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ELLIPTIC BOUNDARY VALUE PROBLEMS IN THE SPACE OF DISTRIBUTIONS

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Introduction

Elliptic boundary value problems have their own long history. For the general system they were, however, first clearly fomulated microlocally by M. Kashiwara and T. Kawai [K-K]. Their theorem has enjoyed many applications, for example, to solvability of operators of simple characteristics, hypoelliptic operators, and tangential Cauchy-Riemann systems. The theorem does not give, however, much information if we restrict ourselves in the space of distributions. This note aims at giving an analogous theorem of Kashiwara-Kawai type in case function spaces are tempered. See Theorem 3 in Section 1 for the main theorem. By this theorem, we can obtain many application to distribution boundary values of holomorphic functions (e.g. M. Uchida[U]). The result of this note was obtained while the second author was staying in Univ. de Paris VI and Univ. Paris XIII.

1. Main theorem

Let M be a real analytic manifolfd of dimension n with a complex neighborhood X. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$ module on X and assume that $\mathcal{M}$ is elliptic on M, *i.e.*

(1) $$\mathrm{char}(\mathcal{M}) \cap T^*_M X \subset T^*_X X.$$

Let N be a real analytic submanifold of M of codimension $d \geq 1$ in M, and Y be a complexification of N in X. We assume that Y is non-characteristic for $\mathcal{M}$, *i.e.*

(2) $$\mathrm{char}(\mathcal{M}) \cap T^*_Y X \subset T^*_X X.$$

In this situation, we have the canonical morphisms

$$T^*_N M \underset{\rho}{\longleftarrow} T^*_N X \underset{\varpi}{\overset{\simeq}{\longrightarrow}} T^*_N X.$$

Under the above notation we have

THEOREM 1. *The natural morphism*

$$\mathbf{R}\rho_* \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{C}^f_{N|X}) \leftarrow \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, T\text{--}\mu_N(\mathcal{D}b_M)) \otimes or_{N/M}$$

is an isomorphism.

In the above therem $or_{N/M}$ denotes the relativs orientation sheaf of N in M. The sheaf $\mathcal{C}^f_{N|X}$ on $T^*_N X$ is the tempered version of $\mathcal{C}_{N|X}$ and is given, with the tempered microlocalization due to E. Andronikof[An], by

$$\mathcal{C}^f_{N|X} := T\text{--}\mu_N(\mathcal{O}_X) \otimes or_M[n].$$

We remark that the above object in the derived category is concentrated in degree 0. For a point $\mathring{x} \in T^*_N X$, the stalk of $\mathcal{C}^f_{N|X}$ at $\mathring{x}$ is given, with the aid of local cohomology with bounds, by

$$\mathcal{C}^f_{N|X,\mathring{x}} \simeq \varinjlim \underline{\mathrm{H}}^n_{[Z]}(\mathcal{O}_X)_{\pi_X(\mathring{x})}.$$

Here π_X denotes the projection $\pi_X : T^*X \longrightarrow X$ and the inductive limit is taken for all closed subanalytic sets Z in X satisfying the property

$$C_N(Z)_{\pi_X(\mathring{x})} \subset \{v \in T_N X;\ < \mathring{x}, v >< 0\} \cup \{0\}.$$

Refer here to Kashiwara-Schapira[K-S2] for the notion of normal cones $C_N(\cdot)$. The sheaf $T\text{-}\mu_N(\mathcal{D}b_M)$ on T_N^*M is also constructed by E. Andronikof[An]. We just explain that its stalk at $\overset{\circ}{x} \in T_N^*M$ is given by the isomorphism

$$T\text{-}\mu_N(\mathcal{D}b_M)_{\overset{\circ}{x}} \simeq \varinjlim_Z \Gamma_Z(\mathcal{D}b_M)_{\pi_M(\overset{\circ}{x})}.$$

Here the inductive limit is taken for any closed subanalytic set Z in M with the property

$$C_N(Z)_{\pi_M(\overset{\circ}{x})} \subset \{v \in T_NM;\ <\overset{\circ}{x}, v>< 0\} \cup \{0\}$$

$(\pi_M :\ T^*M \longrightarrow M)$.

Next we give another theorem, which is analogous to Theorem 6.3.1 of Kashiwara-Shapira [K-S1] (refer also to Kashiwara-Kawai[K-K] where we find the theorem of [K-S1] in its original form).

THEOREM 2. *Let $\tilde{\mathcal{M}} = \mathcal{E}_X \otimes_{\pi_X^{-1}\mathcal{D}_X} \pi_X^{-1}\mathcal{M}$. Then the natural morphism*

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\tilde{\mathcal{M}}, \mathcal{C}^f_{N|X}) \longleftarrow \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\tilde{\mathcal{M}}, \mathcal{E}_{X\leftarrow Y}) \overset{\mathbf{L}}{\underset{End(\mathcal{E}_{X\leftarrow Y})}{\otimes}} \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathcal{E}_{X\leftarrow Y}, \mathcal{C}^f_{N|X})$$

is an isomorphism outside of $T_N^*X \cap T_Y^*X$. This entails an isomorphism

$$\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}^f_{N|X}) \simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{E}_{X\leftarrow Y}) \otimes^{\mathbf{L}}_{p^{-1}\mathcal{E}_Y} p^{-1}\mathcal{C}^f_N$$

on $T_N^*X \setminus T_Y^*X$ where p is the canonical morphism

$$p :\ T_N^*X \setminus T_Y^*X \longrightarrow T_N^*Y.$$

In the above theorem, the object $\mathcal{C}^f_N$ on T_N^*Y is the sheaf of temperate microfunctions. This is a subsheaf of $\mathcal{C}_N$ and describes microlocal analytic singularities of distributions on N. By the notation of E. Andronikof[An], this sheaf is defined as

$$\mathcal{C}^f_N := T\text{-}\mu_N(\mathcal{O}_Y)[n-d] \otimes or_{N/Y}.$$

The proof of this theorem is essentially the same as in Theorem 6.3.1 of [K-S1] and relies on the division theorem of temperate microfunctions with holomorphic parameters with respect to microdifferential operators. We also remark that only the non-charactericity of Y is utilized in its proof.

By combining the above theorems into one, we get the main theorem of this note. Let q denote the restriction of ρ to $\overset{\circ}{T}{}_N^*X \setminus T_M^*X$; $q :\ \overset{\circ}{T}{}_N^*X \setminus T_M^*X \longrightarrow T_N^*M$ and p the projection $\overset{\circ}{T}{}_N^*X \setminus T_Y^*X \longrightarrow \overset{\circ}{T}{}_N^*Y$. Then we have

THEOREM 3. *We have a canonical isomorphism on* $\overset{\circ}{T^*_N}Y$

$$\mathbf{R}q_*\left(\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{E}_X}(\tilde{\mathcal{M}},\mathcal{E}_{X\leftarrow Y}|_{\overset{\circ}{T^*_N}X})\overset{\mathbf{L}}{\underset{p^{-1}\mathcal{E}_Y}{\otimes}}p^{-1}\mathcal{C}^f_N\right)\simeq \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M},T\text{–}\mu_N(\mathcal{D}b_M))\otimes or_{N/M}.$$

2. Idea of Proof

What is left to us is now to construct the morphism in Theorem 1 and to show it an isomorphism.

First we construct a commutative diagram

$$\text{(A)}\qquad \begin{array}{ccc} \mathbf{R}\rho_!\mathcal{C}^f_{N|X}\otimes or_{N/X} & \longrightarrow & T\text{–}\mu_N(\mathcal{A}_M)\\ \downarrow & & \downarrow\\ \mathbf{R}\rho_*\mathcal{C}^f_{N|X}\otimes or_{N/X} & \longleftarrow & T\text{–}\mu_N(\mathcal{D}b_M) \end{array}$$

where $T\text{–}\mu_N(\mathcal{A}_M)$ is the tempered microlocalization of the sheaf $\mathcal{A}_M$ along N and is constructed by E. Andronikof[A]. This object is the Fourier transform of the tempered specialization $T\text{–}\nu_N(\mathcal{A}_M)$ whose stalk at $\overset{\circ}{v}\in T_NM$ is given by

$$T\text{–}\nu_N(\mathcal{A}_M)_{\overset{\circ}{v}}\simeq \varinjlim_U\ \{u\in\mathcal{A}(U);\ u \text{ is tempered on } M \text{ as a distribution}\}.$$

Here U in the inductive limit ranges through any open subanalytic set in M with the property

$$\overset{\circ}{v}\notin C_N(M\setminus U).$$

To construct (A), it is sufficient to construct its image by the inverse Fourier transformation

$$\text{(A')}\qquad \begin{array}{ccc} \iota^{-1}T\text{–}\nu_N(\mathcal{O}_X)\otimes or_{N/X} & \longrightarrow & T\text{–}\nu_N(\mathcal{A}_M)\\ \downarrow & & \downarrow\\ \iota^!T\text{–}\nu_N(\mathcal{O}_X)\otimes or_{N/X} & \longleftarrow & T\text{–}\nu_N(\mathcal{D}b_M). \end{array}$$

Here ι is the canonical embedding

$$\iota:\ T_NM\longrightarrow T_NX,$$

and $T\text{–}\nu_N(\mathcal{O}_X)$ is the tempered specialization of the sheaf $\mathcal{O}_X$ along N, which is concentrated in degree 0. The stalk of $T\text{–}\nu_N(\mathcal{O}_X)$ at $\overset{\circ}{v}\in T_NX$ is given by

$$T\text{–}\nu_N(\mathcal{O}_X)_{\overset{\circ}{v}}\simeq \varinjlim_U\ \{u\in\mathcal{O}(U);\ u \text{ can be extended to } X \text{ as a distribution}\}$$

where U runs through all open subanalytic sets in X with $\overset{\circ}{v} \notin C_N(M \setminus U)$. The diagram (A') can be constructed easily if we scrutinize the construction by E. Andronikof[An].

Next we apply $\mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \cdot)$ to the diagram (A') and obtain the commutative diagram

$$\begin{array}{ccc} \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \iota^{-1}T\text{-}\nu_N(\mathcal{O}_X)) \otimes or_{N/X} & \xrightarrow{\Phi_1} & \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, T\text{-}\nu_N(\mathcal{A}_M)) \\ {\scriptstyle \Phi_4}\downarrow & & \downarrow{\scriptstyle \Phi_2} \\ \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, \iota^{!}T\text{-}\nu_N(\mathcal{O}_X)) \otimes or_{N/X} & \xleftarrow[\Phi_3]{} & \mathbf{R}\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{M}, T\text{-}\nu_N(\mathcal{D}b_M)). \end{array}$$

It is easy to see from the ellipticity of $\mathcal{M}$ that Φ_4 and Φ_2 are isomorphisms. (To show Φ_4 is an isomorphism, it is easier to consider its image by Fourier transformation). Thus to prove that Φ_3 and thus its image by Fourier transformation are isomorphisms, it suffices to show that Φ_1 is an isomorphism. The problem for Φ_1 can be reduced to the case where $\mathcal{M}$ is a single equation; i.e. $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$. Moreover it is sufficient to show that

$$\underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P, \iota^{-1}T\text{-}\nu_M(\mathcal{O}_X)) \otimes or_{N/X} \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{D}_X/\mathcal{D}_X P, T\text{-}\nu_N(\mathcal{A}_M))$$

is surjective. This problem can be solved by using the construction of the elementary solution of P by means of Radon transformation and microdifferential operators.

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