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PLANAR SINGER GROUPS AND GROUPS OF MULTIPLIERS

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Our terminology in group theory is taken from [G], that of projective planes is taken from [HP], and that of difference sets is taken from [B] or [L].

Semiregular collineation groups.

A permutation group is *semi-regular* if its non trivial elements act fixed-point-freely.

THEOREM 1. *Let G be a collineation group acting semi-regularly on the points of a projective plane, with $|G| > 3$. Let the point orbits be $O = O_1, ..., O_w$. Suppose L is a line orbit of G. Let a_i be the number of points in O_i incident with a line in L for $i = 1, ..., w$. Then the substructure (O, L) is a subplane if and only if $a_1 > 1$ and $a_i = 0$ or 1 for $i = 2, ..., w$.*

Remark. We could use the condition $a_1 > 2$ instead of $|G| > 3$. If the condition that $|G| > 3$ is not imposed, then (O, L) could be a triangle.

Singer groups and multipliers.

A planar Singer group is a collineation group of a projective plane acting regularly on the points of the plane. In 1938, Singer [S] proved that a finite Desarguesian plane is a cyclic plane. On the other hand, in 1964, Karzel [K] proved that an infinite cyclic plane is not Desarguesian.

Projective planes in this article are of finite cardinalities. We use the term Singer group to mean planar Singer group in the rest of this article. It has been conjectured that projective plane admitting a Singer group is Desarguesian. An automorphism of a Singer group is a multiplier if it is also a collineation when we identify the points of the plane with the elements of the group. The set of all multipliers is called the multiplier group of the Singer group. The importance of a multiplier group can be seen from Ott's result [O] that a plane admitting a cyclic Singer group is Desarguesian or its full collineation group is a semi-direct product of a cyclic Singer group with its multiplier group.

The planar order of a Singer group is defined to be the order of the projective plane in which this Singer group acts on. Two Singer groups of the same planar order might not be isomorphic to each other and their multiplier groups might have different orders. For example, the multiplier group of a nonabelian Singer group of planar order 4 has order 3, but the multiplier group of an abelian Singer group of planar order 4 has order 6.

For an abelian Singer group, Hall [L] proves that any divisor of its planar order yields a multiplier. The lack of such existence theorem for multipliers for nonabelian Singer groups presents the difficulty in studying the general case. Let $M(G)$ be the multiplier group of a Singer group G.

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THEOREM 2. *An odd order abelian subgroup of the multiplier group of a Singer group of planar order n has at most $n+1$ elements.*

The example of an abelian Singer group of planar order 4 shows that the condition being odd order subgroup is necessary in the above theorem.

Sylow 2-subgroups of a multiplier group.

The group structure of a Singer group has not been determined except that its solvability is guaranteed by the celebrated Feit-Thompson theorem **[FT]**. The next theorem combines several results concerning a Sylow 2-subgroup of a multiplier group. Among other things, it establishes the solvabiltity of a multiplier group.

THEOREM 3. *Let $M = M(S)$ be a multiplier group of a Singer group S of planar order n. Let T be a Sylow 2-subgroup of M. The following conclusions hold.*

(1) *T is a cyclic direct factor of M and M is solvable.*
(2) *Let $T = <\tau>$. For any divisor d of $|T|$, there is an integer k such that $n = k^d$ and $F = C_S(\tau^{|T|/d})$ is Singer group of planar order k. Also $|M|$ is bounded by $d|M(F)|$.*
(3) *Let $n = m^{2^a}$, where m is a nonsquare integer. Then $|T| \le 2^a$. If in addition M is abelian, then $|M| \le 2^a(m+1)$.*

Some remarks are in order. When S is abelian, $|T|$ attains the maximal possible value 2^a. The question, that $|M(S)|$ is maximum when S is abelian, is still unsettled. The bound for $|M|$ in (3) of the above theorem improves the result in **[Ho4]** and $n=4$ is no longer an exceptional case. In **[Ho5]**, we completely determine the case of an abelian Singer group with $|T|$ being maximal in the sense that the index of T in a Sylow 2-subgroup of $Aut(S)$ is 2. For a cyclic Singer group S, we prove that $|S|$ is a prime when $|M(S)|$ attains the value $n+1$ in **[Ho4]**. If one can prove that the only possible values of n (for a multiplier group of a cyclic Singer group to have order $n+1$) are the known ones, namely, $n=2$ and $n=8$, then a finite projective plane admitting a collineation group acting primitively on the points is Desarguesian by an important result of Kantor **[K]**.

The next theorem shows how an involution in the multiplier group affects the Singer group. For any number r, let $v(r) = r^2+r+1$. If a projective plane has order n, then $v(n)$ is the number of points of the plane. A group subplane of a Singer group is a subplane which is also a subgroup.

THEOREM 4. *(**[HO4]**, **[HO5]**) Suppose the multiplier group of a Singer group S of planar order n has an involution α. Then n is a square and the following holds.*

(1) *$S = AB$, where $A = [S,\alpha] = \{s \in S | s^\alpha = s^{-1}\}$ is an abelian normal Hall subgroup of order $v(\sqrt{n}-1)$, which is an arc; and $B = C_S(\alpha)$ is a Hall subgroup of order $v(\sqrt{n})$, which is a Baer subplane. Furhter, $S = A \times B$ except possibly for $n=16$.*
(2) *Each subgroup of S is α-invariant except possibly $n=16$ and S is non-abelian.*
(3) *A group subplane not in B must have square planar order.*

Abelian Singer groups.

The points and lines of a projective plane admitting a Singer group can be identified with the elements of the group. The following theorem gives a characterization of abelian Singer groups.

THEOREM 5. *A Singer group is abelian if and only if the left multiplication and the right multiplication by any element of the group are collineations.*

PROPOSITION 6. *Let S be an abelian Singer group of planar order n. Then the following conclusions hold.*

(1) *Let $n = m^3$. Then no group subplane of order m exists.*
(2) *Suppose S is cyclic and $n = m^a$ for some positive integer a, then a group subplane of order m exists if and only if $(a, 3) = 1$.*

Sylow 3-subgroup of a multiplier group.

In all know examples, the Sylow 3-subgroup of a multiplier group is always cyclic.

PROPOSITION 7. *Let S be an abelian Singer group of planar order n. Let r be the order of a group subplane whose multiplier group has a cyclic Sylow 3-subgroup. Then $n = r$ or $n = r^2$ or $n > (r+1)^2$. If in addition $r|n$, then $n > (r+1)^2$ in the last statement can be replaced by $n \geq r^3 + r^2 + r + 1$.*

A consequence of Hall's multiplier result mentioned above is that the multiplier group of an abelian Singer group of square planar order has an involution. The following theorem concerns a Sylow 3-subgroup of $M(S)$.

THEOREM 8. *For an cyclic Singer group S of square planar order a Sylow 3-subgroup of $M(S)$ is cyclic.*

Type II divisors of a cyclic Singer group.

For any two coprime integers a and b, let $ord_a(b)$ denote the multiplicative order of b modulo a. Let $v = v(n)$. For a cyclic Singer group S of order v, a multiplier σ is always of the form $s \longrightarrow s^t$ for some positive integer $t < v$. Note that $| < \sigma > | = ord_v(t)$. A prime divisor w of v is a type II divisor of n if $| < \sigma > | = ord_w(t)$ for any multipier σ. Type II divisors have been studied by Ostrom. (See, for example, [**B**].) Note that if a type II divisor exists, then the multiplier group is cyclic.

PROPOSITION 9. *Suppose S is a cyclic Singer group of planar order n. Let p be a prime factor of $v(n)$ of the form $1 + 3^a k$ with $a \geq 1$ and $(3, k) = 1$. (Any prime factor different from 3 of $v(n)$ is in this form.) If $n = m^{3^b}$, then $b \leq (a-1)$. If the Sylow 3-subgroup W of M is cyclic, then $|W|$ divides 3^a.*

We remark that for a cyclic Singer group of square planar order $n = m^2$, a divisor of type II must be a divisor of $v(m-1)$. This Singer group may not have any Type II divisors as examples show.

THEOREM 10. *Let S be a cyclic Singer group of planar order $n = m^2$. Suppose $v(m-1) = p^a q^b$ for some primes p, q, and nonnegative integers a, b. Then $M = M(S)$ is cyclic. If $n \neq 4$, then there is is a divisor of type II of $v = v(n)$.*

Remark. From the fact that a prime divisor, different from 3, of $v(n)$ is congruent to 1 modulo 3, we see that a prime divisor different from 3 of $v(n)$ cannot be of the form $1 + 2^k$ (or $1 + 5 \cdot 2^k$). A similar argument show that the primes p, q that appear in 8.2 cannot be twin primes.

COROLLARY 10.1. *Let S be a cyclic Singer group of square planar order n. If $v = v(n)$ is divisible by at most four different primes, then $M(S)$ is cyclic. If in addition $n \neq 4$, then there is a type II divisor of v.*

Singer group of order pq.

We now generalize the concept of type II divisor. Given a Singer group of planar order n, a prime divisor w of $v(n)$ is a type II divisor of S if S has a subgroup W of order w, which is invariant under the multiplier $M(S)$ such that the kernel of the action of $M(S)$ on W is trivial. Thus $M(S)$ is cyclic if S has a type II divisor. A Singer group of prime order certaily has a II divisor. We will prove the following theorem.

THEOREM 11. *Let S be a Singer group of planar order n and group order pq, where p and q are two primes. Then $p \neq q$. Suppose $p < q$. Then the following holds.*

(1) *Suppose S is abelain. Then S has a type II divisor if and only if $n \neq 4$.*
(2) *Suppose S is nonabelian. Then $M(S)$ has odd order and q is a type II divisor of S.*
(3) *The multiplier group $M(S)$ of S is always cyclic.*

COROLLARY 11.1. *Let S be a non abelian Singer group of order pq as in Theorem 11. If p divides $|M(S)|$, then the plane admits a cyclic Singer group of order pq.*

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