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THE EXISTENCE AND THE CONTINUATION OF HOLOMORPHIC SOLUTIONS FOR CONVOLUTION EQUATIONS

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1. Problem and notations.

In this paper we denote by $\mathcal{O}_{\mathbb{C}^n}$ the sheaf of holomorphic functions on $\mathbb{C}^n$ and identify $\mathbb{R}^n \times \sqrt{-1}\mathbb{R}^n$ with $\mathbb{C}^n$. Denoting the projection $\mathbb{R}^n \times \sqrt{-1}\mathbb{R}^n \to \sqrt{-1}\mathbb{R}^n$ by τ, we set

$$\mathcal{O}^\tau := R\tau_* \mathcal{O}_{\mathbb{C}^n} \tag{1.1}$$

which is, in fact, concentrated at degree 0, so is considered as a sheaf on $\sqrt{-1}\mathbb{R}^n$. Let $\mu(x)$ be a hyperfunction on $\mathbb{R}^n$ with compact support. Then the convolution operator $P := \mu *$ operates on the sheaf $\mathcal{O}^\tau$, and we consider the complex

$$\mathcal{S} : 0 \to \mathcal{O}^\tau \xrightarrow{\mu *} \mathcal{O}^\tau \to 0. \tag{1.2}$$

Now our problem is stated as follows.

Problem 1.3. *Estimate the micro-support* SS($\mathcal{S}$) *by "the characteristic set" Char(P) of $P = \mu *$.*

We refer to Kashiwara-Schapira [KS], for terminologies above.

For an analytic functional $T \in \mathcal{O}(\mathbb{C}^n)'$, we denote by $\hat{T}(\zeta)$ its Fourier-Borel transform

$$\hat{T}(\zeta) = < T, e^{z\cdot\zeta} >_z,$$

which is an entire function of exponential type satisfying the following estimate (the theorem of Polyà-Ehrenpreis-Martineau). If T is supported by compact set $K \subset \mathbb{C}^n$, for every $\varepsilon > 0$, we can take a constant $C_\varepsilon > 0$ such that

$$|\hat{T}(\zeta)| \le C_\varepsilon \exp(H_K(\zeta) + \varepsilon|\zeta|),$$

where $H_K(\zeta) := \sup_{z\in K} \operatorname{Re} < z, \zeta >$ is the supporting function of K. In particular, if μ is a hyperfunction with compact support, its Fourier-Borel transform $\hat{\mu}(\zeta)$ is of infra-exponential growth on $\sqrt{-1}\mathbb{R}^n$.

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In the sequel, we often write as $\zeta = \xi + \sqrt{-1}\eta$ the decomposition into the real and the imaginary parts and denote for $R > 0$ and $\zeta_0 \in \mathbb{C}^n$ by $B(\zeta_0; R)$ the open ball of center ζ_0 with radius R in $\mathbb{C}^n$.

In this paper, we will suppose the following condition (S) due to T. Kawai [Kw] for the entire function $f(\zeta) = \hat{\mu}(\zeta)$.

$$
\text{(S)} \qquad \begin{cases} \text{For every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that} \\ \text{for any } \sqrt{-1}\eta \in \sqrt{-1}\mathbb{R}^n \text{ with } |\eta| > N, \\ \text{we can find } \zeta \in \mathbb{C}^n, \text{ which satisfies} \\ |\sqrt{-1}\eta - \zeta| < \varepsilon|\eta|, \\ |f(\zeta)| \geq e^{-\varepsilon|\eta|}. \end{cases}
$$

In the next section, we will investigate the meaning of this condition (S).

2. A division Lemma.

First we give the following lemma which ensures the division of analytic functionals under the condition (S).

Lemma 2.1. *Let f, g and h be entire functions satisfying $fg = h$, and M and K be two compact convex sets in $\mathbb{R}^n$ and in $\mathbb{C}^n$ respectively. We suppose that for every $\varepsilon > 0$, f and h satisfy the following estimates (2.1) and (2.2) with constants $A_\varepsilon > 0$ and $B_\varepsilon > 0$,*

$$
\log|f(\zeta)| \leq A_\varepsilon + H_M(\zeta) + \varepsilon|\zeta|, \tag{2.1}
$$

$$
\log|h(\zeta)| \leq B_\varepsilon + H_K(\zeta) + \varepsilon|\zeta|. \tag{2.2}
$$

We also assume that f satisfies the condition (S). Then for any $\varepsilon > 0$, there exists a compact set $L = L_\varepsilon \subset \mathbb{C}^n$ and $C_\varepsilon > 0$ such that

$$
\begin{gathered} \tau(L) \subset \tau(K + B(0;\varepsilon)), \\ \log|g(\zeta)| \leq C_\varepsilon + H_L(\zeta). \end{gathered} \tag{2.3}
$$

To prove this lemma, we use the following lemma of Harnack-Malgrange-Hörmander ([H], Lemma3.1).

Lemma 2.2. *Let $F(\zeta)$, $H(\zeta)$ and $G(\zeta) = H(\zeta)/F(\zeta)$ be three holomorphic functions in the open ball $B(0;R)$. If $|F(\zeta)| < A$ and $|H(\zeta)| < B$ holds on $B(0;R)$, then the estimate*

$$
|G(\zeta)| \leq BA^{\frac{2|\zeta|}{R-|\zeta|}}|F(0)|^{-\frac{R+|\zeta|}{R-|\zeta|}}
$$

holds for all $\zeta \in B(0;R)$.

3. The existence of holomorphic solutions.

For the first cohomology group of the complex $\mathcal{S}$, we will give the surjectivity theorem under the condition (S).

Theorem 3.1. *Let $\mu(x)$ be a hyperfunction with compact support. Assume that $\hat{\mu}(\zeta)$ satisfies the condition (S). Then for any open set $\sqrt{-1}\omega \subset \sqrt{-1}\mathbb{R}^n$, the operator*

$$\mu* : \mathcal{O}^\tau(\sqrt{-1}\omega) \to \mathcal{O}^\tau(\sqrt{-1}\omega)$$

is surjective.

Corollary 3.2. $H^1(\mathcal{S}) = 0$ *and so* $\mathrm{SS}(H^1(\mathcal{S})) = \emptyset$.

4. The characteristic set and the continuation of homogeneous solutions.

For the 0-th cohomology group of the complex $\mathcal{S}$, under the condition (S) for $\mu \in \mathcal{B}_c(\mathbb{R}^n)$ a hyperfunction with compact support, we shall now solve the problem of continuation for $\mathcal{O}^\tau$-solutions of the homogeneous equation $\mu * g = 0$ by the method of Kiselman [Ki] - Sébbar [Sé]. We denote the sheaf of $\mathcal{O}^\tau$-solutions by $\mathcal{N}$, that is, for any open set $\sqrt{-1}\omega \subset \sqrt{-1}\mathbb{R}^n$, we set

$$\mathcal{N}(\sqrt{-1}\omega) := \{g \in \mathcal{O}^\tau(\sqrt{-1}\omega); \mu * g = 0\}$$

For an open set $\sqrt{-1}\Omega \subset \sqrt{-1}\mathbb{R}^n$ with $\sqrt{-1}\omega \subset \sqrt{-1}\Omega$, the problem is formulated as to get the condition so that the restriction map $r : \mathcal{N}(\sqrt{-1}\Omega) \to \mathcal{N}(\sqrt{-1}\omega)$ is surjective.

We note that the subspace $\mathcal{N}(\sqrt{-1}\omega)$ of the (FS) space $\mathcal{O}^\tau(\sqrt{-1}\omega)$, endowed with the induced topology is a closed subspace. We have

Proposition 4.1. *Let $\sqrt{-1}\omega$ and $\sqrt{-1}\Omega$ be two open sets of $\sqrt{-1}\mathbb{R}^n$ with $\sqrt{-1}\omega \subset \sqrt{-1}\Omega$. Assume that μ satisfies (S), then the image $r(\mathcal{N}(\sqrt{-1}\Omega))$ of the restriction map r is dense in $\mathcal{N}(\sqrt{-1}\omega)$.*

We can prove this theorem by using the following lemma.

Lemma 4.2. *(B. Malgrange [M]) $T \in \mathcal{O}(\mathbb{C}^n)'$ belongs to E° if and only if there exists an entire function $s(\zeta)$ satisfying*

$$\hat{T}(\zeta) = s(\zeta)\hat{\mu}(-\zeta).$$

In order to describe the theorem of continuation, we will prepare the notion of characteristics which is a natural generalization of the case of usual differential operators of finite order with constant coefficients. We define the sphere at infinity S_∞^{2n-1} by $(\mathbb{C}^n \setminus \{0\})/\mathbb{R}_+$ and consider the compactification with directions $\mathbb{D}^{2n} = \mathbb{C}^n \sqcup S_\infty^{2n-1}$ of $\mathbb{C}^n$. For $\zeta \in \mathbb{C}^n \setminus \{0\}$, we write $\zeta\infty \in S_\infty^{2n-1}$ the class represented by ζ, i.e. $\{\zeta\infty\} = ($ the closure of $\{t\zeta; t > 0\}$ in $\mathbb{D}^{2n}) \cap S_\infty^{2n-1}$. We denote by $\sqrt{-1}S_\infty^{n-1}$ the pure imaginary sphere at infinity $\{(\xi + \sqrt{-1}\eta)\infty \in S_\infty^{2n-1}; \xi = 0\}$, which is a closed subset of S_∞^{2n-1}.

For a hyperfunction μ with compact support, using the terms of the modulus of the Fourier-Borel transform $f = \hat{\mu}$ of μ, we define the characteristics $\mathrm{Char}_\infty(\mu*)$ as follows. For $\varepsilon > 0$ we set

$$V_f(\varepsilon) := \{\zeta \in \mathbb{C}^n; e^{\varepsilon|\zeta|}|f(\zeta)| < 1\},$$
$$W_f(\varepsilon) := \sqrt{-1}S_\infty^{n-1} \cap (\text{ the closure of } V_f(\varepsilon) \text{ in } \mathbb{D}^{2n}).$$

Now we define the characteristic set of $\mu*$.

Definition 4.3. With the above notation, we define the *characteristics* of $\mu*$ (at infinity)

$$\mathrm{Char}_\infty(\mu*) := \text{ the closure of } \bigcup_{\varepsilon>0} W_f(\varepsilon),$$

a closed set in $\sqrt{-1}S_\infty^{n-1}$.

We remark that if $P = \mu*$ defines a finite order differential operator with constant coefficients, the characteristic set of $\mu*$ coincides with the usual characteristics of P defined as zeros of its principal symbol. Even if P is a differential operator of infinite order, it can be shown by a similar argument to the proof of Lemma 2.1, that $\mathrm{Char}(\mu*)$ coincides with accumulating directions of zeros of the total symbol. (See for example T. Kawai [Kw]).

We note that the direction $\sqrt{-1}\rho\infty \in \sqrt{-1}S_\infty^{n-1}$ does not belong to $\mathrm{Char}_\infty(\mu*)$ if and only if for any $\varepsilon > 0$, there exist a conical neighborhood $\Gamma \subset \mathbb{C}^n$ of $\sqrt{-1}\rho$ and a positive N such that $f(\zeta)$ satisfies the following estimate

$$|f(\zeta)| > e^{-\varepsilon|\zeta|} \tag{4.1}$$

on $\Gamma \cap \{|\zeta| > N\}$.

For this $\mathrm{Char}_\infty(\mu*)$ and an open convex set $\sqrt{-1}\omega \subset \sqrt{-1}\mathbb{R}^n$, now we put

$$\sqrt{-1}\Omega := \text{ the interior of}$$
$$\left(\bigcap_{\sqrt{-1}\eta\infty\in\mathrm{Char}_\infty(\mu*)^a} \{\sqrt{-1}y \in \sqrt{-1}\mathbb{R}^n; -y\cdot\eta \le H_{\sqrt{-1}\omega}(\sqrt{-1}\eta)\}\right),$$

here ^{a} means the antipodal. Then $\sqrt{-1}\Omega$ becomes an open convex subset in $\sqrt{-1}\mathbb{R}^n$ containing $\sqrt{-1}\omega$. Moreover for any compact set L in $\mathbb{R}^n \times \sqrt{-1}\Omega$, we can take compact set K in $\mathbb{R}^n \times \sqrt{-1}\omega$ such that

$$H_L(\sqrt{-1}\eta) \le H_K(\sqrt{-1}\eta) \quad \text{for any } \sqrt{-1}\eta\infty \in \mathrm{Char}_\infty(\mu*)^a. \tag{4.2}$$

Under the above situation we are ready to state the main theorem of this section.

Theorem 4.4. *Let μ be a hyperfunction with compact support, and a pair of open sets $\sqrt{-1}\omega \subset \sqrt{-1}\Omega$ as above. Assume that $\hat{\mu}$ satisfies the condition (S). Then the restriction $r : \mathcal{N}(\sqrt{-1}\Omega) \to \mathcal{N}(\sqrt{-1}\omega)$ is surjective.*

Proof. Remarking that the restriction $r : \mathcal{N}(\sqrt{-1}\Omega) \to \mathcal{N}(\sqrt{-1}\omega)$ has a dense image, the transpose ${}^t r : \mathcal{N}(\sqrt{-1}\Omega)' \leftarrow \mathcal{N}(\sqrt{-1}\omega)'$ is injective and we must show only its surjectivity. By Hahn-Banach's theorem, any functional $T \in \mathcal{N}(\sqrt{-1}\Omega)'$ has an extension $T_1 \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\Omega)'$, and we prove that we can choose $R \in \mathcal{O}(\mathbb{R}^n \times \sqrt{-1}\omega)'$ so that T_1 and R coincide on $\mathcal{N}(\sqrt{-1}\Omega)$. To do this, we use the following division lemma.

Lemma 4.5. *Let L and K be a pair of compact subsets of $\mathbb{C}^n$ satisfying the estimate (4.2) on* $\mathrm{Char}_\infty(\mu*)$ *and $p(\zeta)$ be an entire function satisfying the estimate*

$$\log|p(\zeta)| \le H_L(\zeta).$$

Then for any $\varepsilon > 0$, there exist constants $C_\varepsilon > 0$, $k_\varepsilon > 0$ and entire functions $q(\zeta)$ and $r(\zeta)$ which satisfy

$$\begin{aligned} &p(\zeta) = \hat{\mu}(-\zeta)q(\zeta) + r(\zeta),\\ &\log|q(\zeta)| \le H_{L\cup K}(\zeta) + k_\varepsilon|\operatorname{Re}\zeta| + \varepsilon|\zeta| + C_\varepsilon,\\ &\log|r(\zeta)| \le H_K(\zeta) + k_\varepsilon|\operatorname{Re}\zeta| + \varepsilon|\zeta| + C_\varepsilon. \end{aligned}$$

Sketch of the proof. In this proof, we write $f(\zeta) := \hat{\mu}(-\zeta)$. For any constant $\delta > 0$, consider the following set

$$\Lambda_1 := \{\zeta\infty \in S^{2n-1}_\infty;\ H_L(\zeta) < H_K(\zeta) + \delta|\zeta|\},$$

which is a neighborhood of $\mathrm{Char}(\mu*)^a$ in S^{2n-1}_∞. Since $\sqrt{-1}S^{n-1}_\infty \setminus \Lambda_1$ is compact and does not meet $\mathrm{Char}(\mu*)^a$, we can take a neighborhood Λ_2 of $\sqrt{-1}S^{n-1}_\infty \setminus \Lambda_1$ in S^{2n-1}_∞ and a constant $N > 1$ such that

$$\zeta\infty \in \Lambda_2, |\zeta| > N \text{ implies } |f(\zeta)| > e^{-\delta|\zeta|}.$$

Moreover if we remark that $\Lambda_1 \cup \Lambda_2$ is a neighborhood of $\sqrt{-1}S^{n-1}_\infty$ in S^{2n-1}_∞, we can take a positive δ' such that

$$\zeta \in \mathbb{C}^n, |\operatorname{Re}\zeta| < \delta'|\operatorname{Im}\zeta| \text{ implies } \zeta\infty \in \Lambda_1 \cup \Lambda_2.$$

According to the above, we get the covering $\bigcup_j \gamma_j$ of $\mathbb{C}^n$ by

$$\begin{aligned} &\gamma_1 = \{\zeta \in \mathbb{C}^n;\ \zeta\infty \in \Lambda_1, |\zeta| > N\},\\ &\gamma_2 = \{\zeta \in \mathbb{C}^n;\ \zeta\infty \in \Lambda_2, |\zeta| > N\},\\ &\gamma_3 = \{\zeta \in \mathbb{C}^n;\ |\operatorname{Re}\zeta| \ge \delta'|\operatorname{Im}\zeta|, |\zeta| > N\},\\ &\gamma_4 = \{\zeta \in \mathbb{C}^n;\ |\zeta| \le N\}. \end{aligned}$$

Now we construct $q(\zeta)$ and $r(\zeta)$ as the following form

$$\begin{cases} q(\zeta) := \dfrac{p(\zeta)}{f(\zeta)}(1 - \varphi(e^{\lambda(|\zeta|)}f(\zeta))) + v(\zeta),\\ r(\zeta) := p(\zeta)\varphi(e^{\lambda(|\zeta|)}f(\zeta)) - f(\zeta)v(\zeta), \end{cases}$$

by choosing a suitable C^∞ function $v(\zeta)$ on $\mathbb{C}^n$, with fixed real-valued functions $\varphi(\tau) \in C^\infty(\mathbb{C})$ and $\lambda(t) \in C^\infty([0,\infty[)$ so that $0 \le \varphi(\tau) \le 1$, $0 \le \lambda(t) \le \delta t$, $0 \le \lambda'(t) \le 1$ and that

$$\varphi(\tau) = \begin{cases} 1 & (|\tau| \le \frac{1}{2}),\\ 0 & (|\tau| \ge 1), \end{cases}$$

$$\lambda(t) = \begin{cases} 0 & (t \le \frac{1}{2}),\\ \delta t & (t \ge 1). \end{cases}$$

The condition for q and r to be holomorphic, is given as

$$\bar{\partial}v = \frac{p(\zeta)}{f(\zeta)}e^{\lambda(|\zeta|)}\{(\frac{\partial\varphi}{\partial\tau}f(\zeta) + \frac{\partial\varphi}{\partial\bar{\tau}}\bar{f}(\zeta))\frac{\lambda'(|\zeta|)}{2|\zeta|}\zeta \cdot d\bar{\zeta} + \frac{\partial\varphi}{\partial\bar{\tau}}\bar{\partial}\bar{f}\}. \tag{4.3}$$

We denote the right hand side by w which satisfies $\bar{\partial}w = 0$, and solve the $\bar{\partial}$-problem $\bar{\partial}v = w$ with growth condition.

5. Main theorem.

In this section, we define *the characteristic set* $\mathrm{Char}(\mu*)$ of $\mu*$ as the closed conic set in $T^*(\sqrt{-1}\mathbb{R}^n)$ defined by $\mathrm{Char}_\infty(\mu*)$ i.e.

Definition 5.1.

$$\mathrm{Char}(\mu*) := \{(\sqrt{-1}y, \sqrt{-1}\eta) \in T^*(\sqrt{-1}\mathbb{R}^n); \sqrt{-1}\eta\infty \in \mathrm{Char}_\infty(\mu*) \text{ or } \sqrt{-1}\eta = 0\}.$$

By the definition of micro-support (see [KS], Definition 5.1.2), the Theorem 4.4 implies that

$$\mathrm{SS}(H^0(\mathcal{S})) \subset \mathrm{Char}(\mu*). \tag{5.1}$$

So recalling the Remark 5.1.4 in [KS], by this estimate (5.1) and by the Corollary 3.2, we resume our results of preceding sections as following final theorem

Theorem 5.2. *Under the condition (S), we have*

$$\mathrm{SS}(\mathcal{S}) \subset \mathrm{Char}(\mu*).$$

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