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Interaction of two vortex filament with different strength

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1. Introduction.

In this paper, we examine the three-dimensional interactions of two vortex filament with different strength. The goal of our investigations is to find a possible mechanism to explain finite time singularity formation in inviscid ideal flow. This problem may be stated as follows. Are there any smooth three-dimensional velocity fields of inviscid fluid with finite energy such that there is a critical time $T < \infty$ for which these solutions become singular as $t \rightarrow T$?

By the work of Beale, Kato and Majda [1], we know that for maximum norm of vorticity

$$|\omega|_{L^\infty} = \max_{x \in R^3} |\omega(x)|.$$

the interval $[0,T)$ with $T < \infty$ is a maximal interval of smooth existence if and only if the vorticity accumulates so rapidly that

$$\int_0^t |\omega|_{L^\infty}(s)ds \rightarrow +\infty \quad \text{as} \quad t \rightarrow T.$$

This mathematical result assure us to treat the problem in the framework of vorticity dynamics.

Siggia [2] proposed a model to explain finite time singularity formation using a notion of collapsing vortex pair. The scenario for a finite-time singularity is like that : an isolated vortex filament can fold upon itself such that oppositely signed pieces of the filament form a pair, advance under their mutual induced velocity fields and cause stretching of filament. Because of the constraint of volume conservation the core size of the filament will shrink as the filament stretches. There are several difficulties with this scenario. It is observed that the interfilament spacing decreases more rapidly than core size indicating overlap of core. Pumir and Siggia [3] have performed careful numerical simulations of collapsing solutions to 3-D Euler equations and have shown that maximum vorticity grows only exponentially.

2. Biot-Savart simulations.

Here, we examine the interaction of two vortex filaments with strength Γ_1 and Γ_2, respectively. Suppose that a curved filament Γ_1 (hereafter we call filament 1) approaches to a straight filament Γ_2 (filament 2) at some initial instant $t=0$(see Fig. 1). Due to the three dimensional interaction of two filaments, two filaments may approach and deform for $t>0$. If $\Gamma_1 = \Gamma_2$, the deformation may be schematically like Fig. 1 (*a*). Near the point of closest approaches O an antiparallel pair of vortex filaments is formed. Mutual distance of the pair decreases as time proceeds. But if Γ_1 is sufficiently small compared with Γ_2, then strong filament Γ_2 may stay almost straight while the filament Γ_1 may wind round to Γ_2 as in Fig. 1 (*b*).

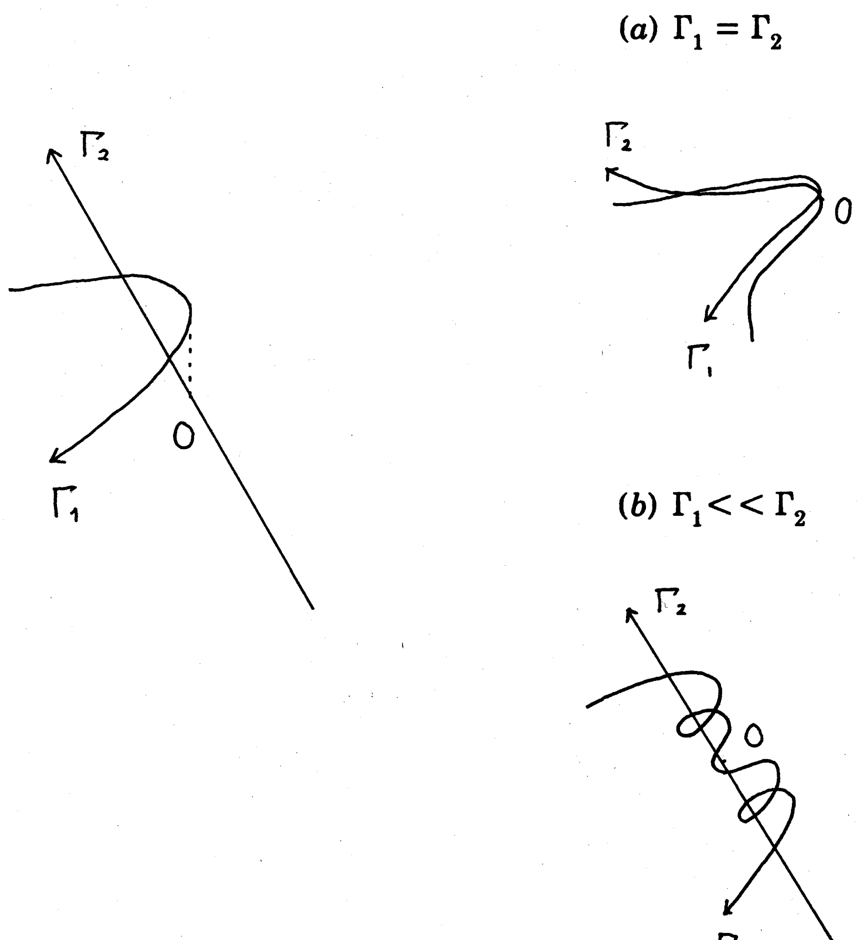


Fig. 1

Case (*a*) corresponds to antiparallel pair considered by Siggia.We consider the situation (*b*) here. We perform numerical simulations using the Biot-Savart's law. If the vorticity is distributed into two isolated filaments then Biot-Savart law may be given as

$$\mathbf{v}(\mathbf{r}) = -\frac{\Gamma_1}{4\pi}\int \frac{(\mathbf{r}-\mathbf{r}')\times\omega(\mathbf{r}')}{[|\mathbf{r}-\mathbf{r}'|^2+\sigma(\mathbf{r}')^2]^{3/2}}\,d\mathbf{r}' - \frac{\Gamma_2}{4\pi}\int \frac{(\mathbf{r}-\mathbf{r}')\times\omega(\mathbf{r}')}{[|\mathbf{r}-\mathbf{r}'|^2+\sigma(\mathbf{r}')^2]^{3/2}}\,d\mathbf{r}'$$

where σ is core diameter of the filaments. In inviscid flow, the tube volume of a filament is conserved which is a consequence of the Helmholtz laws. In numerical simulations, there are many possible core laws to express the tube volume conservation. In this paper we use a core law like

$$\sigma^2 |\delta\mathbf{r}| = const.$$

where $\delta\mathbf{r}$ is a small segment of the tube. In numerical simulations, each filament is represented by an ensemble of N_i segments with length $\delta\mathbf{r}_i$ and diameter σ_i. Then the Biot-Savart law is modeled by

$$\mathbf{v}(\mathbf{r}) = -\frac{1}{4\pi}\sum_{i=1}^{2}\Gamma_i\sum_{k=1}^{N_i}\frac{\mathbf{a}_k^{(i)}\times\delta\mathbf{r}_k^{(i)}}{\{|\mathbf{a}_k^{(i)}|^2+(\sigma_k^{(i)})^2\}^{3/2}}$$

$$\mathbf{a}_k^{(i)} = \frac{1}{2}(\mathbf{r}_{k+1}^{(i)}-\mathbf{r}_k^{(i)})-\mathbf{r},\quad \delta\mathbf{r}_k^{(i)} = \mathbf{r}_{k+1}^{(i)}-\mathbf{r}_k^{(i)}\quad (i=1,2)$$

As time proceeds, each filament may be deformed and be stretched and two filaments may get entangled. In this situation, we need appropriate subdivisions of segments to assure the accuracy of the calculations.

3. Numerical results.

We consider two cases:

Case1 : $\Gamma_1 = -1.0, \ \Gamma_2 = 1.0$

Case2 : $\Gamma_1 = -0.05, \ \Gamma_2 = 1.0$

Initial situation of two filaments is shown in Fig. 2. The mutual distance d_0 at $t=0$ is, thus $d_0=0.3$. Initial core diameter σ_0 is $\sigma_0=d_0/30$ for all cases.

In Fig 2 (a) xy-projections of two filaments at $t=0$ and $t=0.47745$ are shown and in Fig.2 (b) yz-projections are shown for case 1. At t=0.47745 the minimum distance d of two filament is $d=0.002762$ where as the core diameter σ at the point of closest approach O is $\sigma/\sigma_0=0.2079$ for filament 1 and $\sigma/\sigma_0=0.2284$ for filament 2.

In Fig. 3(a) and (b), xy- and yz-projections of two filaments at t=3.8842 are shown. We see that filament 2 stays almost straight while filament 1 winds round to filament 1. At $t=3.8842$, minimum distance $d=0.1067$ whereas core diameter at O is $\sigma/\sigma_0=0.6630$ for filament 1 and $\sigma/\sigma_0=0.5535$ for filament 2.

Fig.4 shows the stretching of filament 2 along the filament at different time. The ordinate is the ratio of the core diameter $\sigma(s)$ to the initial core diameter. The abscissa is the distance s from O along the filament. Each curve corresponds to time $t=0$, 2.4228, 2.9560, and 3.8842, respectively. We see that at the center O of the strong filament 2 which stays almost straight efficient stretching occurs. This efficient stretching is due to the induced velocity of filament 1 which winds round to filament 2.

In Fig. 5, we compare the development in time of the ratio of core diameter to minimum distance normalized to 1 at the initial instant. The curve 1 corresponds to Case 1, curve 2 to Case 2 and curve 3 corresponds to Case 3 ($\Gamma_1=-0.001$, *i.e.* very weak filament and $\Gamma_2=1.0$). In Case 1, the minimum distance d decreases very rapidly compared to the core diameter σ resulting to overlap of core. In Case 2, though this ratio becomes larger than 1 but much slowly compared to Case 1. In Case 3, this ratio stays almost 1 which means that core size decreases almost same rate as minimum distance of two filament thus overlap of core does not occur and thin filament approximation remains valid.

4. Simple model

From the numerical results above, we found that at the center O of strong straight filament an efficient stretching is take place due to the induced velocity of weak filament which wind round to straight filament. A flow field near the center O of the strong filament may be approximated by a flow due to vortex rings of opposite direction (Fig. 6)

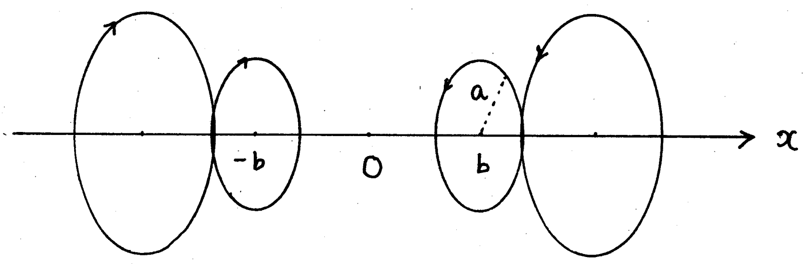


Fig. 6

We take x axis along the straight filament. Then the induced velocity on the x axis due to two vortex rings near the center of straight filament is given by

$$u_x = \frac{\Gamma a^2 b}{(a^2 + b^2)^{5/2}} x + O(x^2)$$

here, a is the radius of vortex ring and b is the distance of the ring from O. If two vortex rings stay fixed relative to O, then motion of a fluid particle on the x axis near O may be governed by

$$\frac{dx}{dt} = \frac{\Gamma a^2 b}{(a^2 + b^2)^{5/2}} x$$

Suppose a small portion $(-x_0, x_0)$ of the filament $(x_0 \ll b)$. The volume of this vortex segment is $V = \pi\sigma^2 x_0$. Fluid particles which initially contained within this portion flow out of the portion through the sections at $x = x_0$ and $x = -x_0$. Then,

$$\frac{dV}{dt} = -\frac{\Gamma a^2 b}{(a^2 + b^2)^{5/2}} V$$

Then decrease of V and thus decrease of core diameter at O is at most exponential.

In our situation, weak filament continues to wind round to straight filament. In the context of the model considered here, this means that smaller and smaller size of vortex rings continue to generate around O. Here, we assume a similarity of generation of rings in a form

$$\frac{a(t)}{b(t)} = const.$$

In this case, the motion of a fluid particle on the x axis (*i.e.* on the straight filament) is governed by the equation of the form

$$\frac{dx}{dt} = \frac{x}{c^2(t)}$$

here $c(t)$ is an effective distance of innermost vortex ring from the center O. $c(t)$ is a decreasing function of t. If $c(t)$ decrease as

$$c(t) = c_0 (T - t)^q$$

where q is some positive constant. Let x_0 be initial position of a fluid particle. In the limit $x_0 \to 0$, we have

$$\text{If } q > \frac{1}{2}, \quad \frac{x(t)}{x_0} \sim \exp\left[\frac{const.}{(T-t)^{2q-1}} \right]$$

$$\text{If } q = \frac{1}{2}, \quad \frac{x(t)}{x_0} \sim (T-t)^{-1/c_0}$$

$$\text{If } q < \frac{1}{2}, \quad \frac{x(t)}{x_0} \to const. \quad \text{as} \quad t \to T$$

Thus, if $q \geqq 1/2$ infinite stretching occurs at the center O of the filament as $t \to T$.

References

[1] T.Beale, T.Kato and A.Majda, Commun. Math. Phys. **94**, 61 (1984)
[2] E.D.Siggia, Phys. Fluids, **28**, 794 (1985)
[3] A.Pumir and E.D.Siggia, Phys. Fluids, **A2**, 220 (1990)

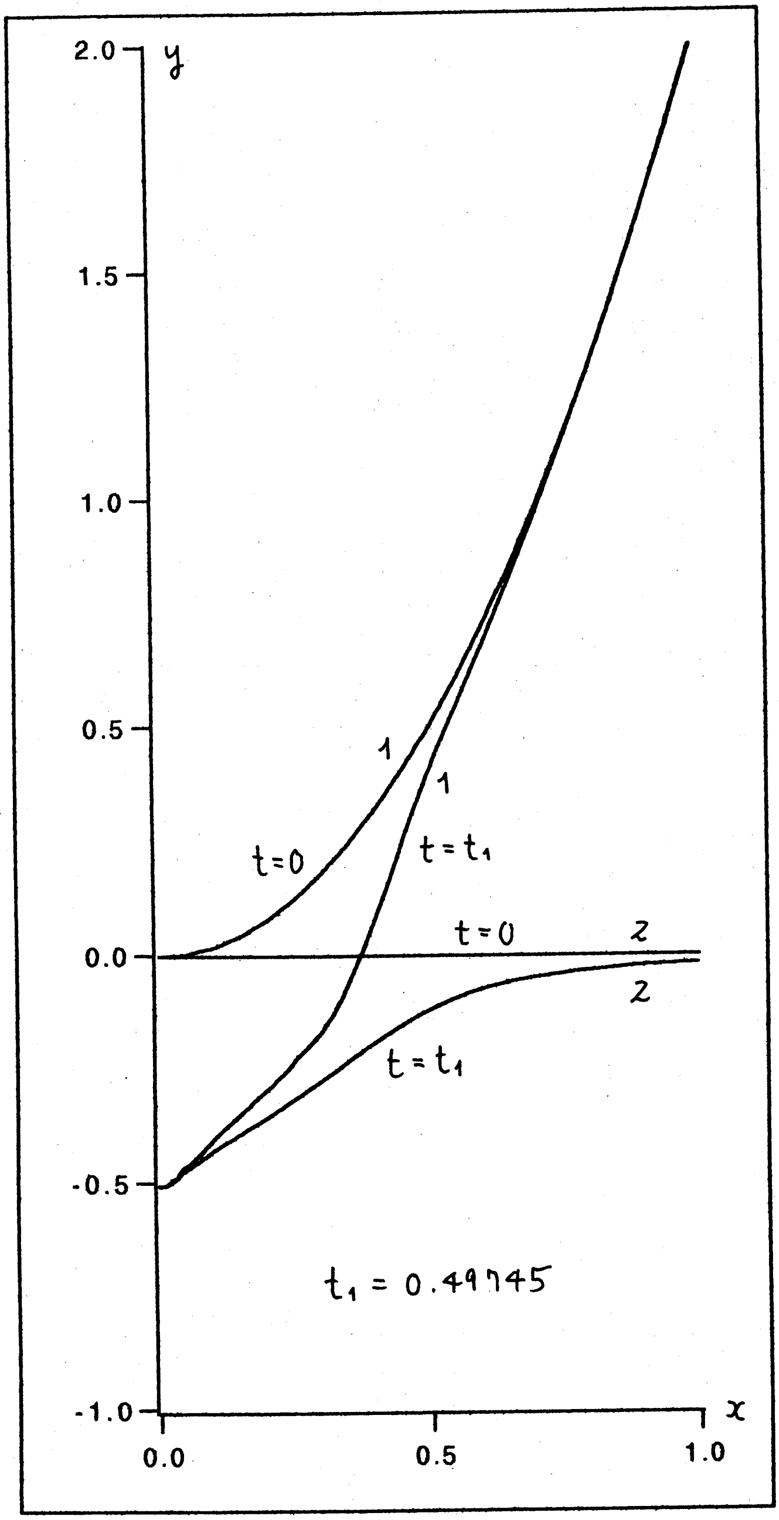


Fig 2 (a) Case 1.

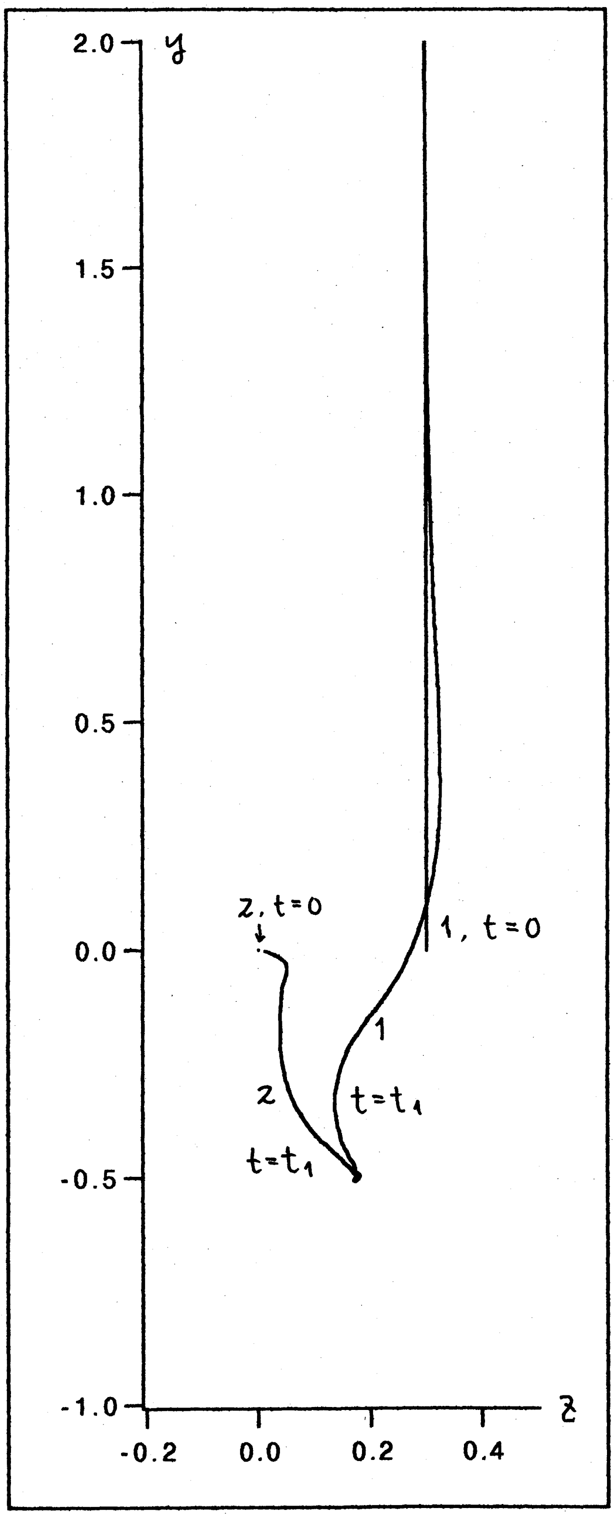


Fig 2 (b) Case 2

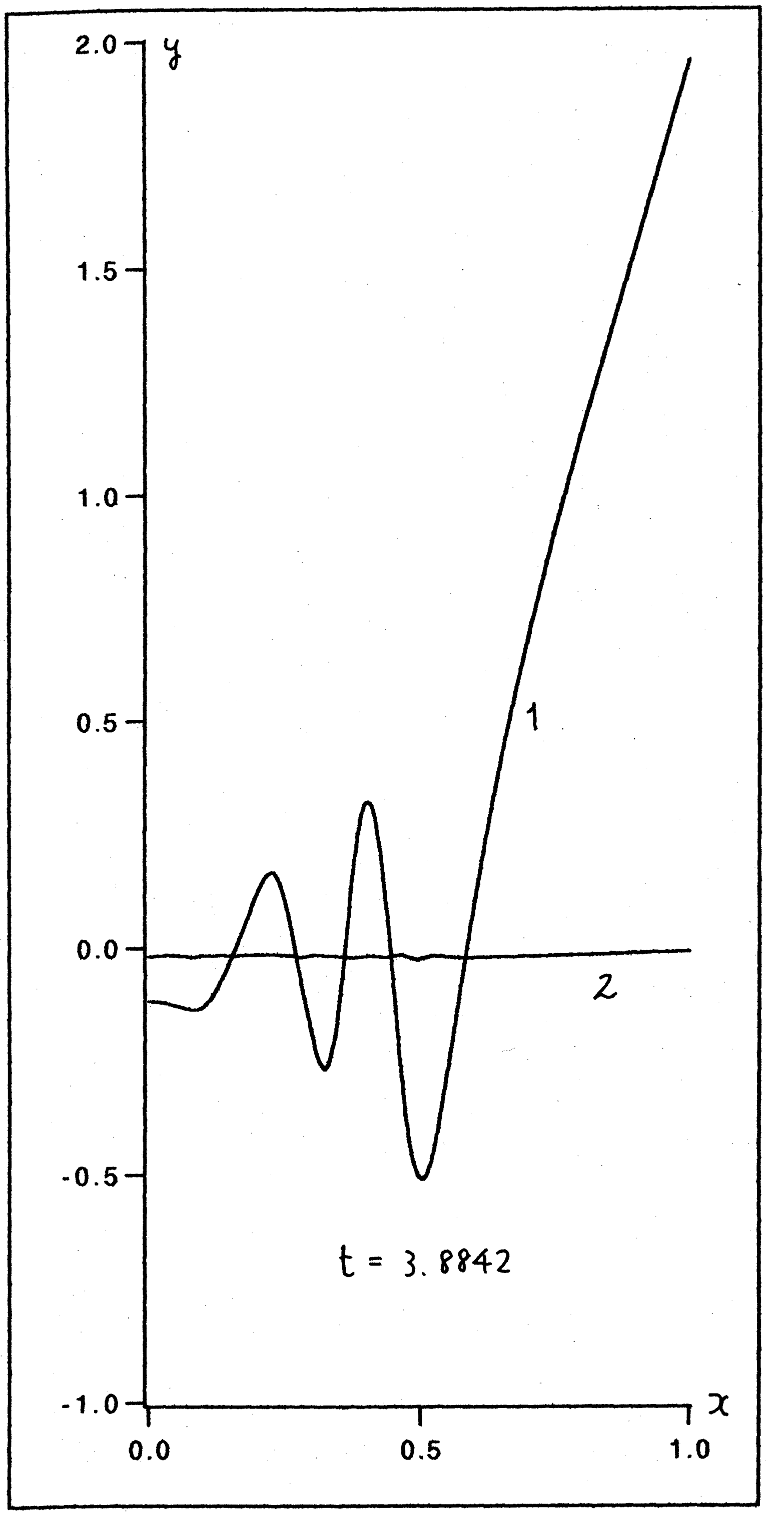


Fig 3 (a) Case 2

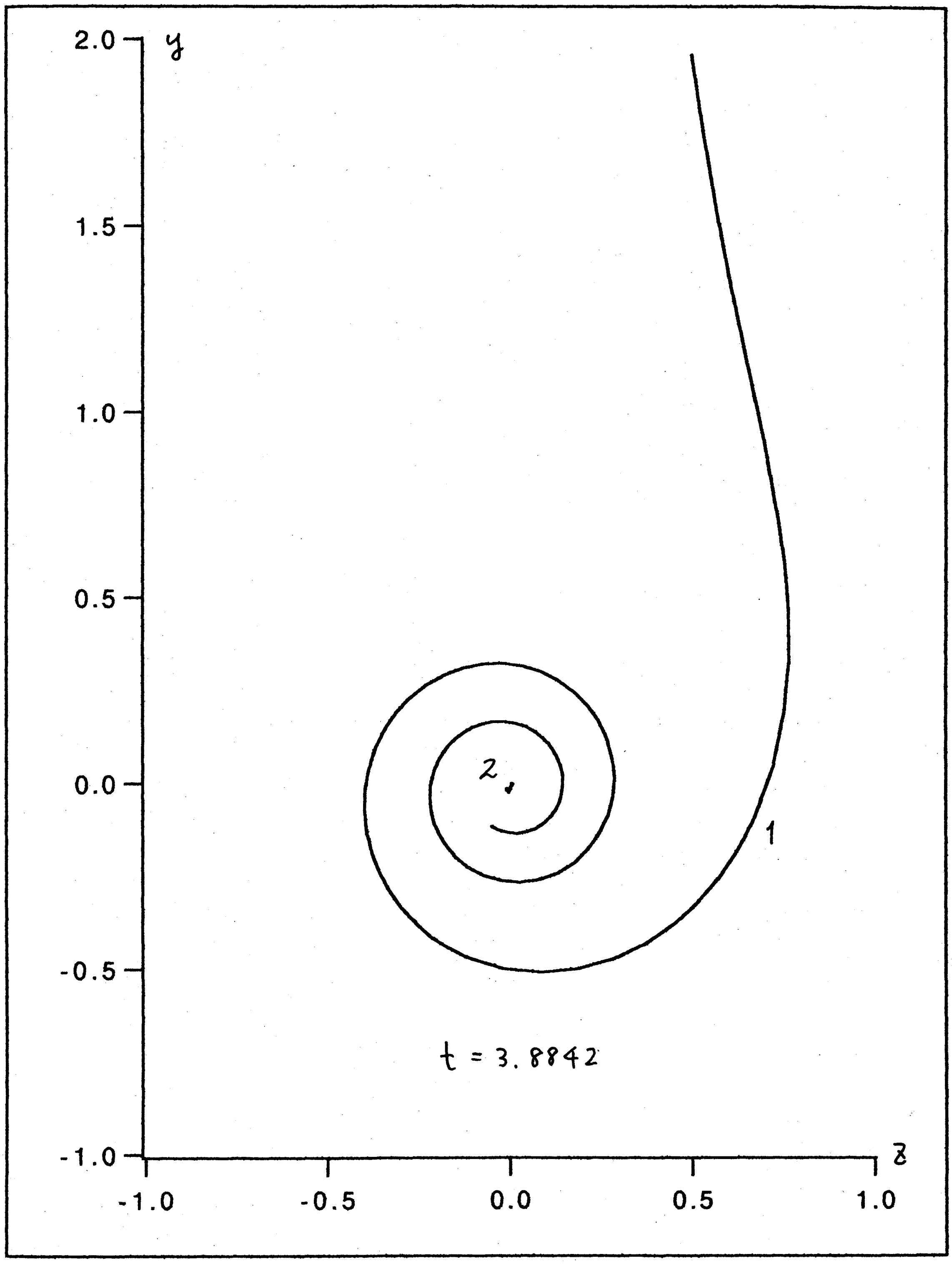
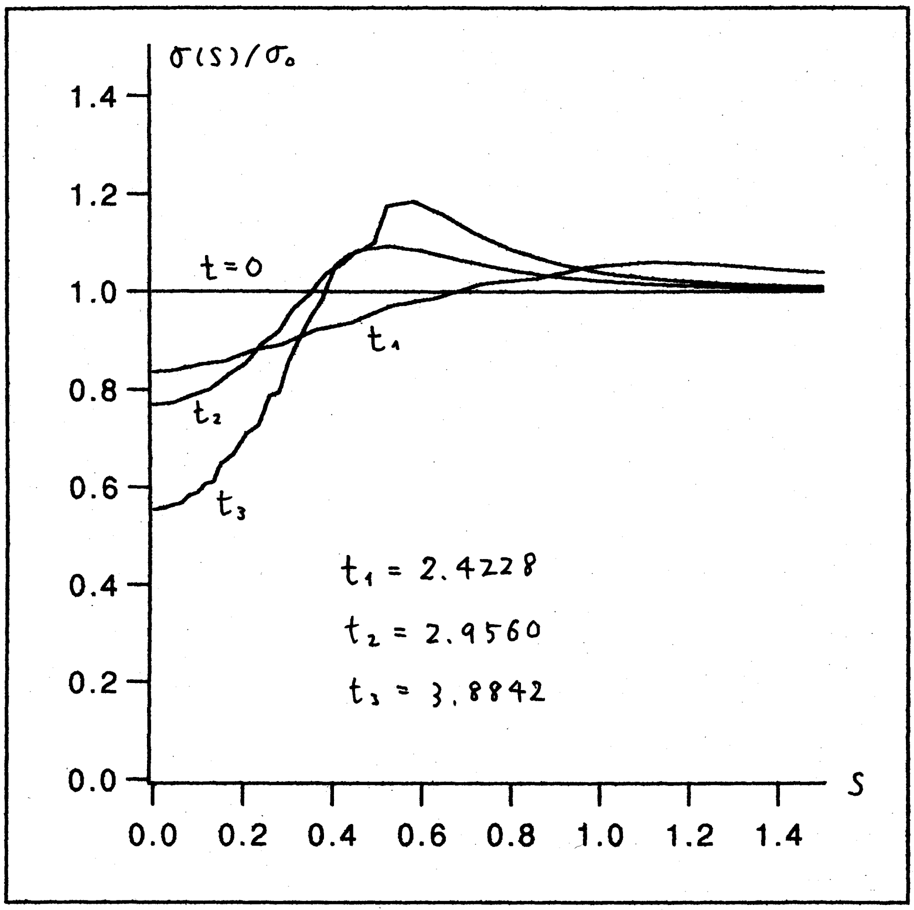


Fig 3 (b) Case 2



$\sigma(s)/\sigma_0$
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
t=0
t_1
t_2
t_3
$t_1 = 2.4228$
$t_2 = 2.9560$
$t_3 = 3.8842$
s
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4

Fig 4

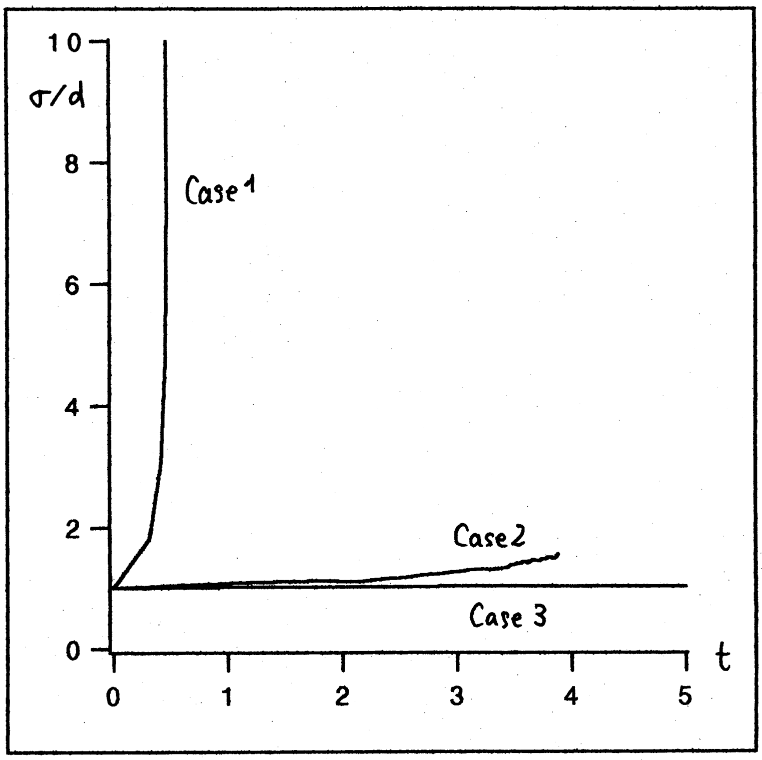


Fig 5