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完全２組グラフのＳ$_5$因子分解
(S_5-FACTORIZATION OF COMPLETE BIPARTITE GRAPHS)

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In this paper, trivial necessary conditions for the existence of an S_5-factorization of $K_{m,n}$ are given. Several types of construction algorithms of S_5-factorization of $K_{m,n}$ are also given.

1. Introduction

Let S_5 be a ***star*** on 5 vertices and $K_{m,n}$ be a ***complete bipartite graph*** with partite sets V_1 and V_2 of m and n vertices each. A spanning subgraph F of $K_{m,n}$ is called an S_5-***factor*** if each component of F is isomorphic to S_5. If $K_{m,n}$ is expressed as an edge-disjoint sum of S_5-factors, then this sum is called an S_5-***factorization*** of $K_{m,n}$.
In this paper, trivial necessary conditions for the existence of an S_5-factorization of $K_{m,n}$ are given. Several types of construction algorithms of S_5-factorization of $K_{m,n}$ are also given.

2. S_5-factor of $K_{m,n}$

The following theorem is on the existence of S_5-factors of $K_{m,n}$.

Theorem 1. $K_{m,n}$ has an S_5-factor if and only if (i) $m+n \equiv 0 \pmod 5$, (ii) $4n-m \equiv 0 \pmod{15}$, (iii) $4m-n \equiv 0 \pmod{15}$, (iv) $m \leqq 4n$ and (v) $n \leqq 4m$.

Proof. Suppose that $K_{m,n}$ has an S_5-factor F. Let t be the number of components of F. Then $t=(m+n)/5$. Hence, Condition (i) is necessary. Among these t components, let x and y be the number of components whose endvertices are in V_2 and V_1, respectively. Then, since F is a spanning subgraph of $K_{m,n}$, we have $x+4y=m$ and $4x+y=n$. Hence $x=(4n-m)/15$ and $y=(4m-n)/15$. From $0 \leqq x \leqq m$ and $0 \leqq y \leqq n$, we must have $m \leqq 4n$ and $n \leqq 4m$. Conditions (ii)-(v) are, therefore, necessary.
For those parameters m and n satisfying (i)-(v), let $x=(4n-m)/15$ and $y=(4m-n)/15$. Then x and y are integers such that $0 \leqq x \leqq m$ and $0 \leqq y \leqq n$.

Hence, $x+4y=m$ and $4x+y=n$. Using x vertices in V_1 and $4x$ vertices in V_2, consider x S_5's whose endvertices are in V_2. Using the remaining $4y$ vertices in V_1 and the remaining y vertices in V_2, consider y S_5's whose endvertices are in V_1. Then these $x+y$ S_5's are edge-disjoint and they form an S_5-factor of $K_{m,n}$. □

Corollary 1. $K_{n,n}$ has an S_5-factor if and only if $n \equiv 0 \pmod 5$.

3. S_5-factorization of $K_{m,n}$

We use the following notations.

Notation 1. r,t,b : number of S_5-factors, number of S_5-components of each S_5-factor, and total number of S_5-components, respectively, in an S_5-factorization of $K_{m,n}$.

t_1 (t_2) : number of components whose centers are in V_1 (V_2), respectively, among t S_5-components of each S_5-factor.

$r_1(u)$ ($r_2(v)$) : number of components whose centers are all u (v) for any u (v) in V_1 (V_2), respectively, among b S_5-components.

3.1. *Trivial necessary conditions of S_5-factorization of $K_{m,n}$*

We give the following trivial necessary conditions for the existence of S_5-factorization of $K_{m,n}$.

Theorem 2. If $K_{m,n}$ has an S_5-factorization then (i) $b=mn/4$, (ii) $t=(m+n)/5$, (iii) $r=5mn/4(m+n)$, (iv) $t_1=(4n-m)/15$, (v) $t_2=(4m-n)/15$, (vi) $r_1=(4n-m)n/12(m+n)$, (vii) $r_2=(4m-n)m/12(m+n)$, (viii) $m \leqq 4n$ and (ix) $n \leqq 4m$.

Proof. Suppose that $K_{m,n}$ has an S_5-factorization. Then it holds that $b=mn/4$, $t=(m+n)/5$, $r=b/t=5mn/4(m+n)$, $t_1=(4n-m)/15$, $t_2=(4m-n)/15$, $m \leqq 4n$ and $n \leqq 4m$. Let $s_1(u)$ ($s_2(v)$) be the number of components which have endvertex u (v) for any u (v) in V_1 (V_2), respectively, among b S_5-componets. Then it holds that $r_1(u)+s_1(u)=r$, $4r_1(u)+s_1(u)=n$, $r_2(v)+s_2(v)=r$ and $4r_2(v)+s_2(v)=m$. Hence we have $r_1(u)=(4n-m)n/12(m+n)$ and $r_2(v)=(4m-n)m/12(m+n)$. $r_1(u)$ ($r_2(v)$) doesn't depend on u (v), respectively. Therefore, Conditions (i)-(ix) are necessary. □

Corollary 2. If $K_{n,n}$ has an S_5-factorization then $n \equiv 0 \pmod{40}$.

3.2. *Extension theorem of S_5-factorization of $K_{m,n}$*

We prove the following extension theorem, which we use later in this paper.

Theorem 3. If $K_{m,n}$ has an S_5-factorization, then $K_{sm,sn}$ has an S_5-factorization for every positive integer s.

Proof. Let V_1, V_2 be the independent sets of $K_{sm,sn}$, where $|V_1|$ =sm and $|V_2|$ =sn. Divide V_1 and V_2 into s subsets of m and n vertices each, respectively. Construct a new graph G with a vertex set consisting of the subsets which were just constructed. In this graph, two vertices are adjacent if and only if the subsets come from disjoint independent sets of $K_{sm,sn}$. G is a complete bipartite graph $K_{s,s}$. Noting that the cardinality of each subset identified with a vertex set of G is m or n and that $K_{s,s}$ has a 1-factorization, we see that the desired result is obtained. 1-factorization of $K_{s,s}$ is discussed in [1,3]. □

3.3. *Sufficient conditions of S_5-factorization of $K_{m,n}$*

We consider the following three cases.

Case (1) m=4n: In this case, from Theorem 3, $K_{4n,n}$ has an S_5-factorization since $K_{4,1}$ is just S_5.

Case (2) n=4m: Obviously, $K_{m,4m}$ has an S_5-factorization.

Case (3) m<4n and n<4m: In this case, let $x=(4n-m)/15$ and $y=(4m-n)/15$. Then from Conditions (iv)-(v), x and y are integers such that $0<x<m$ and $0<y<n$. We have $x+4y=m$ and $4x+y=n$. Hence it holds that $b=(x^2+4xy+y^2)+xy/4$, $t=x+y$, $r=(x+y)+9xy/4(x+y)$, $t_1=x$, $t_2=y$, $r_1=x-3xy/4(x+y)$ and $r_2=y-3xy/4(x+y)$. Let $z=3xy/4(x+y)$, which is a positive integer. And let $(x,4y)=d$, $x=dp$, $4y=dq$, where $(p,q)=1$. Then $dq/4$ is an integer and $z=3dpq/4(4p+q)$. The following lemmas can be verified.

Lemma 1. $(p,q)=1 \Longrightarrow (pq,p+q)=1$.

Lemma 2. $(p,q)=1 \Longrightarrow (pq,4p+q)=1$ (q is an odd integer) , 2 (q/2 is an odd integer) and 4 (q/4 is an integer).

Using these p,q,d, the parameters m and n satisfying Conditions (i)-(ix) are expressed as follows:

Lemma 3. (p,q)=1 and 3dpq/4(4p+q) is an integer
===> (I) m=4(p+q)(4p+q)s, n=(16p+q)(4p+q)s ((4p+q)/3:not integer)
or m=4(p+q)(4p+q)s/3, n=(16p+q)(4p+q)s/3 ((4p+q)/3:integer)
when q is an odd integer,
(II) m=4(p+2q')(2p+q')s, n=2(8p+q')(2p+q')s ((2p+q')/3:not integer)
or m=4(p+2q')(2p+q')s/3, n=2(8p+q')(2p+q')s/3 ((2p+q')/3:integer)
when q=2q' and q' is an odd integer,
(III) m=4(p+4q")(p+q")s, n=4(4p+q")(p+q")s ((p+q")/3:not integer)
or m=4(p+4q")(p+q")s/3, n=4(4p+q")(p+q")s/3 ((p+q")/3:integer)
when q=4q",
where s is a positive integer.

We use the following notations for sequences.

Notation 2. Let A and B be two sequences of the same size such as
A: $a_1,a_2,...,a_u$
B: $b_1,b_2,...,b_u$.
If $b_i=a_i+c$ (i=1,2,...,u), then we write **B=A+c**. If **b_i=((a_i+c) mod w)** (i=1,2,...,u), then we write **B=A+c mod w**, where the residuals a_i+c mod w are integers in the set {1,2,...,w}.

Lemma 4. (p,q)=1 and q is an odd integer
m=4(p+q)(4p+q)s, n=(16p+q)(4p+q)s, where s is a positive integer
===> $K_{m,n}$ has an S_5-factorization.

Proof. When s=1, the proof is by construction (**Algorithm I**). Let x=(4n-m)/15, y=(4m-n)/15, t=(m+n)/5, r=5mn/4(m+n). Then we have x=4p(4p+q), y=q(4p+q), t=$(4p+q)^2$, r=(p+q)(16p+q). Let r_m=p+q, r_n=16p+q, $m_0=m/r_m$=4(4p+q), $n_0=n/r_n$=4p+q. Consider two sequences R and C of the same size 16(4p+q).
R: 1,1,1,1,2,2,2,2,...,4(4p+q),4(4p+q),4(4p+q),4(4p+q)
C: 1,2,...,16(4p+q)-1,16(4p+q).
Construct p sequences R_i such that R_i=R+4(i-1)(4p+q) (i=1,2,...,p).
Construct p sequences C_i such that C_i=(C+4(i-1) mod 16(4p+q))+16(i-1)(4p+q)

(i=1,2,...,p). Consider two sequences R' and C' of the same size 4(4p+q).

R': $r_1,r_2,\dots,r_{4(4p+q)}$, where $r_i=(i-1)p+1 \bmod 4(4p+q)$ (i=1,2,...,4(4p+q))

C': $c_1,c_2,\dots,c_{4(4p+q)}$, where $c_i=n-(i-1)q \bmod q(4p+q)$ (i=1,2,...,4(4p+q)).

Construct q sequences R_i' such that $R_i'=R'+4(i-1)(4p+q)+4p(4p+q)$ (i=1,2,...,q). Construct q sequences C_i' such that $C_i'=(C'-(i-1) \bmod q(4p+q))+16p(4p+q)$ (i=1,2,...,q). Consider two sequences I and J of the same size.

I: $R_1,R_2,\dots,R_p,R_1',R_2',\dots,R_q'$

J: $C_1,C_2,\dots,C_p,C_1',C_2',\dots,C_q'$.

Then the size of I or J is 4t. Let i_k and j_k be the k-th element of I and J, respectively (k=1,2,...,4t). Join two vertices i_k in V_1 and j_k in V_2 with an edge (i_k,j_k) (k=1,2,...,4t). Construct a graph F with two vertex sets $\{i_k\}$ and $\{j_k\}$ and an edge set $\{(i_k,j_k)\}$. Then F is an S_5-factor of $K_{m,n}$. This graph is called an ***S_5-factor constructed with two sequences I and J.***

Construct r_m sequences I_i such that $I_i=I+(i-1)m_0 \bmod m$ ($i=1,2,\dots,r_m$).

Construct r_n sequences J_j such that $J_j=J+(j-1)n_0 \bmod n$ ($j=1,2,\dots,r_n$).

Construct $r_m r_n$ S_5-factors F_{ij} with I_i and J_j ($i=1,2,\dots,r_m;j=1,2,\dots,r_n$). Then it is easy to show that F_{ij} are edge-disjoint and that their sum is an S_5-factorization of $K_{m,n}$. By Theorem 3, $K_{m,n}$ has an S_5-factorization for every positive integer s. □

Lemma 5. (p,q)=1 and q=2q' (q' is an odd integer)
m=4(p+2q')(2p+q')s, n=2(8p+q')(2p+q')s, where s is a positive integer
===> $K_{m,n}$ has an S_5-factorization.

Proof. When s=1, the proof is by construction (Algorithm II). Let x=(4n-m)/15, y=(4m-n)/15, t=(m+n)/5, r=5mn/4(m+n). Then we have x=4p(2p+q'), y=2q'(2p+q'), $t=2(2p+q')^2$, r=(p+2q')(8p+q'). Let $r_m=p+2q'$, $r_n=8p+q'$, $m_0=m/r_m=4(2p+q')$, $n_0=n/r_n=2(2p+q')$. Consider two sequences R and C of the same size 16(2p+q').

R: 1,1,1,1,2,2,2,2,...,4(2p+q'),4(2p+q'),4(2p+q'),4(2p+q')

C: 1,2,...,16(2p+q')-1,16(2p+q').

Construct p sequences R_i such that $R_i=R+4(i-1)(2p+q')$ (i=1,2,...,p).

Construct p sequences C_i such that $C_i=(C+4(i-1) \bmod 16(2p+q'))+16(i-1)(2p+q')$ (i=1,2,...,p). Consider two sequences R' and C' of the same size 4(2p+q').

R': $r_1,r_2,\dots,r_{4(2p+q')}$, where $r_i=(i-1)p+1 \bmod 4(2p+q')$ (i=1,2,...,4(2p+q'))

C': $c_1,c_2,\dots,c_{4(2p+q')}$, where $c_i=n-2(i-1)q' \bmod 2q'(2p+q')$ (i=1,2,...,4(2p+q')).

Construct 2q' sequences R_i' such that $R_i'=R'+4(i-1)(2p+q')+4p(2p+q')$ (i=1,2,...,2q'). Construct 2q' sequences C_i' such that $C_i'=(C'-(i-1) \bmod 2q'(2p+q'))+16p(2p+q')$ (i=1,2,...,2q'). Consider two sequences I and J of the same size 4t.

I: $R_1,R_2,\ldots,R_p,R_1',R_2',\ldots,R_{2q'}'$

J: $C_1,C_2,\ldots,C_p,C_1',C_2',\ldots,C_{2q'}'$.

Construct r_m sequences I_i such that $I_i=I+(i-1)m_0$ mod m $(i=1,2,\ldots,r_m)$.

Construct r_n sequences J_j such that $J_j=J+(j-1)n_0$ mod n $(j=1,2,\ldots,r_n)$.

Construct $r_m r_n$ S_5-factors F_{ij} with I_i and J_j $(i=1,2,\ldots,r_m;j=1,2,\ldots,r_n)$. Then it is easy to show that F_{ij} are edge-disjoint and that their sum is an S_5-factorization of $K_{m,n}$. By Theorem 3, $K_{m,n}$ has an S_5-factorization for every positive integer s. □

Lemma 6. $(p,q)=1$ and $q=4q''$
$m=4(p+4q'')(p+q'')s$, $n=4(4p+q'')(p+q'')s$, where s is a positive integer
===> $K_{m,n}$ has an S_5-factorization.

Proof. When s=1, the proof is by construction (Algorithm III). Let $x=(4n-m)/15$, $y=(4m-n)/15$, $t=(m+n)/5$, $r=5mn/4(m+n)$. Then we have $x=4p(p+q'')$, $y=4q''(p+q'')$, $t=4(p+q'')^2$, $r=(p+4q'')(4p+q'')$. Let $r_m=p+4q''$, $r_n=4p+q''$, $m_0=m/r_m=4(p+q'')$, $n_0=n/r_n=4(p+q'')$. Consider two sequences R and C of the same size $16(p+q'')$.

R: $1,1,1,1,2,2,2,2,\ldots,4(p+q''),4(p+q''),4(p+q''),4(p+q'')$

C: $1,2,\ldots,16(p+q'')-1,16(p+q'')$.

Construct p sequences R_i such that $R_i=R+4(i-1)(p+q'')$ $(i=1,2,\ldots,p)$.

Construct p sequences C_i such that $C_i=(C+4(i-1) \bmod 16(p+q''))+16(i-1)(p+q'')$ $(i=1,2,\ldots,p)$. Consider two sequences R' and C' of the same size $4(p+q'')$.

R': $r_1,r_2,\ldots,r_{4(p+q'')}$, where $r_i=(i-1)p+1 \bmod 4(p+q'')$ $(i=1,2,\ldots,4(p+q''))$

C': $c_1,c_2,\ldots,c_{4(p+q'')}$, where $c_i=n-4q''(i-1) \bmod 4q''(p+q'')$ $(i=1,2,\ldots,4(p+q''))$.

Construct 4q" sequences R_i' such that $R_i'=R'+4(i-1)(p+q'')+4p(p+q'')$ $(i=1,2,\ldots,4q'')$. Construct 4q" sequences C_i' such that $C_i'=(C'-(i-1) \bmod 4q''(p+q''))+16p(p+q'')$ $(i=1,2,\ldots,4q'')$. Consider two sequences I and J of the same size 4t.

I: $R_1,R_2,\ldots,R_p,R_1',R_2',\ldots,R_{4q''}'$

J: $C_1,C_2,\ldots,C_p,C_1',C_2',\ldots,C_{4q''}'$.

Construct r_m sequences I_i such that $I_i=I+(i-1)m_0$ mod m $(i=1,2,\ldots,r_m)$.

Construct r_n sequences J_j such that $J_j=J+(j-1)n_0$ mod n $(j=1,2,\ldots,r_n)$.

Construct $r_m r_n$ S_5-factors F_{ij} with I_i and J_j $(i=1,2,\ldots,r_m;j=1,2,\ldots,r_n)$. Then it is easy to show that F_{ij} are edge-disjoint and that their sum is an S_5-factorization of $K_{m,n}$. By Theorem 3, $K_{m,n}$ has an S_5-factorization for every positive integer s. □

In Lemma 6, put p=1, q=4q"=4. Then we have the following example.

Example 1. $K_{408,408}$ has an S_5-factorization.

By Corollary 2 and Example 1, we have the following theorem.

Theorem 4. $K_{n,n}$ has an S_5-factorization if and only if $n \equiv 0 \pmod{40}$.

Conjecture 1. (p,q)=1, q is an odd integer and (4p+q)/3 is an integer
m=4(p+q)(4p+q)s/3, n=(16p+q)(4p+q)s/3,
where s is a positive integer and s/3 is not an integer
===> $K_{m,n}$ has an S_5-factorization.

Conjecture 2. (p,q)=1, q=2q' (q' is an odd integer) and (2p+q)/3 is an integer
m=4(p+2q')(2p+q')s/3, n=2(8p+q')(2p+q')s/3,
where s is a positive integer and s/3 is not an integer
===> $K_{m,n}$ has an S_5-factorization.

Conjecture 3. (p,q)=1, q=4q" and (p+q")/3 is an integer
m=4(p+4q")(p+q")s/3, n=4(4p+q")(p+q")s/3,
where s is a positive integer and s/3 is not an integer
===> $K_{m,n}$ has an S_5-factorization.

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