数理解析研究所講究録
第 844 巻 1993 年 190-196

Analogy between Drinfeld modules and $\mathcal{D}$-modules

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Introduction

The Trinity has long been an attractive motif in the creation of art, especially in the Western world. One of the most marvelous constructions which employ this principle extensively is perhaps "Messe in h-moll" by J. S. Bach [2]. More generally, one has various kinds of trinities, such as 真善美, 天地人, 知情意, 宇宙 · 素数 · ζ ([0]), etc., which are well worth our contemplation. Recently, there has emerged a new trinity consisting of three D's: Drinfeld modules, $\mathcal{D}$-modules, and Dieudonné modules.

The formal analogy of Drinfeld modules to $\mathcal{D}$-modules has already been realized by Drinfeld since the very beginnig of the history of Drinfeld modules [5], and was explained in [6] and [11]. The relation of Drinfeld modules and Dieudonné modules has been formulated, for example, by Drinfeld ([7] etc.) as the relation of Drinfeld modules and shtukas (or F-sheaves), and by Anderson ([1]) as the relation of abelian t-modules and t-motives. Finally, Dieudonné modules are originally the positive characteristic analogue of the Lie algebras of Lie groups — the Galois groups of $\mathcal{D}$-modules. Furthermore, over a more general base scheme than a perfect field, Dieudonné modules themselves must be furnished with connections.

Number theory is the most interesting when differentials appear. How cannot our Trinity be in Gloria? In this note, we concentrate on φ-modules, which arise naturally in the theory of Drinfeld modules and have a similar formalism to $\mathcal{D}$-modules — similarity summarized as follows:

Drinfeld modules	$\mathcal{D}$-modules
$\phi_t = a_0 + a_1\sigma + \cdots + a_n\sigma^n$	$P = a_0 + a_1\partial + \cdots + a_n\partial^n$
$a_i \in K = \mathbf{F}_q(t)$	$a_i \in K = \mathbf{C}(t)$
$\sigma : x \mapsto x^q$	$\partial = \frac{d}{dt}$
$K[\sigma] = \mathrm{End}_{\mathbf{F}_q\text{-linear}}(\mathbf{G}_a/K)$	$K[\partial] = \mathrm{Der}_{\mathbf{C}\text{-linear}}(K)$
$\sigma x = x^q\sigma$ $(x \in K)$	$\partial x = x\partial + \frac{dx}{dt}$ $(x \in K)$
φ-module structures $\varphi : D^{(q)} \to D$	connections $\nabla : D \to D \otimes \Omega^1_{K/\mathbf{C}}$
Galois representations	local systems of horizontal sections
Galois extensions by φ-modules	Picard-Vessiot extensions of differential fields
solvable extensions	Liouville extensions
(Artin-Schreier, Kummer)	($\int f$, $\exp(\int f)$)
tame ramification	regular singularity
wild ramification	irregular singularity
???	·········
! ! !	Sato's theory

In §1, we give a generality on φ-moduels. §2 is devoted to an exposition of results on regular singularity of φ-modules over a local field.

For a field K, let K^{sep} denote a fixed separable closure of K, and G_K the absolute Galois group $\text{Gal}(K^{\text{sep}}/K)$.

1. φ-modules and Galois representations

Let (S,σ) be a couple of a commutative ring S and an endomorphism σ of S; $\lambda \mapsto \lambda^\sigma$.

DEFINITION (1.1). A *φ-module* (D,φ) (or simply, D) over (S,σ) (or simply, S) is an S-module which is endowed with a σ-semi-linear map $\varphi : D \to D$ (i.e. φ is additive and $\varphi(\lambda x) = \lambda^\sigma \varphi(x)$ for all $\lambda \in S$ and $x \in D$). A *morphism* of φ-modules over (S,σ) is an S-module homomorphism which commutes with the φ's.

The σ-semi-linear map φ can be viewed as an S-linear map $\varphi : D^{(\sigma)} \to D$, where $D^{(\sigma)}$ is the base extension of D by $\sigma : S \to S$.

A φ-module D is said to be étale if D is of finite type over S and $\varphi : D^{(\sigma)} \to D$ is an isomorphism.

φ-modules arise, for example, as follows. Let R be an $\mathbb{F}_q$-algebra endowed with an $\mathbb{F}_q$-algebra homomorphism $\alpha : \mathbb{F}_q[[\pi]] \to R$, where π is an indeterminate. Let S be the formal power series ring $R[[\pi]]$, with an endomorphism $\sigma : \sum r_i \pi^i \mapsto \sum r_i^q \pi^i$. Then the π-adic v-sheaf $\mathcal{E}_G$ associated with a π-divisible group G over R ([12], §6) can be regarded as a φ-module over S, where the map φ is induced by the Frobenius morphism of G. This kind of φ-modules $\mathcal{E}_G$ are free over S if R is local, and φ acts on $\det_S \mathcal{E}_G$ by multiplication by (power of $(\pi - \alpha(\pi))) \times$ (unit).

In the following, we *assume all φ-modules are étale* over a *field* $S = K$ containing the finite field $\mathbb{F}_q$ of q elements, and σ is the q-th power Frobenius. Thus D is a finite dimensional K-vector space, whose dimension is called the *rank* of D, and $D^{(\sigma)}$ is written as $D^{(q)}$.

We define the tensor product $(D,\varphi) = (D_1,\varphi_1) \otimes (D_2,\varphi_2)$ of two φ-modules (D_1,φ_1) and (D_2,φ_2) by setting $D := D_1 \otimes_K D_2$ and defining $\varphi : D \to D$ to be the map $\varphi_1 \otimes \varphi_2$. With this tensor product, $\mathcal{F}_K$ becomes a $\otimes$-category.

For any φ-module (D,φ) over K and any field extension L/K, we make $D_L := L \otimes_K D$ a φ-module over L by defining $\varphi : D_L \to D_L$ to be the map

$$\sum a \otimes x \mapsto \sum a^q \otimes \varphi(x).$$

If the extension is Galois, then the Galois group acts on D_L via the first factor.

For a φ-module (D,φ) over K, put

$$V(D) := (K^{\text{sep}} \otimes_K D)^\varphi,$$

the set of fixed points of $D_{K^{\text{sep}}}$ by φ. It is clear that $V(D)$ is an $\mathbb{F}_q$-vector space which is stable under the action of G_K on $D_{K^{\text{sep}}}$. We have thus an $\mathbb{F}_q$-linear representation $V(D)$ of G_K.

Conversely, if V is a finite dimensional $\mathbb{F}_q$-linear representation of G_K, put

$$D(V) := (K^{\text{sep}} \otimes_{\mathbb{F}_q} V)^{G_K},$$

the set of points of $K^{\text{sep}} \otimes_{\mathbb{F}_q} V$ which are fixed by the diagonal action of G_K. Clearly $D(V)$ is a K-vector space, which we make a φ-module by defining $\varphi : D(V) \to D(V)$ to be the map

$$\sum a \otimes x \mapsto \sum a^q \otimes x.$$

The following lemma holds in fact in much greater generality; both the base scheme Spec K and the coefficient $\mathbb{F}_q$ can be generalized (cf. [13], §0; [8], §A.1; [7], Proposition 2.1; [12], Proposition (1.7)).

LEMMA (1.2). *Let $\mathcal{F}_K$ (resp. $\mathcal{G}_K$) be the category of φ-modules over K (resp. the category of finite dimensional $\mathbb{F}_q$-linear representaions of G_K). Then by the construction explained above, we have a $\otimes$-equivalence of $\otimes$-categories $V : \mathcal{F}_K \to \mathcal{G}_K$, with a quasi-inverse $D : \mathcal{G}_K \to \mathcal{F}_K$.*

This correspondence is the most primitive version of the φ-module analogue of the correspondence of $\mathcal{D}$-modules and local systems of horizontal sections.

In what follows, n denotes the rank of the φ-module under consideration.

A vector x of a φ-module D is said to be *cyclic* if the n vectors $x, \varphi(x), \cdots, \varphi^{n-1}(x)$ form a K-base of D. As in Lemme 1.3, Chapitre II of [3], we have

LEMMA (1.3). *If the base field K is infinite, there exists a cyclic vector for (D, φ).*

From now on, we assume the φ-module (D, φ) has a cyclic vector x. With x is associated a polynomial $P_x(X) \in K[X]$ as follows: if

$$a_0 x + a_1 \varphi(x) + \cdots + a_{n-1}\varphi^{n-1}(x) + \varphi^n(x) = 0 \qquad \text{with } a_i \in K,$$

then put

$$P_x(X) := a_0 X + a_1 X^q + \cdots + a_{n-1} X^{q^{n-1}} + X^{q^n}.$$

Thus we have recovered a presentation of "classical" appearance. This polynomial is determined by x uniquely. Multiplying x by a scalar $a \in K^\times$ yields:

$$\begin{aligned} P_{ax}(X) &= a^{q^n} P_x(a^{-1}X) \\ &= a_0 a^{q^n - 1} X + a_1 a^{q^n - q} X^q + \cdots + a_{n-1} a^{q^n - q^{n-1}} X^{q^{n-1}} + X^{q^n}. \end{aligned}$$

We also define

$$\check{V}_x(D) := \{\alpha \in K^{\text{sep}};\ P_x(\alpha) = 0\}.$$

This is clearly an $\mathbb{F}_q$-vector space on which G_K acts.

Recall that we have a canonical inclusion $D \subset D_{K^{\text{sep}}}$ (by definition) and a canonical identification $D_{K^{\text{sep}}} = K^{\text{sep}} \otimes_{\mathbb{F}_q} V(D)$ (by Lemma (1.2)).

LEMMA (1.4). *Suppose that x is expressed by a column vector ${}^t(x_0, \cdots, x_{n-1})$, $x_i \in K^{\mathrm{sep}}$, with respect to an $\mathbb{F}_q$-base $(e_i)_{0 \le i \le n-1}$ of $V(D)$. Then the n elements $x_0, \cdots, x_{n-1}$ form an $\mathbb{F}_q$-base of $\check{V}_x(D)$, so $\check{V}_x(D)$ is an n-dimensional $\mathbb{F}_q$-linear representation of G_K. The two representations of G_K, $V(D)$ and $\check{V}_x(D)$, are contragredient to each other.*

All this explained above reminds us of the theory of Picard-Vessiot on differential Galois extensions. A formulation à la Deligne ([4], §9) of our theory will be as follows: The functor $\omega : \mathcal{F}_K \to \mathrm{Vect}(K)$ which associates with a φ-module over K its underlying K-vector space is a fibre functor. So the category $\mathcal{F}_K$ is Tannakian. For a φ-module X, we set $X^\vee := \mathrm{Hom}_K(X, K)$, which we make a φ-module by definig $\varphi^\vee : X^{\vee(q)} \to X^\vee$ to be the map $f \mapsto f \circ \varphi^{-1}$. Note that $X^{\vee(q)}$ is canonically isomorphic to $\mathrm{Hom}_K(X^{(q)}, K)$ and that $\varphi : X^{(q)} \to X$ is an isomorphism since we are assuming X is étale. Let $\langle X \rangle_\otimes$ denote the full subcategory of $\mathcal{F}_K$ whose objects are subquotients of sums of $X^{\otimes l} \otimes X^{\vee \otimes m}$. This is again a Tannakian category.

Suppose that $\omega_\circ : \langle X \rangle_\otimes \to \mathrm{Vect}(\mathbb{F}_q)$ is a fibre functor (e.g. $\omega_\circ(D) = V(D) = (K^{\mathrm{sep}} \otimes_K D)^\varphi$). Put $\underline{G} := \underline{\mathrm{Aut}}^\otimes(\omega_\circ) \subset \mathrm{GL}_n(\omega_\circ(X))$ (i.e. intuitively, the algebraic subgroup of automorphisms of the $\mathbb{F}_q$-vector space $\omega_\circ(X)$ which commute with φ-module endomorphisms of X), $\underline{G}_K := \underline{G} \otimes_{\mathbb{F}_q} K$, and $\underline{P}$ the $\underline{G}_K$-torsor $\underline{\mathrm{Isom}}_K^\otimes(\omega_\circ \otimes_{\mathbb{F}_q} K,\ \omega|_{\langle X \rangle_\otimes})$. $\underline{P}$ is a K-subscheme of the $\mathrm{GL}_n(\omega_\circ(X))$-torsor $\mathrm{Isom}_K(\omega_\circ(X) \otimes_{\mathbb{F}_q} K,\ X)$. (It is the locus where "the φ-module structure is preserved".) The functor $\mathrm{Rep}(\underline{G}) \to \langle X \rangle_\otimes$; $V \mapsto (V \otimes_{\mathbb{F}_q} K)^{\underline{P}}$ is an equivalence of $\otimes$-categories (cf. Lemma (1.2)). Here the φ-module structure of $(V \otimes_{\mathbb{F}_q} K)^{\underline{P}}$ is induced by the geometric (q-th power) Frobenius $\underline{P} \to \underline{P}^{(q)}$ over K.

Now we take $\omega_\circ(D)$ to be $V(D)$ as our fibre functor. Then $\underline{G}(\mathbb{F}_q)$ is the bicommutant of the image of G_K in $\mathrm{GL}_n(V(X))$, and $(V \otimes_{\mathbb{F}_q} K)^{\underline{P}} = D(V)$. If L is a subfield of K^{sep} such that G_L acts trivially on $V(X)$ (i.e. the φ-module X is trivialized over L) then for any object D of $\langle X \rangle_\otimes$, one has an isomorphism of φ-modules over L:

$$(D \otimes_K L)^\varphi \otimes_{\mathbb{F}_q} L \xrightarrow{\ \simeq\ } D \otimes_K L.$$

This gives an L-valued point of $\underline{P}$:

$$\mathrm{Spec}\ L \to \underline{\mathrm{Isom}}_K^\otimes(\omega_\circ \otimes_{\mathbb{F}_q} K,\ \omega|_{\langle X \rangle_\otimes}).$$

As in [4], §9, we have

PROPOSITION (1.5). *The K-scheme $\underline{P}$ is étale and connected. The subfield L of K^{sep} which corresponds to the kernel of the action of G_K on $V(X)$ is the function field of $\underline{P}$.*

2. Regular singularity of φ-modules

In this section, we present some results on regular singularity of φ-modules, in analogy with the classical theory of ordinary differential equations (see e.g. [10], [3], [9]). Let K be a complete discrete valuation field containing $\mathbb{F}_q$, with valuation v and residue field k. Let p be the characteristic of K.

DEFINITION (2.1). A polynomial $f(X) = a_0X + a_1X^p + \cdots + a_nX^{p^n} \in K[X]$ is said to be *regular* (at v) if $a_0 \neq 0$, $a_n \neq 0$, and

$$v(a_i) - v(a_n) \geq \frac{p^n - p^i}{p^n - 1}(v(a_0) - v(a_n)) \tag{2.1.1}$$

for all $i = 1, \cdots, n-1$.

A regular polynomial $f(X)$ is separable because $f'(X) = a_0 \neq 0$. The condition (2.1.1) is saying that the Newton polygon of $f(X)$ is a straight line. This is equivalent to that all non-zero roots of $f(X)$ have the same valuation $(v(a_0) - v(a_n))/(p^n - 1)$.

Regularity of $f(X)$ is invariant by multiplying $f(X)$ by an element of $K^\times$. If a_n is a unit (i.e. $v(a_n) = 0$), then (2.1.1) is simply

$$v(a_i) \geq \frac{p^n - p^i}{p^n - 1}v(a_0). \tag{2.1.2}$$

Regularity of $f(X)$ is invariant also under the change of variable $X \mapsto aX$ with $a \in K^\times$.

For any separable polynomial $f(X) \in K[X]$, we denote by K_f the minimal splitting field of f contained in K^{sep}.

PROPOSITION (2.2). *Let f be a regular polynomial over K. Then the extension K_f/K is tamely ramified at v.*

We shall interpret the regularity (in the sense of (2.1)) of the polynomial P_x of §1 in terms of lattices, Galois actions, and connections. For any algebraic extension L/K, the valuation v of K extend uniquely to L, which are again denoted v. We denote by $\mathcal{O}_L$ the valuation ring of L.

Let (D, φ) be a φ-module over K. An $\mathcal{O}_K$-lattice D^0 of D is said to be φ-*stable* if $\varphi(D^0) \subset D^0$ and $\mathcal{O}_K \cdot \varphi(D^0) = D^0$ (i.e. if (D^0, φ) is an étale φ-module over $\mathcal{O}_K$). The existence of a φ-stable $\mathcal{O}_K$-lattice means that $V(D)$ is "finite étale" over $\mathcal{O}_K$;

LEMMA (2.3). *Let (D, φ) be a φ-module over K.*

(i) The following conditions are equivalent:

(1) There exists a φ-stable $\mathcal{O}_K$-lattice D^0 in D.

(2) The representation of G_K on $V(D)$ is unramified.

(ii) If G_K acts on $V(D)$ trivially, then the φ-stable $\mathcal{O}_K$-lattice is the $\mathcal{O}_K$-submodule of D spanned by $V(D)$.

(iii) There exists a finite separable extension L/K such that D_L has a φ-stable $\mathcal{O}_L$-lattice.

DEFINITION (2.4). A φ-module (D,φ) over K is said to have *regular singularity* (at v) if it is the direct sum of φ-submodules (D_i,φ_i) each of which has a cyclic vector x_i such that the associated polynomial P_{x_i} is regular in the sense of (2.1).

When the residue field k of K is separably closed, the D_i in the above definition can be taken to be irreducible if once (D,φ) has regular singularity.

THEOREM (2.5). *Assume the residue field k of K is separably closed. Let (D,φ) be a φ-module over K. Then the following conditions are equivalent:*

(1) The φ-module (D,φ) is regular.

(2) There exists a finite tamely ramified extension L/K such that D_L has a φ-stable $\mathcal{O}_L$-lattice D_L^0.

(3) The Galois representation $V(D)$ is tamely ramified.

Now we turn our attention to the connection associated with a φ-module D. Recall (e.g. [8], A.2.2) that there exists on D a unique connection $\nabla : D \to D \otimes_K \Omega^1_{K/k}$ for which $\varphi : D \to D$ is horizontal; $\nabla \circ \varphi = (\varphi \otimes \mathrm{id}) \circ \nabla$. If the Galois representation $V(D)$ is trivial, then $M^\nabla (:= \mathrm{Ker}(\nabla)) = k \otimes_{\mathbb{F}_q} V(D)$. So if $x \in D$ is expressed by a column vector ${}^t(x_0, \cdots, x_{n-1})$, $x_j \in K$, with respect to an $\mathbb{F}_q$-base of $V(D)$, then we have

$$\nabla(x) = {}^t(dx_0, \cdots, dx_{n-1}).$$

The connection may also be regarded as a K-linear map

$$\nabla : \mathrm{Der}_k(K) \to \mathrm{End}_k(D)$$

such that, for all $\partial \in \mathrm{Der}_k(K) \simeq \mathrm{Hom}_K(\Omega^1_{K/k}, K)$, one has $\nabla(\partial) = (1 \otimes \partial) \circ \nabla$.

Let $\|\cdot\|$ be the norm on $D_{K^{\mathrm{sep}}}$ for which the unit ball is the φ-stable $\mathcal{O}_{K^{\mathrm{sep}}}$-lattice $D^0_{K^{\mathrm{sep}}} = \mathcal{O}_{K^{\mathrm{sep}}} \cdot V(D)$.

THEOREM (2.6). *Assume the residue field k of K is separably closed. Let t be a uniformizer of K. Then the following conditions are equivalent:*

(1) The φ-module D has regular singularity;

(2) For any $x \in D$, we have $\|\nabla(t\frac{d}{dt})(x)\| \le \|x\|$.

The condition (2) may be rephrased that the *norm* of ∇, $\|\nabla\| := \sup_{x \in D-0} \|\nabla(t\frac{d}{dt})(x)\|/\|x\|$, equals 1 (We have always $\|\nabla\| \ge 1$). Also, it may be rephrased that there exists in D a $\nabla(t\frac{d}{dt})$-stable $\mathcal{O}_K$-lattice which is a "proper ball" with respect to the norm $\|\cdot\|$.

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