1,\ell}^{\alpha} b_{j-1}).$$

Case $\alpha = 1$:

(i,j)	p_{ij}^1
(0,1), (1,0), (3,4), (4,3)	1
(1,2), (2,1), (2,3), (3,2)	$4m-1$

Condition (C3) becomes

$$t_0 t_1^{-1} + t_1 t_0^{-1} + t_3 t_4^{-1} + t_4 t_3^{-1} + (4m-1)(t_1 t_2^{-1} + t_2 t_1^{-1} + t_2 t_3^{-1} + t_3 t_2^{-1}) = 0.$$

This holds by $t_3 = -t_1$ and $t_0 = t_4$.

Case $\alpha = 2$:

(i,j)	p_{ij}^2
(0,2), (2,0), (2,4), (4,2)	1
(1,1), (1,3), (3,1), (3,3)	$2m$
(2,2)	$8m-4$

(C3) becomes

$$t_0 t_2^{-1} + t_2 t_0^{-1} + t_2 t_4^{-1} + t_4 t_2^{-1} + 2m(t_1 t_1^{-1} + t_1 t_3^{-1} + t_3 t_1^{-1} + t_3 t_3^{-1}) + (8m-4) = 0.$$

This is implied by $t_3 = -t_1$, $t_0 = t_4$, $t_2 = s t_0$ and $s^2 + 2(2m-1)s + 1 = 0$.

Case $\alpha = 3$:

(i,j)	p_{ij}^3
(0,3), (3,0), (1,4), (4,1)	1
(1,2), (2,1), (2,3), (3,2)	$4m-1$

$$t_0 t_3^{-1} + t_3 t_0^{-1} + t_1 t_4^{-1} + t_4 t_1^{-1} + (4m-1)(t_1 t_2^{-1} + t_2 t_1^{-1} + t_2 t_3^{-1} + t_3 t_2^{-1}) = 0.$$

This holds by $t_3 = -t_1$ and $t_0 = t_4$.

Case $\alpha = 4$:

(i,j)	p_{ij}^4
(0,4), (4,0)	1
(1,3), (3,1)	$4m$
(2,2)	$8m-2$

$$t_0t_4^{-1} + t_4t_0^{-1} + 4m(t_1t_3^{-1} + t_3t_1^{-1}) + (8m-2)t_2t_2^{-1} = 0.$$

Clearly this holds.

Now we consider condition (C4). Since (C4) is symmetric in u, v, we may assume $\partial(u,w) \le \partial(v,w)$. Fix three vertices u, v, w. Put $\partial(u,v) = \alpha$, $\partial(u,w) = \beta$, $\partial(v,w) = \gamma$ and $P_{ij\ell} = P_{ij\ell}(u,v,w)$. If $\beta = 0$, we have $u = w$, $\alpha = \gamma$, and $P_{ij\ell} = 0$ for $i \ne \ell$. Therefore

$$\sum_{i,j,\ell} P_{ij\ell}t_it_jt_\ell^{-1} = \sum_j \sum_i P_{iji}t_j = \sum_j k_jt_j,$$

and (C4) is equivalent to (C1) in the case $\beta = 0$. So we must verify (C4) in each of the following cases of (α,β,γ):

$$\begin{array}{lllll} (0,1,1) & (0,2,2) & (0,3,3) & (0,4,4) & \\ (1,1,2) & (1,2,3) & (1,3,4) & & \\ (2,1,1) & (2,1,3) & (2,2,2) & (2,2,4) & (2,3,3) \\ (3,1,2) & (3,1,4) & (3,2,3) & & \\ (4,1,3) & (4,2,2) & & & \end{array}$$

In the case $(\alpha,\beta,\gamma) \ne (2,2,2)$, the values of $P_{ij\ell}$ are easily computed. We need the following Lemma for the case $(\alpha,\beta,\gamma) = (2,2,2)$.

Lemma 4 *If $\partial(u,v) = \partial(u,w) = \partial(v,w) = 2$, then w has precisely m neighbours in $\Gamma_1(u) \cap \Gamma_1(v)$.*

Proof. Put $D_j^i = \Gamma_i(u) \cap \Gamma_j(v)$. We have $w \in D_2^2$. Put $e(w, D_1^1) = r$, $e(w, D_3^1) = s$, $e(w, D_1^3) = s'$, $e(w, D_3^3) = r'$. Notice that every vertex $x \in X$ has the unique *opposite* vertex x' such that $\partial(x,x') = 4$, since we have $k_4 = 1$. Sine the opposite vertex x' of $x \in D_1^1 \cap \Gamma_1(w)$ is in D_3^3, we get $r' \le |D_3^3| - r = 2m - r$. Similarly we get $s' \le 2m - s$. On the other hand, we have $r + s = 2m$ since w has precisely $2m$ neighbours in $\Gamma_1(u)$. We have also $s + r' = 2m$ since w has $2m$ neighbours in $\Gamma_3(v)$. These imply $r = r'$. By the same reason, we get $s = s'$. Therefore we must have $r = s = r' = s' = m$.

Case $(\alpha,\beta,\gamma) = (0,1,1)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,0,1), (1,1,0), (3,3,4), (4,4,3)	1
(1,1,2), (2,2,1), (2,2,3), (3,3,2)	$4m-1$

So, condition (C4) becomes

$$t_0^2t_1^{-1} + t_1^2t_0^{-1} + t_3^2t_4^{-1} + t_4^2t_3^{-1} + (4m-1)(t_1^2t_2^{-1} + t_2^2t_1^{-1} + t_2^2t_3^{-1} + t_3^2t_2^{-1}) = Dt_0t_1^{-2},$$

$$2t_1^2t_0^{-1} + (8m-2)t_1^2t_2^{-1} = Dt_0t_1^{-2}.$$

By $t_1^4 = 1$, this is equivalent to (C2).

Case $(\alpha, \beta, \gamma) = (0, 2, 2)$:

(i, j, ℓ)	$P_{ij\ell}$
$(0,0,2)$, $(2,2,0)$, $(2,2,4)$, $(4,4,2)$	1
$(1,1,1)$, $(1,1,3)$, $(3,3,1)$, $(3,3,3)$	$2m$
$(2,2,2)$	$8m-4$

Then condition (C4) becomes

$$2(t_0^2t_2^{-1} + t_2^2t_0^{-1}) + (8m-4)t_2 = Dt_0t_2^{-2},$$

$$s^{-1} + s^2 + (4m-2)s = 2\sqrt{m}\, s^{-2}t_0^{-2}.$$

By the assumption $t_0^2 = 2\sqrt{m}((4m-1)s+1)^{-1}$, this becomes

$$s^{-1} + s^2 + (4m-2)s = (4m-1)s^{-1} + s^{-2}.$$

This is implied by the assumption $s^2 + 2(2m-1)s + 1 = 0$.

Case $(\alpha, \beta, \gamma) = (0, 3, 3)$:

(i, j, ℓ)	$P_{ij\ell}$
$(0,0,3)$, $(1,1,4)$, $(3,3,0)$, $(4,4,1)$	1
$(1,1,2)$, $(2,2,1)$, $(2,2,3)$, $(3,3,2)$	$4m-1$

(C4) becomes

$$t_0^2t_3^{-1} + t_1^2t_4^{-1} + t_3^2t_0^{-1} + t_4^2t_1^{-1} + (4m-1)(t_1^2t_2^{-1} + t_2^2t_1^{-1} + t_2^2t_3^{-1} + t_3^2t_2^{-1}) = Dt_0t_3^{-2}.$$

This is equivalent to Case $(\alpha, \beta, \gamma) = (0, 1, 1)$.

Case $(\alpha, \beta, \gamma) = (0, 4, 4)$:

(i, j, ℓ)	$P_{ij\ell}$
$(0,0,4)$, $(4,4,0)$	1
$(1,1,3)$, $(3,3,1)$	$4m$
$(2,2,2)$	$8m-2$

$$t_0^2t_4^{-1} + t_4^2t_0^{-1} + 4m(t_1^2t_3^{-1} + t_3^2t_1^{-1}) + (8m-2)t_2^2t_2^{-1} = Dt_0t_4^{-2}.$$

Case $(\alpha, \beta, \gamma) = (1, 1, 2)$:

(i, j, ℓ)	$P_{ij\ell}$
$(0,1,1)$, $(1,0,2)$, $(1,2,0)$, $(3,2,4)$, $(3,4,2)$, $(4,3,3)$	1
$(2,1,1)$, $(2,3,3)$	$2m-1$
$(2,1,3)$, $(2,3,1)$	$2m$
$(1,2,2)$, $(3,2,2)$	$4m-2$

$$t_0 + t_4 + t_0t_1t_2^{-1} + t_1t_2t_0^{-1} + t_2t_3t_4^{-1} + t_3t_4t_2^{-1} + 2m(t_1t_2t_3^{-1} + t_2t_3t_1^{-1})$$
$$+(4m-2)(t_1 + t_2 + t_3) = Dt_2^{-1}.$$

Case $(\alpha, \beta, \gamma) = (1, 2, 3)$:

(i, j, ℓ)	$P_{ij\ell}$
$(0,1,2)$, $(1,0,3)$, $(2,1,4)$, $(2,3,0)$, $(3,4,1)$, $(4,3,2)$	1
$(1,2,3)$, $(3,2,1)$	$2m-1$
$(1,2,1)$, $(3,2,3)$	$2m$
$(2,1,2)$, $(2,3,2)$	$4m-2$

$$t_0t_1t_2^{-1} + t_0t_1t_3^{-1} + t_1t_2t_4^{-1} + t_2t_3t_0^{-1} + t_3t_4t_1^{-1} + t_3t_4t_2^{-1}$$
$$+(2m-1)(t_1t_2t_3^{-1} + t_2t_3t_1^{-1}) + (4m-2)(t_1 + t_3) + 4mt_2 = Dt_1t_2^{-1}t_3^{-1}.$$

Case $(\alpha, \beta, \gamma) = (1, 3, 4)$:

(i, j, ℓ)	$P_{ij\ell}$
$(0,1,3)$, $(1,0,4)$, $(3,4,0)$, $(4,3,1)$	1
$(1,2,2)$, $(2,1,3)$, $(2,3,1)$, $(3,2,2)$	$4m-1$

$$t_0t_1t_3^{-1} + t_0t_1t_4^{-1} + t_3t_4t_0^{-1} + t_3t_4t_1^{-1} + (4m-1)(t_1 + t_3 + t_1t_2t_3^{-1} + t_2t_3t_1^{-1})$$
$$= Dt_1t_3^{-1}t_4^{-1}.$$

Case $(\alpha, \beta, \gamma) = (2, 1, 1)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,2,1), (2,0,1), (2,4,3), (4,2,3), (1,1,0), (3,3,4)	1
(1,1,2), (3,3,2)	$2m-1$
(1,3,2), (3,1,2)	$2m$
(2,2,1), (2,2,3)	$4m-2$

$$t_1^2t_0^{-1} + t_3^2t_4^{-1} + 2(t_0t_2t_1^{-1} + t_2t_4t_3^{-1}) + (2m-1)(t_1^2t_2^{-1} + t_3^2t_2^{-1})$$
$$+(4m-2)(t_2^2t_1^{-1} + t_2^2t_3^{-1}) + 4mt_1t_3t_2^{-1} = Dt_2t_1^{-2}.$$

Case $(\alpha,\beta,\gamma) = (2,1,3)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,2,1), (2,0,3), (2,4,1), (4,2,3), (1,3,0), (3,1,4)	1
(1,3,2), (3,1,2)	$2m-1$
(1,1,2), (3,3,2)	$2m$
(2,2,1), (2,2,3)	$4m-2$

$$t_0t_2t_1^{-1} + t_0t_2t_3^{-1} + t_2t_4t_1^{-1} + t_2t_4t_3^{-1} + t_1t_3t_0^{-1} + t_1t_3t_4^{-1} + 2m(t_1^2t_2^{-1} + t_3^2t_2^{-1})$$
$$+(4m-2)(t_1t_3t_2^{-1} + +t_2^2t_1^{-1} + t_2^2t_3^{-1}) = Dt_2t_1^{-1}t_3^{-1}.$$

Case $(\alpha,\beta,\gamma) = (2,2,2)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,2,2), (2,0,2), (2,2,0), (2,2,4), (2,4,2), (4,2,2)	1
(1,1,1), (1,1,3), (1,3,1), (1,3,3)	m
(3,1,1), (3,1,3), (3,3,1), (3,3,3)	m
(2,2,2)	$8m-6$

$$t_2^2t_0^{-1} + t_2^2t_4^{-1} + 2(t_0 + t_4) + m(t_1^2t_3^{-1} + t_3^2t_1^{-1}) + 3m(t_1 + t_3)$$
$$+(8m-6)t_2 = Dt_2^{-1}.$$

Case $(\alpha,\beta,\gamma) = (2,2,4)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,2,2), (2,0,4), (2,4,0), (4,2,2)	1
(1,1,3), (1,3,1), (3,1,3), (3,3,1)	$2m$
(2,2,2)	$8m-4$

$$t_0 + t_4 + t_0t_2t_4^{-1} + t_2t_4t_0^{-1} + 2m(t_1 + t_3 + t_1^2t_3^{-1} + t_3^2t_1^{-1}) + (8m-4)t_2 = Dt_4^{-1}.$$

Case $(\alpha,\beta,\gamma) = (2,3,3)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,2,3), (1,1,4), (2,0,3), (2,4,1), (3,3,0), (4,2,1)	1
(1,1,2), (3,3,2)	$2m-1$
(1,3,2), (3,1,2)	$2m$
(2,2,1), (2,2,3)	$4m-2$

$$t_1^2t_4^{-1}+t_3^2t_0+2(t_0t_2t_3^{-1}+t_2t_4t_1^{-1})+(2m-1)(t_1^2t_2^{-1}+t_3^2t_2^{-1})$$
$$+(4m-2)(t_2^2t_1^{-1}+t_2^2t_3^{-1})+4mt_1t_3t_2^{-1}=Dt_2t_3^{-2}.$$

Case $(\alpha,\beta,\gamma)=(3,1,2)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,3,1), (1,2,0), (3,0,2), (1,4,2), (3,2,4), (4,1,3)	1
(2,1,3), (2,3,1)	$2m-1$
(2,1,1), (2,3,3)	$2m$
(1,2,2), (3,2,2)	$4m-2$

$$t_0t_3t_1^{-1}+t_0t_3t_2^{-1}+t_1t_2t_0^{-1}+t_1t_4t_2^{-1}+t_1t_4t_3^{-1}+t_2t_3t_4^{-1}$$
$$+(2m-1)(t_1t_2t_3^{-1}+t_2t_3t_1^{-1})+(4m-2)(t_1+t_3)+4mt_2=Dt_3t_1^{-1}t_2^{-1}.$$

Case $(\alpha,\beta,\gamma)=(3,1,4)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,3,1), (3,0,4), (1,4,0), (4,1,3)	1
(1,2,2), (2,1,3), (2,3,1), (3,2,2)	$4m-1$

$$t_0t_3t_1^{-1}+t_0t_3t_4^{-1}+t_1t_4t_0^{-1}+t_1t_4t_3^{-1}+(4m-1)(t_1+t_3)$$
$$+(4m-1)(t_1t_2t_3^{-1}+t_2t_3t_1^{-1})=Dt_3t_1^{-1}t_4^{-1}.$$

Case $(\alpha,\beta,\gamma)=(3,2,3)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,3,2), (2,1,4), (3,0,3), (1,4,1), (2,3,0), (4,1,2)	1
(1,2,1), (3,2,3)	$2m-1$
(1,2,3), (3,2,1)	$2m$
(2,1,2), (2,3,2)	$4m-2$

$$t_0+t_4+t_0t_3t_2^{-1}+t_2t_3t_0^{-1}+t_1t_2t_4^{-1}+t_1t_4t_2^{-1}+2m(t_1t_2t_3^{-1}+t_2t_3t_1^{-1})$$
$$+(4m-2)(t_1+t_2+t_3)=Dt_2^{-1}.$$

Case $(\alpha,\beta,\gamma)=(4,1,3)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,4,1), (1,3,0), (3,1,4), (4,0,3)	1
(1,3,2), (2,2,1), (2,2,3), (3,1,2)	$4m-1$

$$t_0t_4t_1^{-1}+t_0t_4t_3^{-1}+t_1t_3t_0^{-1}+t_1t_3t_4^{-1}+(4m-1)(t_2^2t_1^{-1}+t_2^2t_3^{-1})$$
$$+(8m-2)t_1t_3t_2^{-1}=Dt_4t_1^{-1}t_3^{-1}.$$

Case $(\alpha,\beta,\gamma)=(4,2,2)$:

(i,j,ℓ)	$P_{ij\ell}$
(0,4,2), (2,2,0), (2,2,4), (4,0,2)	1
(1,3,1), (1,3,3), (3,1,1), (3,1,3)	$2m$
(2,2,2)	$8m-4$

$$t_2^2t_0^{-1}+t_2^2t_4^{-1}+2t_0t_4t_2^{-1}+4m(t_1+t_3)+(8m-4)t_2=Dt_4t_2^{-2}.$$

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