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On Q-structures of quasi-symmetric domains

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0. It is well known that in the theory of automorphic forms, especially in the discussion of cusp singularities, the upper half-plane is more convenient than the unit disc. The tube domain is a natural generalization of the upper half-plane to the higher dimensional case. However, there are certain types of (bounded) symmetric domains, natural generalizations of the unit disc, which can not be realized as a tube domain. In the 1960's Piatetski-Shapiro introduced the notion of Siegel domains of the 1st$\sim$3rd kind and obtained the following results:

1) Every homogeneous bounded domain can be realized as a Siegel domain of the 2nd kind.

2) Every symmetric domain of classical type with a given boundary component can be realized as a Siegel domain of the 3rd kind having that boundary component as the base space.

The result 2) has been generalized to the case of arbitrary symmetric domains by Korányi and Wolf. (For these references, see [S4].)

To be more precise, we give a definition of a Siegel domain of the 2nd kind. Let U and V be a real resp. complex vector space of dimension n resp. m: $U \simeq \mathbf{R}^n$, $V \simeq \mathbf{C}^m$. The space V will be viewed as a real vector space of dimension $2m$

endowed with a complex structure I. Setting $V(\mathbf{C}) = V \otimes_{\mathbf{R}} \mathbf{C}$ and

$$V_{\pm} = \{v \in V(\mathbf{C}) \mid Iv = \pm iv\},$$

one has

$$V(\mathbf{C}) = V_+ \oplus V_-, \quad \bar{V}_+ = V_-, \quad (V, I) \simeq V_+.$$

Let $\mathcal{C}$ be an open convex cone in U (with the vertex at the origin 0), which is "non-degenerate" in the sense that $\bar{\mathcal{C}} \cap (-\bar{\mathcal{C}}) = \{0\}$. Let A be an alternating bilinear map from $V \times V$ into U, which is "$\mathcal{C}$-positive" in the sense that

$$(1) \qquad \begin{cases} A(v, Iv') \text{ is symmetric in } v, v' \in V, \\ A(v, Iv) \in \bar{\mathcal{C}}, \text{ and } A(v, Iv) = 0 \Longleftrightarrow v = 0. \end{cases}$$

For a data $(U, V, \mathcal{C}, A, I)$ satisfying these conditions, the *Siegel domain* (of the 2nd kind) $\mathcal{D}$ defined by it is a domain in $U(\mathbf{C}) \times V_+ \simeq \mathbf{C}^{n+m}$ given by

$$(2) \qquad \mathcal{D} = \{(u, w) \in U(\mathbf{C}) \times V_+ \mid \operatorname{Im} u - \frac{1}{2i} A(w, \bar{w}) \in \mathcal{C}\}.$$

When $V = \{0\}$, one has $\mathcal{D} = U + i\mathcal{C}$, which is called a tube domain, or a Siegel domain of the 1st kind. In what follows, we will consider only the case where $V \neq \{0\}$.

Now, making use of the results on the automorphism group of a homogeneous Siegel domain due to Kaup-Matsushima-Ochiai and Murakami, we have shown that the (bounded) symmetric domains are characterized among homogeneous Siegel domains by three conditions (i), (ii), (iii) (see [S4] , Ch.V). While the first two conditions (i), (ii) are very natural, the last one (iii) looks rather artificial. We call a Siegel domain $\mathcal{D}$ *quasi-symmetric* if it satisfies the conditions (i), (ii).

As is shown in the text, the conditions (i), (ii) can be stated with only $(U, V, \mathcal{C}, A)$ without referring to the complex structure I. When $(U, V, \mathcal{C}, A)$ is fixed, the set $\mathcal{S}$

of all complex structures I satisfying the condition (1) is viewed as a "deformation space" of $\mathcal{D}$. It is known that $\mathcal{S}$ becomes always a symmetric domain of classical type ([S3]).

As mentioned above, any symmetric domain with a fixed boundary component $\mathcal{F}$ has a structure of a fibre space (a Siegel domain of the 3rd kind) over $\mathcal{F}$. In this expression, a fibre $\mathcal{D}$ over a point in $\mathcal{F}$ is a quasi-symmetric domain, whose deformation space $\mathcal{S}$ coincides with $\mathcal{F}$. It is known that all quasi-symmetric domain of "standard" type is obtained in this manner (for definition, see **4**). However, there are quasi-symmetric domains of non-standard type that are not obtained in this form.

All the results on quasi-symmetric domains mentioned above had already been obtained some fifteen years ago (in the 1970's). But the reason why we want to revive it now is the following.

1. To show that one can easily determine all $\mathbf{Q}$-structures of a quasi-symmetric domain $\mathcal{D}$.

2. In case $\mathcal{D}$ is a symmetric tube domain, the cusp singularities (of the first kind) arising from $\mathcal{D}$ have been studied by many mathematicians. In particular, there has been a remarkable progress concerning the relationship between the geometric invariants of the cusp and the zero-value of the zeta functions associated with the cone $\mathcal{C}$, starting with the "Hirzebruch conjecture" (1974) in the Hilbert modular case, proved (analytically) by Atiyah-Donnelly-Singer (1983) and Müller (1984). Recently (1991), an extension of this conjecture to the most general case was obtained (algebraically) by Ogata and Ishida (cf. [SO1], [O1], [I1]). But

as a further generalization, it seems interesting to consider the cusp singularities of the *second* kind arising from quasi-symmetric domains to relate their geometric invariants with the special values at negative integers of the above-mentioned zeta functions.

In this report we will summarize the results on 1 and give a brief account on 2 in the end.

1. Automorphism groups. We start with explaining a Heisenberg group $\widetilde{V}$ defined by the data (U, V, A). By definition $\widetilde{V}$ is the direct product $U \times V$ (as a topological space) endowed with a product given by

$$(u, v) \cdot (u', v') = (u + u' - \frac{1}{2}A(v, v'), v + v').$$

Then, with the natural homomorphisms, one has an exact sequence

$$1 \longrightarrow U \longrightarrow \widetilde{V} \longrightarrow V \longrightarrow 1. \tag{3}$$

When there exists $(\mathcal{C}, I)$ satisfying (1), U coincides with the center of $\widetilde{V}$.

We are also concerned with the following linear groups:

$$\begin{cases} G_1 = \mathrm{Aut}\,(U, \mathcal{C}) = \{g_1 \in GL(U) \mid g_1\mathcal{C} = \mathcal{C}\}, \\ G = \{(g_1, g_2) \in G_1 \times GL(V) \mid g_1 \circ A = A \circ (g_2 \times g_2)\}, \\ G_2 = Sp(V, A) = \{g_2 \in GL(V) \mid (1, g_2) \in G\}. \end{cases} \tag{4}$$

Again with the natural homomorphisms one has an exact sequence

$$1 \longrightarrow G_2 \longrightarrow G \xrightarrow{\rho_1} G_1. \tag{5}$$

Clearly one has $G \subset \mathrm{Aut}\,\widetilde{V}$, so that one can form a semidirect product $G \cdot \widetilde{V}$.

Among these groups, notice that G_2 is a reductive algebraic group with $I \in G_2$ and the map $g_2 \mapsto I^{-1}g_2I$ is a Cartan involution of G_2. Thus

$$G_{2I} = G_2 \cap GL(V, I)$$

is a maximal compact subgroup of G_2 and the symmetric space G_2/G_{2I} associated with G_2 is identified with

$$\mathcal{S} = \mathcal{S}(V, A, \mathcal{C}) = \{\text{complex structures } I' \text{ of } V \text{ satisfying (1)}\}, \tag{6}$$

which may be viewed as the *deformation space* of $\mathcal{D}$. It is known that $\mathcal{S}$ becomes always a symmetric domain of classical type (see [S3]).

Now, considering the complexification of $\tilde{V}$, one has

$$\tilde{V}(\mathbf{C}) = V_+ \cdot U(\mathbf{C}) \cdot V_- \quad \text{(topological direct product)},$$
$$\tilde{V} \cap V_+ \cdot V(\mathbf{C}) = U.$$

Hence one can define a natural action (and automorphy factor) of $\tilde{V}$ on $V_+ \times V(\mathbf{C})$, which preserves $\mathcal{D}$ ([S4], Ch.III, §5). On the other hand, the group G_I defined by

$$G_I = \{(g_1, g_2) \in G \mid g_2 \in GL(V, I)\}$$

acts naturally on $\mathcal{D}$, the action being compatible with that of $\tilde{V}$ mentioned above. Hence the semidirect product $G_I \cdot \tilde{V}$ acts on $\mathcal{D}$. It can be shown easily that the group of affine automorphisms $\mathrm{Aff}(\mathcal{D})$ of $\mathcal{D}$ coincides with $G_I \cdot \tilde{V}$.

2. Quasi-symmetric domains. A Siegel domain $\mathcal{D}$ is called *quasi-symmetric* if the following two conditions (i), (ii) are satisfied.

(i) *$\mathcal{C}$ is self-dual homogeneous cone.*

$\mathcal{C}$ being homogeneous means of course that G_1 acts transitively on $\mathcal{C}$. In general, given a (positive definite) inner product $< >$ on U, one sets

$$\mathcal{C}^* = \left\{ u \in U \mid \; <u, u'> > 0 \text{ for all } u' \in \bar{\mathcal{C}} - \{0\} \right\}.$$

Then $\mathcal{C}^*$ is also a non-degenerate open convex cone in U. $\mathcal{C}$ is called *self-dual* if there exists an inner product on U such that $\mathcal{C} = \mathcal{C}^*$.

Under the condition (i) one has $G_1 = {}^tG_1$, whence follows that G_1 is a reductive "algebraic" group (in a weaker sense), i.e. the identity connected component G_1° of G_1 coincides with that of an algebraic group defined over $\mathbf{R}$.

(ii) *In the sequence* (5) *the projection map* ρ_1 *is "surjective" (in a weaker sense), i.e. one has* $\rho_1(G^\circ) = G_1^\circ$.

Under the assumptions (i) , (ii) the group G is also "algebraic" (in a weaker sense). Since one has $(G/G_2)^\circ \simeq G_1^\circ$ and both G_1 and G_2 are reductive, so is G. It follows that there exists a (unique) connected "algebraic " normal subgroup G_1' of G such that one has

$$G^\circ = G_1' \cdot G_2^\circ, \quad G_1' \cap G_2 = \text{(finite)}. \tag{7}$$

Then the map $\rho_1 \mid G_1' : G_1' \to G_1$ is an isogeny (local isomorphism). Hence, denoting the Lie algebras of G, G_1, G_1', G_2 by the corresponding lower case bold letters, one has

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_1' \oplus \mathfrak{g}_2, \quad \mathfrak{g}_1' \simeq \mathfrak{g}_1, \\ \mathfrak{g}_1' &= \{(x, \beta(x)) \mid x \in \mathfrak{g}_1\}, \end{aligned} \tag{8}$$

where β is a representation of $\mathfrak{g}_1$ on V. Since $I \in G_2^\circ$, one has by (7) $[G_1', I] = 0$, which means that $\beta(\mathfrak{g}_1) \subset \mathfrak{gl}(V, I)$.

Now, fixing an inner product $< >$ on U satisfying (i), one denotes the adjoint of $x \in \operatorname{End} U$ by ${}^t x$. Then the map $x \mapsto -{}^t x$ is a Cartan involution of $\mathbf{g}_1$; let $\mathbf{g}_1 = \mathbf{k}_1 \oplus \mathbf{p}_1$ be the corresponding Cartan decomposition. It is known that, for a suitable $e \in \mathcal{C}$, one has

$$\text{(9)} \qquad \mathbf{k}_1 = \{x \in \mathbf{g}_1 \mid xe = 0\}$$

(see Example below). Hence by a natural correspondence one has $\mathbf{p}_1 \simeq \mathbf{g}_1/\mathbf{k}_1 \simeq U$ (isomorphisms of vector spaces). Let T_u denote the element of $\mathbf{p}_1$ corresponding to $u \in U$. Then T_u is uniquely characterized by the following properties:

$$\text{(10)} \qquad T_u \in \mathbf{g}_1, \quad {}^t T_u = T_u, \quad T_u e = u;$$

especially one has $T_e = 1_U$. (Note that the vector space U endowed with the product defined by $u \circ u' = T_u u'$ is a formally real Jordan algebra. See e.g. [S4], Ch.I, §8.)

3. Admissible triples. We normalize the correspondence between $< >$ and e in a suitable manner so that it becomes bijective. Then one sets

$$\text{(11)} \qquad a(v, v') = < e, A(v, v') > .$$

Clearly a is a non-degenerate alternating bilinear form on V and $a(v, Iv')$ is symmetric and positive definite. (When these conditions are satisfied, a pair (a, I) is called a "hermitian structure" on V.) If one sets

$$h(v, v') = a(v, Iv') + ia(v, v'),$$

then h is a positive definite hermitian form on V. The adjoint with respect to h is denoted by $*$ and the set of self-adjoint (i.e. hermitian) elements in $\operatorname{End}(V, I)$ is denoted by $\operatorname{Her}(V, a, I)$.

Now, for any $u \in U$, there is a uniquely determined element $\varphi(u) \in \operatorname{End} V$ such that one has

$$< u, A(v, v') > = a(v, \varphi(u)v'). \tag{12}$$

As is easily seen, the map φ is linear and one has

$$\varphi : U \longrightarrow \operatorname{Her}(V, a, I), \quad \varphi(e) = 1_V.$$

In view of the fact that $\{(x, \beta(x) \mid x \in \mathfrak{g}_1\}$ is the Lie algebra of an "algebraic" subgroup G_1' of G, one obtains the following relation

$$\begin{cases} \varphi(xu) = \beta(x)\varphi(u) + \varphi(u)\beta(x)^*, \\ \beta({}^t x) = \beta(x)^* \qquad (x \in \mathfrak{g}_1, u \in U). \end{cases} \tag{13}$$

This together with (10) implies

$$\varphi(u) = 2\beta(T_u), \quad \text{especially} \ \ \beta(1_U) = \frac{1}{2} 1_V. \tag{14}$$

The first equation in (13) means that the representation β and the linear map φ are equivariant. A pair (β, φ) satisfying (13) is called *equivariant.* The equation (14) shows β and φ determine each other uniquely. For (β, φ) to be an equivariant pair is a very strong condition, so that one can easily classify all equivariant pairs, which essentially gives also a classification of quasi-symmetric domains $\mathcal{D}$ (cf. [S1], [S2], or [S4], Ch.V, §5).

In general, a triple (e, a, β) formed of $e \in \mathcal{C}$, a non-degenerate alternating bilinear form a on V, and a representation $\beta : \mathfrak{g}_1 \to \mathfrak{gl}(V, I)$ is called *admissible,* if there exists a linear map $\varphi : U \to \operatorname{Her}(V, a, I)$ with $\varphi(e) = 1_V$ satisfying (13). (This condition can be stated without referring to the complex structure I.) As explained above, (for any $e \in \mathcal{C}$), the bilinear map A determines an admissible triple (e,a,β), and conversely any admissible triple (e,a,β) determines A. Thus there exists a one-to-one correspondence between A and certain equivalence classes of (e,a,β). Therefore, as a data to define $\mathcal{D}$ one can take $(U,V,\mathcal{C},e,a,\beta)$ instead of $(U,V,\mathcal{C},A)$. Then the deformation space $\mathcal{S}$ can be written as

$$\begin{aligned}\mathcal{S} &= \mathcal{S}(V,a,\beta)\\ &= \{I \mid (a,I) \text{ is a hermitian structure of } V \text{ and } [\beta(\mathfrak{g}_1), I] = 0\}.\end{aligned} \tag{15}$$

4. Determination of Q-structures. By a **Q**-*structure* of a quasi-symmetric domain $\mathcal{D}$ we mean **Q**-structures of the real vector spaces U and V such that $(\mathcal{C},A)$ is defined over **Q** and $I \in \mathcal{S}$ is "rational" in the sense to be specified below. By what we mentioned in **3**, such a **Q**-structure can be determined in the following steps.

1) To determine **Q**-forms of $(U,\mathcal{C})$, that is to say, **Q**-forms of $(U,\mathfrak{g}_1)$.

2) To construct the representation (V,β) defined over **Q**. (This can be done immediately by the result of [S1].)

3) To determine (e,a) defined over **Q**.

4) To determine "rational points" I in $\mathcal{S}$. (This was given in [S5].)

We note that a "complete reducibility" holds on the data $(U,V,\mathcal{C},e,a,\beta,I)$ defining a quasi-symmetric domain $\mathcal{D}$. First, let

$$U = \bigoplus_{i=1}^{\ell} U^{(i)}, \quad \mathcal{C} = \prod \mathcal{C}^{(i)}, \quad \mathcal{C}^{(i)} \subset U^{(i)} \tag{16a}$$

be the direct decomposition of $(U,\mathcal{C})$ into irreducible factors. Then one has the corresponding decomposition

$$\mathfrak{g}_1 = \bigoplus_{i=1}^{\ell} \mathfrak{g}_1^{(i)}, \quad \mathfrak{g}_1^{(i)} = \{1_{U^{(i)}}\}_{\mathbf{R}} \oplus \mathfrak{g}_1^{(i)s}, \tag{16b}$$

where $\mathfrak{g}_1^{(i)s}$ is simple or $= \{0\}$. One has also the decompositions

$$\text{(16c)} \qquad \begin{cases} V = \bigoplus_{i=1}^{\ell} V^{(i)}, & \beta = \bigoplus \beta^{(i)}, \\ e = \sum e^{(i)}, & a = \sum a^{(i)}, \\ I = \sum I^{(i)}, \end{cases}$$

where $e^{(i)} \in \mathcal{C}^{(i)}$, $(a^{(i)}, I^{(i)})$ is a hermitian structure of $V^{(i)}$, $\beta^{(i)}$ is a representation $\mathfrak{g}_1^{(i)} \to \mathfrak{gl}(V^{(i)}, I^{(i)})$, and $(e^{(i)}, a^{(i)}, \beta^{(i)})$ is admissible. It follows that the classification of quasi-symmetric domains $\mathcal{D}$ over $\mathbf{R}$ reduces to the case where $(U, \mathcal{C})$ is irreducible and the classification over $\mathbf{Q}$ to the case where $(U, \mathcal{C})$ is $\mathbf{Q}$-irreducible. Suppose now that $(U, \mathcal{C})$, i.e. $(U, \mathfrak{g}_1)$, is defined over $\mathbf{Q}$ and $\mathbf{Q}$-irreducible. Then there exists a totally real number field F of degree ℓ such that $(U^{(1)}, \mathfrak{g}_1^{(1)})$ is defined over F and one has

$$U = R_{F/\mathbf{Q}}(U^{(1)}), \quad \mathfrak{g}_1 = R_{F/\mathbf{Q}}(\mathfrak{g}_1^{(1)}),$$

where $R_{F/\mathbf{Q}}$ is the Weil functor. Thus the determination of $\mathbf{Q}$-forms of $(U, \mathcal{C})$ is equivalent to determining F-forms of $(U^{(1)}, \mathcal{C}^{(1)})$. If moreover (V, β) is defined over $\mathbf{Q}$, then $(V^{(1)}, \beta^{(1)})$ is defined over F and one has

$$(V, \beta) = R_{F/\mathbf{Q}}(V^{(1)}, \beta^{(1)}).$$

Example: $(III_{m_1;m_2})$. Let us first explain the construction of an irreducible $\mathbf{R}$-form denoted by this symbol. Let m_1, m_2 be positive integers and let

$$n = \frac{1}{2} m_1(m_1 + 1), \quad m = 2m_1m_2,$$

$$U = \mathrm{Sym}_{m_1}(\mathbf{R}), \quad \mathcal{C} = \mathcal{P}_{m_1}(\mathbf{R}), \quad V = \mathbf{R}^{2m_1m_2}.$$

To be more intrinsic, one prepares two real vector spaces $V_1 \simeq \mathbf{R}^{m_1}$ and $V_2 \simeq \mathbf{R}^{2m_2}$ and sets

$$U = S(V_1 \otimes V_1), \quad V = V_1 \otimes V_2, \tag{17}$$

S denoting the symmetrizer. Thus U is the space of symmetric contravariant tensors of degree 2. The Lie algebra $\mathfrak{g}_1 = \mathfrak{gl}(V_1) \simeq \mathfrak{gl}_{m_1}(\mathbf{R})$ acts on U in a natural manner, so that one can make an identification $\mathfrak{g}_1 = \mathrm{Lie}\,\mathrm{Aut}(U, \mathcal{C})$. On the other hand, let β_1 denote the action of $\mathfrak{g}_1$ on V_1 and define the representation $\beta : \mathfrak{g}_1 \to \mathfrak{gl}(V)$ by setting $\beta = \beta_1 \otimes 1$. Then β satisfies the condition stated in **3**. It is known that in this case all the representation β satisfying this condition is obtained in this form ([S1]).

Now, if one fixes $e \in \mathcal{C} \subset U = S(V_1 \otimes V_1) \subset \mathrm{Hom}(V_1^*, V_1)$, then for its inverse e^{-1} (as a mapping) one has

$$e^{-1} \in \mathcal{C}^* \subset U^* = \mathrm{Sym}(V_1) \subset \mathrm{Hom}(V_1, V_1^*).$$

From the fact that a and I are invariant under $\beta(\mathbf{k}_1) = \beta_1(\mathbf{k}_1) \otimes 1$ one has

$$a = e^{-1} \otimes a_2, \quad I = 1 \otimes I_2,$$

where (a_2, I_2) is a hermitian structure of V_2. Therefore one has

$$G_2 \simeq Sp(V_2, a_2), \quad \mathcal{S} \simeq \mathcal{S}(V_2, a_2) \ \text{(Siegel space)}. \tag{18}$$

From these one can deduce that the alternating bilinear map $A : V \times V \to U$ corresponding to the tripe (e, a, β) is given by

$$A(v_1 \otimes v_2, v_1' \otimes v_2') = S(v_1 \otimes v_1') \cdot a_2(v_2, v_2'). \tag{19}$$

Now, since the above construction is algebraic, one has that, starting from V_1 and V_2 defined over $\mathbf{Q}$ and defining the $\mathbf{Q}$-structures of U, $\mathfrak{g}_1, V$ in a natural manner, one gets the representation β defined over $\mathbf{Q}$. If one takes $e \in \mathcal{C}$ and a_2 to be $\mathbf{Q}$-rational, then one gets the bilinear map A defined over $\mathbf{Q}$. Finally, taking $I_2 \in \mathcal{S}(V_2, a_2)$ to be "rational" in the sense of [S5], one obtains a data $(U, V, \mathcal{C}, A, I)$ over $\mathbf{Q}$. More generally, let F be a totally real number field of degree ℓ. Taking V_1, V_2, β, e, and a_2 defined over F, one obtains a data $(U, V, \mathcal{C}, A, I)$ over F in a similar way. Then, pulling down the field of definition by the Weil functor $R_{F/\mathbf{Q}}$, one obtains a $\mathbf{Q}$-irreducible $\mathbf{Q}$-form of $(III_{m_1;m_2})^\ell$. It should be noted that in this case there exists another type of $\mathbf{Q}$-forms obtained from a totally indefinite division quaternion algebra D_0 over F.

Remark 1. The domain of type $(III_{m_1;0})$ is nothing but a Siegel upper half space (III_{m_1}). The only case for $(III_{m_1;m_2})$ with $m_2 > 0$ to be symmetric is the case $(III_{1;m_2}) \simeq (I_{m_2+1,1})$.

Remark 2. By definition $I \in \mathcal{S}$ is "rational" if the corresponding Cartan involution $y \mapsto I^{-1}yI$ of $\mathfrak{g}_2$ is defined over $\mathbf{Q}$. This condition does not necessarily mean that I is $\mathbf{Q}$-rational. In the case of the above Example, there surely exist "rational points" in $\mathcal{S}$. But there are also cases where $\mathcal{S}$ has no rational points and so $\mathcal{D}$ has no $\mathbf{Q}$-structures! (cf. [S5])

In general, when $\mathcal{C}$ is isomorphic to the product of $\mathcal{P}_{m_1}(D_1)$, $D_1 = \mathbf{R}$, $\mathbf{H}$, $\mathbf{C}$, $\mathcal{C}$ and $\mathcal{D}$ are called of *standard* type. In that case, replacing the totally real number field F in the above construction by a suitable division algebra over $\mathbf{Q}$ with a positive involution (i.e. a totally indefinite or totally definite division quaternion algebra D_0 over a totally real number field F, or a central division algebra D_0 over a CM-extension Z over F with an involution of the 2nd kind), one can do a similar construction. It is known that all $\mathbf{Q}$-irreducible $\mathbf{Q}$-forms of a quasi-symmetric domain of standard type are obtained in these manners ([S6]).

In the non-standard case, where $\mathcal{C}$ is a cone defined by a quadratic form, one can construct $\mathbf{Q}$-forms of $\mathcal{D}$ in a similar way by using the spin representation(s) β_1 and the corresponding Clifford algebra.

5. Cusp singularities of the 2nd kind. For a given $\mathbf{Q}$-structure of the quasi-symmetric domain $\mathcal{D}$, one can consider arithmetic subgroups of $\tilde{V}, G$, etc. First, let $\tilde{L}$ denote an arithmetic subgroup of $\tilde{V}$ and set $M = \tilde{L} \cap U$, $L = \tilde{L} + U/U$. Then M and L are a lattice in U and V, respectively, in the usual sense, and one has an exact sequence

$$1 \longrightarrow M \longrightarrow \tilde{L} \longrightarrow L \longrightarrow 1. \tag{20}$$

The quotient $\tilde{V}/\tilde{L}$ is compact. (Note that $\tilde{L}$ is determined by M, L, and a quadratic character $L \rightarrow \frac{1}{2}M/M$.)

Next, let Γ be an arithmetic subgroup of G_I and set $\Gamma_2 = \Gamma \cap G_2$, $\Gamma_1 = \Gamma G_2/G_2$. Then Γ_2 and Γ_1 are an arithmetic subgroup of G_{2I} and G_1, respectively, and one has an exact sequence

$$1 \longrightarrow \Gamma_2 \longrightarrow \Gamma \longrightarrow \Gamma_1 \longrightarrow 1.$$

Note that Γ_2 is a finite group. Suppose now that Γ is torsion-free and one has $\Gamma \cdot \tilde{L} = \tilde{L}$. Then one has $\Gamma_2 = \{1\}$ and $\Gamma \simeq \Gamma_1$.

For such Γ and $\widetilde{L}$, the semidirect product $\widetilde{\Gamma} = \Gamma \cdot \widetilde{L}$ is an arithmetic subgroup of $G_I \cdot \widetilde{V} = \mathrm{Aff}\,\mathcal{D}$, acting properly discontinuously and freely on $\mathcal{D}$. Hence one can consider the quotient space $\widetilde{\Gamma}\backslash\mathcal{D}$.

Especially, when $\mathbf{Q}$-rank $G_1 = 1$, the double coset space $\Gamma_1\backslash\mathcal{C}\,/\,\mathbf{R}_+^\times$ being compact, one can complete locally the space $\widetilde{\Gamma}\backslash\mathcal{D}$ by adding a point p (a point at infinity). If one sets

$$X = \widetilde{\Gamma}\backslash\mathcal{D} \cup \{p\}, \tag{21}$$

then X has a structure of normal analytic space with an isolated singularity at p. The point p is called a cusp of the *second* kind.

The Mumford's method of toroidal desingularization can also be applied to this case to yield a smooth completion

$$\widetilde{X} = \widetilde{\Gamma}\backslash\mathcal{D} \cup \bigcup_i D_i, \tag{22}$$

where D_i's are toric bundles with an abelian variety V/L as a common base space. The 2-cohomology class of $\widetilde{X}$ determined by D_i is denoted by δ_i.

One defines the contribution of the cusp p to the Todd genus of X by

$$\chi_\infty(\widetilde{\Gamma}\backslash\mathcal{D}, p) = \left(\prod_i \frac{\delta_i}{1 - e^{-\delta_i}}\right)_{n+m} [\widetilde{X}]. \tag{23}$$

This is a rational number determined independently of the choice of the desingularization (22).

On the other hand, let $\mathbf{R}$-rank $G_1 = r$ and define the norm of $u \in \mathcal{C}$ by

$$N(e) = 1, \quad N(g_1 u) = (\det g_1)^{\frac{r}{n}} N(u) \quad (g_1 \in G_1^\circ, u \in \mathcal{C}).$$

Then $N(u)$ extends to a homogeneous polynomial function of degree r on U. One then defines the zeta function of $\mathcal{C}$ relative to $\Gamma_1 \cdot M$ by

$$Z(\mathcal{C}, \Gamma_1 \cdot M; s) = \sum_{x:\Gamma_1 \backslash \mathcal{C} \cap M} N(x)^{-s}. \tag{24}$$

This is a special case of the Sato-Shintani zeta functions of prehomogeneous vector spaces (see [SO1]). The properties of this kind of zeta functions are still not well known except for the functional equations. (In this direction, a remarkable result was obtained recently by Ibukiyama-Saito.)

For the special values of this zeta function, it seems natural to conjecture the following relation

$$Z\left(\mathcal{C}, {}^t\Gamma_1 \cdot M^*; -\frac{m}{r}\right) = c\,\frac{\chi_\infty(\tilde{\Gamma}\backslash \mathcal{D}, p)}{\mathrm{vol}(V/L)}, \tag{25}$$

where M^* is the dual lattice of M and the volume $\mathrm{vol}(V/L)$ of the abelian variety V/L is measured by a (i.e. it is the Pfaffian of a with respect to L). c is a universal constant independent of $\Gamma_1 \cdot M$ (perhaps one has $c = \pm 1$).

In the case $V = 0$, i.e. when $\mathcal{D}$ is a tube domain, the validity of (25) was proved (algebraically) by Ishida [I1]. This may be regarded as a generalization of the "Hirzebruch conjecture" (proved analytically by A.-D.-S. and M.) ([SO1], [O1]). The simplest case with $V \neq 0$ is given by $(III_{1;m_2}) \simeq (I_{m_2+1,1})$, $F = \mathbf{Q}$, in which case one has $m_1 = n = r = 1$ and the above zeta is nothing but the Riemann zeta function. In this case the validity of (25) with $c = 1$ has been well known. The same case with an arbitrary F was treated by [O2] recently. It is desirable to obtain an algebraic proof of (25) in the general case independently of the classification.

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