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Toward Harmonic Analysis on Gaussian Space

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Introduction

As is well known, there are two important aspects of Gaussian space: stochastic analysis and quantum field theory. Needless to say, in these theories most principal roles are played respectively by Brownian motion and Fock space both of which are realized on Gaussian space. Thus it is widely accepted that Gaussian space is one of the most important concepts of infinite dimensional analysis such as Euclidean space in finite dimensional case. Moreover, since 1970s distribution theories on Gaussian space have developed considerably into a most flourishing field of mathematics.

On the other hand, it is remarkable that some pioneering works were made by japanese mathematicians in 1960s toward "harmonic analysis on Gaussian space." A central object was perhaps the infinite dimensional rotation group $O(E;H)$ proposed by H. Yoshizawa after his study of unitary representations of free groups. During that decade a series of important works appeared discussing infinite dimensional Laplacian, infinite dimensional rotation group, infinite dimensional motion group, and special functions as matrix elements of their unitary representations, see Kôno **[9]**, **[10]**, Orihara **[26]**, Umemura **[27]**, Umemura and Kôno **[28]**. Furthermore, it was shown by Hida, Kubo, Nomoto and Yoshizawa **[5]** and Yoshizawa **[30]** that the infinite dimensional rotation group plays also an important role in describing projective invariance of Brownian motion, see also **[31]**. However, little progress has been made afterward and, in particular, no special attention has been paid to application of distribution theories born in 1970s.

In recent years the so-called white noise calculus, a distribution theory on Gaussian space initiated by Hida **[3]** and axiomatized to some extent by Kubo and Takenaka **[11]**, has developed considerably keeping a profound contact with stochastic (causal) analysis and Feynman path integrals, see e.g., **[7]**. Meanwhile, establishing a general theory of operators on white noise functionals using integral kernel operators and Fock expansion, we have started a study of harmonic analysis on Gaussian space, see also **[21]**. One of the main consequences of our operator theory is that every continuous linear operator on white noise functionals (this class contains all bounded operators on Fock space) admits an infinite series expansion in terms of creation and annihilation operators. This theory is highlighted in [**22**] and [**23**], see also §6.

The main purpose of this paper is to recapitulate the operator theory on Gaussian space with illustrating application to some questions of harmonic analysis, in particular, to description of rotation-invariant operators.

In his important work [**27**] Umemura showed that "rotation-invariant operators" are generated by a single operator, namely, by the number operator N. However, we can not help ourselves feeling that the structure of the rotation-invariant operators is even poorer, comparing to the finite dimensional case. Moreover, during the derivation of the number operator from finite dimensional Laplacians by limit argument, Umemura abandoned polynomial terms simply by reason of divergence. We shall observe that white noise calculus explains it to some extent. In fact, in our sense the rotation-invariant operators on Gaussian space are gerenated by two Laplacians, the *number operator* N and the *Gross Laplacian* Δ_G. Note, however, that there is no contradiction between Umemura's work and our result. The point is very simple: the Gross Laplacian is not symmetric and the proper L^2-domain of Δ_G^* is $\{0\}$. Furthermore, a white noise analogue of Euclidean norm is given by $R = 2N + \Delta_G + \Delta_G^*$. We shall observe that R is extracted from the "divergent terms" in Umemura's argument. Thus, within white noise calculus the structure of rotation-invariant operators is more similar to the finite dimensional case.

1. Gaussian Space

Let T be a topological space with a Borel measure $\nu(dt) = dt$ and let $H = L^2(T, \nu; \mathbf{R})$ be the real Hilbert space of all ν-square integrable functions on T. The inner product is denoted by $\langle \cdot, \cdot \rangle$ and the norm by $|\cdot|_0$. We often regard T as time-parameter space e.g., when $T = \mathbf{R}, \mathbf{Z}$, and as space-time-parameter space in quantum field theory e.g., when $T = \mathbf{R}^D, \mathbf{Z}^D$.

Let A be a positive selfadjoint operator on H with Hilbert-Schmidt inverse. Then there exist an increasing sequence of positive numbers $0 < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots$ and a complete orthonormal basis $(e_j)_{j=0}^{\infty}$ for H such that $Ae_j = \lambda_j e_j$ and

$$\delta \equiv \left(\sum_{j=0}^{\infty} \lambda_j^{-2} \right)^{1/2} = \left\| A^{-1} \right\|_{\mathrm{HS}} < \infty. \tag{1-1}$$

Let E be the standard CH-space constructed from (H, A), that is, the C^∞-domain of

A equipped with the norms:

$$(1\text{-}2)\qquad |\xi|_p = |A^p\xi|_0 = \left(\sum_{j=0}^{\infty} \lambda_j^{2p} \langle \xi, e_j\rangle^2\right)^{1/2}, \qquad \xi \in E, \quad p \in \mathbf{R}.$$

Since A^{-1} is of Hilbert-Schmidt type by assumption, E becomes a nuclear Fréchet space and hence

$$(1\text{-}3)\qquad E \subset H = L^2(T, \nu; \mathbf{R}) \subset E^*$$

becomes a Gelfand triple. The canonical bilinear form on $E^* \times E$ is also denoted by $\langle \cdot, \cdot \rangle$. The dual space E^* is always assumed to be equipped with the strong dual topology.

By the Bochner-Minlos theorem there exists a unique probability measure μ on E^* (equipped with the Borel σ-field) such that

$$(1\text{-}4)\qquad \exp\left(-\frac{1}{2}|\xi|_0^2\right) = \int_{E^*} e^{i\langle x, \xi\rangle} \mu(dx), \qquad \xi \in E.$$

This μ is called the *Gaussian measure* and the probability space (E^*, μ) is called the *Gaussian space*.

In a different context Gaussian space would mean merely a real (usually infinite dimensional) vector space equipped with Gaussian measure. In fact, L^2-theory on Gaussian space is free not only from the particular construction of Gelfand triple (standard CH-space) but also from the underlying space T. However, those particular structures together with the assumptions below are indispensable for our effective theory of distributions.

By construction each $\xi \in E$ is a function on T determined up to ν-null functions. This hinders us from introducing a delta-function which is essential to our discussion. Accordingly we are led to the following:

(H1) For each $\xi \in E$ there exists a unique continuous function $\tilde{\xi}$ on T such that $\xi(t) = \tilde{\xi}(t)$ for ν-a.e. $t \in T$.

Once this is satisfied, we always assume that every element in E is a continuous function on T and do not use the symbol $\tilde{\xi}$. We further need:

(H2) For each $t \in T$ a linear functional $\delta_t : \xi \mapsto \xi(t)$, $\xi \in E$, is continuous, i.e., $\delta_t \in E^*$;

(H3) The map $t \mapsto \delta_t \in E^*$, $t \in T$, is continuous. (Recall that E^* carries the strong dual topology.)

Under (H1)-(H2) the convergence in E implies the pointwise convergence as functions on T. If we have (H3) in addition, the convergence is uniform on every compact subset of T. Moreover, it is noted that the properties (H1)-(H3) are preserved under forming tensor products. By another reason (see §4) we need one more assumption:

(S) $\lambda_0 = \inf \mathrm{Spec}\,(A) > 1$.

The constant number

$$0 < \rho \equiv \lambda_0^{-1} = \left\| A^{-1} \right\|_{\mathrm{OP}} < 1 \tag{1-5}$$

is important as well as δ defined in (1-1) to derive various inequalities, though we do not use them explicitly in this paper.

2. Wiener-Itô-Segal Isomorphism

For simplicity we put

$$(L^2) = L^2(E^*, \mu; \mathbf{C}).$$

In this section we recapitulate the famous Wiener-Itô-Segal isomorphism between (L^2) and the so-called Boson Fock space over $H_{\mathbf{C}}$.

The canonical bilinear form on $(E^{\otimes n})^* \times (E^{\otimes n})$ is denoted by $\langle \cdot, \cdot \rangle$ again and its bilinear extension to $(E_{\mathbf{C}}^{\otimes n})^* \times (E_{\mathbf{C}}^{\otimes n})$ is also denoted by the same symbol. We now define $\tau \in (E \otimes E)^*$ by

$$\langle \tau, \xi \otimes \eta \rangle = \langle \xi, \eta \rangle, \qquad \xi, \eta \in E. \tag{2-1}$$

In other words,

$$\langle \tau, \omega \rangle = \int_T \omega(t,t)\, dt, \qquad \omega \in E \otimes E. \tag{2-2}$$

The fact that any $\omega \in E \otimes E$ is a continuous function on $T \times T$ follows from (H1)-(H3). This distribution is called *trace*.

For $x \in E^*$ we define $:x^{\otimes n}: \in (E^{\otimes n})^*_{\mathrm{sym}}$ inductively as follows:

$$\begin{cases} :x^{\otimes 0}: &= 1, \\ :x^{\otimes 1}: &= x, \\ :x^{\otimes n}: &= x \widehat{\otimes} :x^{\otimes (n-1)}: - (n-1)\tau \widehat{\otimes} :x^{\otimes (n-2)}:, \qquad n \ge 2. \end{cases} \tag{2-3}$$

In other words, $:x^{\otimes n}:$ is defined as a unique element in $(E^{\otimes n})^*_{\mathrm{sym}}$ satisfying

$$\left\langle :x^{\otimes n}:, \xi^{\otimes n} \right\rangle = \frac{|\xi|_0^n}{2^{n/2}} H_n\left(\frac{\langle x, \xi \rangle}{\sqrt{2}\, |\xi|_0} \right), \qquad \xi \in E, \quad \xi \ne 0, \tag{2-4}$$

where H_n denotes the Hermite polynomial of degree n. Or equivalently, $:x^{\otimes n}:$ is defined by generating function:

$$\phi_\xi(x) \equiv \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \frac{\xi^{\otimes n}}{n!} \right\rangle = \exp\left(\langle x, \xi \rangle - \frac{1}{2} \langle \xi, \xi \rangle \right), \qquad \xi \in E. \tag{2-5}$$

Note that the right hand side of (2-5) is a "normalized" exponential function and the identity is valid also for $\xi \in E_{\mathbf{C}}$. We call ϕ_ξ an *exponential vector.*

The orthogonal relation of Hermite polynomials leads us to the following

$$\int_{E^*} \left\langle :x^{\otimes m}:, f_m \right\rangle \left\langle :x^{\otimes n}:, g_n \right\rangle \mu(dx) = n! \langle f_m, g_n \rangle \delta_{mn}, \quad f_m \in E_{\mathbf{C}}^{\widehat{\otimes} m}, \, g_n \in E_{\mathbf{C}}^{\widehat{\otimes} n}. \tag{2-6}$$

Then by usual L^2-approximation one can define a function $x \mapsto \left\langle :x^{\otimes n}:, f \right\rangle$, $x \in E^*$, for any $f \in H_{\mathbf{C}}^{\widehat{\otimes} n}$ in L^2-sense. Let $\mathcal{H}_n(\mathbf{C})$ be the space of all such functions. Then they become mutually orthogonal closed subspaces of (L^2). Since the polynomials, namely the algebra generated by $\{\langle x, \xi \rangle \,; \xi \in E_{\mathbf{C}}\}$ is dense in (L^2), we come to the following

THEOREM 2.1 (WIENER-ITÔ-SEGAL). *The Hilbert space (L^2) admits an orthogonal sum decomposition:*

$$(L^2) = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n(\mathbf{C}). \tag{2-7}$$

More precisely, for each $\phi \in (L^2)$ there exists a unique sequence $f_n \in H_{\mathbf{C}}^{\widehat{\otimes} n}$, $n = 0, 1, 2, \cdots$, such that

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f_n \right\rangle, \qquad x \in E^*, \tag{2-8}$$

where each $\left\langle :x^{\otimes n}:, f_n \right\rangle$ is a function in $\mathcal{H}_n(\mathbf{C})$ and the series is an orthogonal direct sum. In that case

$$\| \phi \|_0^2 \equiv \int_{E^*} |\phi(x)|^2 \mu(dx) = \sum_{n=0}^{\infty} n! \, | f_n |_0^2 . \tag{2-9}$$

Furthermore, the above correspondence $\phi \leftrightarrow (f_n)_{n=0}^{\infty}$ gives a unitary isomorphism between (L^2) and the Boson Fock space over $H_{\mathbf{C}}$.

According to the Wiener-Itô decomposition (2-7) we define an operator N by

$$N\phi = n\phi, \qquad \phi \in \mathcal{H}_n(\mathbf{C}), \quad n = 0, 1, 2, \cdots . \tag{2-10}$$

Equipped with the maximal domain, N becomes a selfadjoint operator in (L^2). This operator is called the *number operator* (because $\mathcal{H}_n(\mathbf{C})$ stands for the Hilbert space of n Bose particles in physical interpretation) and is one of the infinite dimensional Laplacians on Gaussian space. We shall come back to this topic in §6.

3. Infinite Dimensional Rotation Group

Let $O(E;H)$ be the group of all linear homeomorphisms from E onto itself preserving the norm $|\cdot|_0$, namely $|g\xi|_0 = |\xi|_0$ for $\xi \in E$. In other words, $O(E;H)$ is the group of automorphisms of the Gelfand triple $E \subset H \subset E^*$. While, since each $g \in O(E;H)$ is extended to an orthogonal operator on the Hilbert space H, we may regard $O(E;H)$ as a subgroup of the full orthogonal group $O(H)$. The group $O(E;H)$ is called the *infinite dimensional rotation group* (associated with the Gelfand triple $E \subset H \subset E^*$).

The infinite dimensional rotation group $O(E;H)$ acts on the Gaussian space E^* in an obvious manner:

$$\langle x, g\xi \rangle = \langle g^* x, \xi \rangle, \qquad x \in E^*, \quad \xi \in E. \tag{3-1}$$

As is seen immediately from (1-4), the characteristic functional of μ is invariant under the action of $O(E;H)$. Hence the uniqueness of a characteristic functional implies that the Gaussian measure μ is invariant under the action $x \mapsto g^*x$, $x \in E^*$, $g \in O(E;H)$. We then come to a natural unitary representation of $O(E;H)$ on (L^2):

$$(\Gamma(g)\phi)(x) = \phi(g^*x), \qquad \phi \in (L^2), \quad g \in O(E;H). \tag{3-2}$$

As is easily verified, if $\phi \in (L^2)$ is expressed as in (2-8), we have

$$(\Gamma(g)\phi)(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \, g^{\otimes n} f_n \right\rangle. \tag{3-3}$$

Hence each $\mathcal{H}_n(\mathbf{C})$ in the Wiener-Itô decomposition is an invariant subspace. Moreover,

THEOREM 3.1. *The Wiener-Itô decomposition $(L^2) = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n(\mathbf{C})$ is an irreducible decomposition of the unitary representation $(\Gamma, (L^2))$ of $O(E;H)$. Furthermore, all irreducible subspaces $\mathcal{H}_n(\mathbf{C})$ are mutually inequivalent.*

This is a simple consequence of the following fundamental result.

THEOREM 3.2 (UMEMURA [**27**]). *Let Ξ be a symmetric operator on $(L^2) = L^2(E^*, \mu)$ which is invariant under the action of $O(E;H)$, and assume that for any $\xi \in E$, $e^{i\langle \cdot, \xi \rangle}$ belongs to the domain of Ξ. Then Ξ can be expressed as a function of N.*

For the precise meaning of "a function of N" see the original paper. Instead we note here a significant consequence: Let Ξ be a selfadjoint operator on (L^2) with domain

Dom (Ξ) containing all exponential functions of the form $e^{i\langle\cdot,\xi\rangle}$, $\xi \in E$. If Ξ is invariant under $O(E;H)$, then there exists a real sequence $(\alpha_n)_{n=0}^{\infty}$ such that $\Xi\phi = \alpha_n\phi$ for $\phi \in \text{Dom}(\Xi) \cap \mathcal{H}_n(\mathbf{C})$.

The irreducible representations mentioned in Theorem 3.1 are characterized by Matsushima, Okamoto and Sakurai [**16**] and by Okamoto and Sakurai [**25**].

4. White Noise Functionals

We first need a second quantized operator $\Gamma(A)$, where A is the same operator as we used in §1 to construct the Gelfand triple $E \subset H = L^2(T,\nu;\mathbf{R}) \subset E^*$ and the Gaussian space (E^*,μ). Suppose that $\phi \in (L^2)$ is given as

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f_n \right\rangle \tag{4-1}$$

according to the Wiener-Itô-Segal isomorphism. We then put

$$\Gamma(A)\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, A^{\otimes n} f_n \right\rangle . \tag{4-2}$$

In the previous section we employed the same symbol Γ for a particular unitary representation of $O(E;H)$. However, there will occur no confusion due to the fact (3-3). It is known that $\Gamma(A)$ equipped with the maximal domain becomes a positive selfadjoint operator on (L^2).

Let (E) be the standard CH-space constructed from the pair $((L^2), \Gamma(A))$. Since $\Gamma(A)$ admits Hilbert-Schmidt inverse by the hypothesis (S) in §1, (E) is a nuclear Fréchet space and

$$(E) \subset (L^2) = L^2(E^*,\mu;\mathbf{C}) \subset (E)^* \tag{4-3}$$

becomes a complex Gelfand triple. Elements in (E) and $(E)^*$ are called a *test (white noise) functional* and a *generalized (white noise) functional*, respectively. We denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the canonical bilinear form on $(E)^* \times (E)$ and by $\|\cdot\|_p$ the norm induced from $\Gamma(A)$, namely,

$$\|\phi\|_p^2 = \|\Gamma(A)^p\phi\|_0^2 = \sum_{n=0}^{\infty} n! \left|(A^{\otimes n})^p f_n\right|_0^2 = \sum_{n=0}^{\infty} n! \, |f_n|_p^2, \qquad \phi \in (E), \tag{4-4}$$

where ϕ and $(f_n)_{n=0}^{\infty}$ are related as in (4-1). This identity is compatible with (2-9).

By construction each $\phi \in (E)$ is defined only up to μ-null functions. However, it follows from Kubo-Yokoi's continuous version theorem [**12**] that for $\phi \in (E)$ the right hand side of (4-1) converges absolutely at each $x \in E^*$ and becomes a unique continuous function on E^* which coincides with $\phi(x)$ for μ-a.e. $x \in E^*$. Thus, (E) is always assumed to be a space of continuous functions on E^* and for $\phi \in (E)$ the right hand side of (4-1) is understood as pointwisely convergent series as well as in the sense of norms $\|\cdot\|_p$.

For a generalized white noise functional $\Phi \in (E)^*$ there exists a unique sequence $F_n \in (E_{\mathbf{C}}^{\otimes n})^*_{\mathrm{sym}}$, $n = 0, 1, 2, \cdots$, such that

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \tag{4-5}$$

for $\phi \in (E)$ given as in (4-1). In that case it holds that

$$\| \Phi \|_{-p}^2 = \sum_{n=0}^{\infty} n! \, | F_n |_{-p}^2 . \tag{4-6}$$

This is finite for all sufficiently large $p \geq 0$ and is compatible with (4-4). It is then convenient to adopt a formal expression:

$$\Phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, F_n \right\rangle . \tag{4-7}$$

Conversely, we agree that (4-7) defines a generalized white noise functional Φ via (4-5) whenever $\sum_{n=0}^{\infty} n! \, | F_n |_{-p}^2 < \infty$ for some $p \geq 0$.

The simplest example of generalized white noise functionals would be *white noise coordinate*. For each $t \in T$, $\Phi_t(x) = \langle :x:, \delta_t \rangle = \langle x, \delta_t \rangle$ belongs to $(E)^*$. For simplicity we put

$$x(t) = \langle x, \delta_t \rangle , \qquad t \in T, \tag{4-8}$$

which may be regarded as white noise analogue of the usual coordinate $(x_1, \cdots, x_D)$ of Euclidean space $\mathbf{R}^D$.

As for exponential vectors (2-5) we remind the following

PROPOSITION 4.1. *$\phi_\xi \in (E)$ for any $\xi \in E_{\mathbf{C}}$ and such exponential vectors span a dense subspace of (E).*

The *S-transform* of $\Phi \in (E)^*$ is a function on $E_{\mathbf{C}}$ defined by

$$S\Phi(\xi) = \left\langle\left\langle \Phi, \phi_\xi \right\rangle\right\rangle = e^{-\langle \xi, \xi \rangle/2} \int_{E^*} \Phi(x) e^{\langle x, \xi \rangle} \mu(dx), \qquad \xi \in E_{\mathbf{C}}. \tag{4-9}$$

While, the *T-transform* is defined by

$$T\Phi(\xi) = \langle\langle \Phi, e^{i\langle\cdot,\xi\rangle} \rangle\rangle = \int_{E^*} \Phi(x) e^{i\langle x,\xi\rangle} \mu(dx), \qquad \xi \in E_{\mathbf{C}}. \tag{4-10}$$

Of course the integral expressions are valid only when the integrands are integrable functions, in paticular when $\Phi \in (E)$. There is a simple relation:

$$T\Phi(\xi) = S\Phi(i\xi)e^{-\langle\xi,\xi\rangle/2}, \qquad S\Phi(\xi) = T\Phi(-i\xi)e^{-\langle\xi,\xi\rangle/2}, \qquad \xi \in E_{\mathbf{C}}. \tag{4-11}$$

5. Integral Kernel Operators and Fock Expansion

For $y \in E^*$ we put

$$D_y\phi(x) = \lim_{\theta\to 0} \frac{\phi(x+\theta y) - \phi(x)}{\theta}, \qquad x \in E^*, \quad \phi \in (E). \tag{5-1}$$

It is known that the limit always exists and D_y becomes a continuous operator (in fact a derivation) from (E) into itself, i.e., $D_y \in \mathcal{L}((E),(E))$. In particular, for $y = \delta_t$, we put

$$\partial_t = D_{\delta_t}, \qquad t \in T. \tag{5-2}$$

In most physical literature ∂_t is called an *annihilation operator* at point $t \in T$ and is understood to be (unbounded) operator-valued distribution. However, in our setup ∂_t is just a *continuos operator* on (E) for itself. The adjoint $\partial_t^* \in \mathcal{L}((E)^*,(E)^*)$ is therefore called *creation operator*. Note also that ∂_t is often called *Hida's differential operator* as well. That we are free from smeared creation and annihilation operators is one of the most significant features of white noise calculus.

The annihilation and creation operators satisfy the canonical commutation relation in a generalized sense:

$$[\partial_s, \partial_t] = 0, \quad [\partial_s^*, \partial_t^*] = 0, \quad [\partial_s, \partial_t^*] = \delta_s(t). \tag{5-3}$$

The precise meaning of the last identity is:

$$[D_y, D_\xi^*] = \langle y, \xi\rangle\, I, \qquad y \in E^*, \quad \xi \in E. \tag{5-4}$$

Note that both D_y and D_ξ^* belong to $\mathcal{L}((E),(E))$ and their compositions are meaningful, see (5-7) and Theorem 5.1 below.

With each $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ we may associate an *integral kernel operator* whose formal expression is given by

$$\Xi_{l,m}(\kappa) = \int_{T^{l+m}} \kappa(s_1, \cdots, s_l, t_1, \cdots, t_m)\partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m, \tag{5-5}$$

where κ is called the *kernel distribution*. More presicely, it is defined through two canonical bilinear forms:

$$\langle\langle \Xi_{l,m}(\kappa)\phi, \psi \rangle\rangle = \langle \kappa, \langle\langle \partial_{s_1}^* \cdots \partial_{s_l}^* \partial_{t_1} \cdots \partial_{t_m}\phi, \psi \rangle\rangle \rangle, \qquad \phi, \psi \in (E). \tag{5-6}$$

It is proved that $\Xi_{l,m}(\kappa) \in \mathcal{L}((E), (E)^*)$, see [8] for further details. For example,

$$\Xi_{0,1}(y) = \int_T y(t)\partial_t dt = D_y, \qquad y \in E^*. \tag{5-7}$$

THEOREM 5.1 ([8]). *Let $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$. Then $\Xi_{l,m}(\kappa) \in \mathcal{L}((E), (E))$ if and only if $\kappa \in (E_{\mathbf{C}}^{\otimes l}) \otimes (E_{\mathbf{C}}^{\otimes m})^* \cong \mathcal{L}(E_{\mathbf{C}}^{\otimes m}, E_{\mathbf{C}}^{\otimes l})$.*

By virtue of (5-3) we may assume that the kernel distribution κ is symmetric with respect to the first l and the last m variables independently. We denote by $(E_{\mathbf{C}}^{\otimes(l+m)})_{\mathrm{sym}(l,m)}^*$ the space of such distributions. The importance of an integral kernel operator is due to the following

THEOREM 5.2 ([20], [21], [22]). *For any $\Xi \in \mathcal{L}((E), (E)^*)$ there exists a unique family of kernel distributions $\kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})_{\mathrm{sym}(l,m)}^*$ such that*

$$\Xi\phi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})\phi, \qquad \phi \in (E), \tag{5-8}$$

where the right hand side converges in $(E)^*$. Moreover, if $\Xi \in \mathcal{L}((E), (E))$, then $\kappa_{l,m} \in ((E_{\mathbf{C}}^{\otimes l}) \otimes (E_{\mathbf{C}}^{\otimes m})^*)_{\mathrm{sym}(l,m)}$ and the infinite series (5-8) converges in (E).

The unique expression of $\Xi \in \mathcal{L}((E), (E)^*)$ given in Theorem 5.2 is called the *Fock expansion* of Ξ and denoted simply by

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}). \tag{5-9}$$

A few simple examples will be found in the rest of the paper, for further discussion see e.g., [20], [21], [22].

For $\Xi \in \mathcal{L}((E),(E)^*)$ a function on $E_{\mathbf{C}} \times E_{\mathbf{C}}$ defined by

$$\widehat{\Xi}(\xi,\eta) = \left\langle\!\left\langle \Xi\phi_\xi,\, \phi_\eta \right\rangle\!\right\rangle, \qquad \xi,\eta \in E_{\mathbf{C}}, \tag{5-10}$$

is called the *symbol* of Ξ. For example, for Ξ with Fock expansion (5-9) we have

$$e^{-\langle \xi,\, \eta\rangle}\widehat{\Xi}(\xi,\eta) = \sum_{l,m=0}^{\infty} \left\langle \kappa_{l,m},\, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle, \qquad \xi,\eta \in E_{\mathbf{C}}. \tag{5-11}$$

Hence, in order to find kernel distributions $\kappa_{l,m}$ from a given $\Xi \in ((E),(E)^*)$ one need only to compute the Taylor expansion of $e^{-\langle \xi,\, \eta\rangle}\widehat{\Xi}(\xi,\eta)$.

6. Infinite Dimensional Laplacians

Consider the following two integral kernel operators:

$$\Delta_G = \Xi_{0,2}(\tau) = \int_{T\times T} \tau(s,t)\partial_s\partial_t ds dt = \int_T \partial_t^2\, dt, \tag{6-1}$$

$$N = \Xi_{1,1}(\tau) = \int_{T\times T} \tau(s,t)\partial_s^*\partial_t ds dt = \int_T \partial_t^*\partial_t\, dt, \tag{6-2}$$

where $\tau \in (E\otimes E)^*_{\mathrm{sym}}$ was defined in (2-1). (It is easily verified that $\Xi_{1,1}(\tau)$ coincides with N introduced in §2.) The operators Δ_G and N are called the *Gross Laplacian* and the *number operator*, respectively. By Theorem 5.1 both Δ_G and N belong to $\mathcal{L}((E),(E))$. In fact, $\tau \in E\otimes E^*$ since the corresponding operator under the isomorphism $\mathcal{L}(E,E) \cong E\otimes E^*$ is the identity. Note that

$$\Delta_G^* = \Xi_{2,0}(\tau) \tag{6-3}$$

and that the number operator is symmetric, i.e., N^* is a continuous extension of N to $(E)^*$.

The action of the above operators on $\psi_n(x) = \left\langle :x^{\otimes n}:,\, \xi^{\otimes n}\right\rangle$, $\xi \in E_{\mathbf{C}}$, $n = 0,1,\cdots$ is easily derived:

$$\begin{aligned} N\psi_n &= N^*\psi_n = n\psi_n, \\ \Delta_G\psi_n &= n(n-1)\,\langle \xi,\, \xi\rangle\, \psi_{n-2}, \\ (\Delta_G^*\psi_n)(x) &= \left\langle :x^{\otimes(n+2)}:,\, \tau\otimes\xi^{\otimes n}\right\rangle. \end{aligned} \tag{6-4}$$

It is also possible to express these Laplacians in terms of discrete coordinate. For simplicity we put

$$D_j = D_{e_j}, \qquad j = 0,1,2,\cdots. \tag{6-5}$$

PROPOSITION 6.1. *For $\phi \in (E)$ we have*

$$\Delta_G \phi = \sum_{j=0}^{\infty} D_j^2 \phi, \qquad N\phi = \sum_{j=0}^{\infty} D_j^* D_j \phi, \tag{6-6}$$

where the right hand sides converge in (E).

By the above result together with the definitions we might be convinced that both Δ_G and N are white noise analogies of a finite dimensional Laplacian.

We then find a relation between two Laplacians N and Δ_G. It is known [**11**] that the pointwise multiplication gives rise to a continuous bilinear map from $(E) \times (E) \to (E)$. Hence each $\Phi \in (E)^*$ is identified with multiplication operator $\phi \to \Phi\phi = \phi\Phi$, $\phi \in (E)$, by

$$\langle\langle \Phi\phi, \psi \rangle\rangle = \langle\langle \Phi, \phi\psi \rangle\rangle, \qquad \phi, \psi \in (E). \tag{6-7}$$

Thus $\Phi \in \mathcal{L}((E), (E)^*)$. Moreover, $\Phi \in \mathcal{L}((E), (E))$ if and only if $\Phi \in (E)$.

LEMMA 6.2. *It holds that*

$$D_j + D_j^* = \langle x, e_j \rangle, \tag{6-8}$$

where the right hand side is identified with multiplication operator.

PROOF. For $\xi, \eta \in E_{\mathbf{C}}$ we have

$$\begin{aligned} \left\langle\left\langle (D_j + D_j^*)\phi_\xi, \phi_\eta \right\rangle\right\rangle &= \left\langle\left\langle D_j\phi_\xi, \phi_\eta \right\rangle\right\rangle + \left\langle\left\langle \phi_\xi, D_j\phi_\eta \right\rangle\right\rangle \\ &= (\langle e_j, \xi \rangle + \langle e_j, \eta \rangle) \left\langle\left\langle \phi_\xi, \phi_\eta \right\rangle\right\rangle \\ &= \langle e_j, \xi + \eta \rangle \, e^{\langle \xi, \eta \rangle}. \end{aligned}$$

On the other hand, since $\phi_\xi \phi_\eta = \phi_{\xi+\eta} e^{\langle \xi, \eta \rangle}$ we see that

$$\left\langle\left\langle \langle x, e_j \rangle \phi_\xi, \phi_\eta \right\rangle\right\rangle = \left\langle\left\langle \langle x, e_j \rangle, \phi_\xi\phi_\eta \right\rangle\right\rangle = \left\langle\left\langle \langle x, e_j \rangle, \phi_{\xi+\eta} \right\rangle\right\rangle e^{\langle \xi, \eta \rangle} = \langle e_j, \xi + \eta \rangle \, e^{\langle \xi, \eta \rangle}.$$

Combining the above two expressions, we come to

$$\left\langle\left\langle (D_j + D_j^*)\phi_\xi, \phi_\eta \right\rangle\right\rangle = \left\langle\left\langle \langle x, e_j \rangle \phi_\xi, \phi_\eta \right\rangle\right\rangle.$$

Then the assertion follows from Proposition 4.1. QED

PROPOSITION 6.3. *It holds that*

$$-N = \Delta_G - \sum_{j=0}^{\infty} \langle x, e_j \rangle D_j, \tag{6-9}$$

where $\langle x, e_j \rangle$ *is regarded as multiplication operator.*

This is an immediate consequence of Proposition 6.1 and Lemma 6.2.

7. Rotation-invariant Operators

The main purpose of this section is to characterize all rotation-invariant operators by means of Fock expansion (Theorem 5.2), though the result was first proved in [**19**] using a weaker form of Fock expansion.

We say that $\Xi \in \mathcal{L}((E),(E)^*)$ is *rotation-invariant* if

$$\Gamma(g)^* \Xi \Gamma(g) = \Xi \qquad \text{for all } g \in O(E;H). \tag{7-1}$$

Note here that $\Gamma(g) \in \mathcal{L}((E),(E))$. By definition, if Ξ is rotation-invariant, so is Ξ^*. The condition (7-1) for $\Xi \in \mathcal{L}((E),(E))$ is equivalent to the following

$$\Gamma(g)\Xi = \Xi\Gamma(g) \qquad \text{for all } g \in O(E;H), \tag{7-2}$$

which is the usual rotation-invariance.

It is rather straightforward to see that N and Δ_G are rotation-invariant. (This fact follows from Lemma 7.4 and Proposition 7.5 below.) Therefore Δ_G^* is also rotation-invariant, while N^* is the extension of N. The goal of this section is the following significant characterization of rotation-invariant operators.

THEOREM 7.1. *Let* $\Xi \in \mathcal{L}((E),(E)^*)$ *and let* $\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m})$ *be its Fock expansion. Then* Ξ *is rotation-invariant if and only if all* $\Xi_{l,m}(\kappa_{l,m})$ *are rotation-invariant.*

THEOREM 7.2. *Let* $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ *and assume that* $\Xi_{l,m}(\kappa)$ *is rotation-invariant. If* $l+m$ *is odd, then* $\Xi_{l,m}(\kappa) = 0$. *If* $l+m$ *is even, then* $\Xi_{l,m}(\kappa)$ *is a linear combination of* $(\Delta_G^*)^\alpha N^\beta \Delta_G^\gamma$ *with* α,β,γ *being non-negative integers such that* $\alpha+\beta+\gamma \le (l+m)/2$.

THEOREM 7.3. *Let* $\kappa \in (E_{\mathbf{C}}^{\otimes l}) \otimes (E_{\mathbf{C}}^{\otimes m})^*$ *and assume that* $\Xi_{l,m}(\kappa)$ *is rotation-invariant. If* $l+m$ *is odd, then* $\Xi_{l,m}(\kappa) = 0$. *If* $l+m$ *is even, then* $\Xi_{l,m}(\kappa)$ *is a linear combination of* $N^\beta \Delta_G^\gamma$ *with* β,γ *being non-negative integers such that* $\beta+\gamma \le (l+m)/2$.

In other words, any rotation-invariant operator $\Xi \in \mathcal{L}((E),(E)^*)$ is generated by Δ_G^*, Δ_G and N, and any rotation-invariant operator $\Xi \in \mathcal{L}((E),(E))$ is generated by Δ_G and N. It is also easily checked that

$$[\Delta_G, N] = 2\Delta_G. \tag{7-3}$$

Hence any product of Δ_G^*, Δ_G and N (whenever it is well defined on (E)) may be rearranged as a sum of $(\Delta_G^*)^\alpha N^\beta \Delta_G^\gamma$ with α, β, γ being non-negative integers. This is also related to the normal ordering of creation and annihilation operators.

PROOF OF THEOREM 7.1. Suppose we are given $\Xi \in \mathcal{L}((E), (E)^*)$ with Fock expansion

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{7-4}$$

where $\kappa_{l,m} \in (E_{\mathbf{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$. Since $\Gamma(g)\phi_\xi = \phi_{g\xi}$ for $\xi \in E_{\mathbf{C}}$, we obtain

$$\begin{aligned}(\Gamma(g)^*\Xi\Gamma(g))^\frown(\xi,\eta) &= \langle\langle \Xi\Gamma(g)\phi_\xi,\, \Gamma(g)\phi_\eta \rangle\rangle \\ &= \langle\langle \Xi\phi_{g\xi},\, \phi_{g\eta} \rangle\rangle \\ &= \widehat{\Xi}(g\xi, g\eta), \qquad \xi,\eta \in E_{\mathbf{C}}.\end{aligned} \tag{7-5}$$

Moreover, from (5-10) we see that

$$\begin{aligned}\widehat{\Xi}(g\xi, g\eta) &= e^{\langle g\xi, g\eta\rangle} \sum_{l,m=0}^{\infty} \left\langle \kappa_{l,m},\, (g\eta)^{\otimes l} \otimes (g\xi)^{\otimes m} \right\rangle \\ &= e^{\langle \xi, \eta\rangle} \sum_{l,m=0}^{\infty} \left\langle (g^{\otimes(l+m)})^*\kappa,\, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle.\end{aligned} \tag{7-6}$$

It then follows from (7-5) and (7-6) that

$$\Gamma(g)^*\Xi\Gamma(g) = \sum_{l,m=0}^{\infty} \Xi_{l,m}\left((g^{\otimes(l+m)})^*\kappa_{l,m}\right)$$

is the Fock expansion. In particular,

$$\Gamma(g)^*\Xi_{l,m}(\kappa)\Gamma(g) = \Xi_{l,m}\left((g^{\otimes(l+m)})^*\kappa\right).$$

It then follows from the uniqueness of the Fock expansion that Ξ is rotation-invariant if and only if $\Xi_{l,m}(\kappa_{l,m})$ is rotation-invariant for all $l, m = 0, 1, 2, \cdots$. QED

We say that $F \in (E_{\mathbf{C}}^{\otimes n})^*$ is *rotation-invariant* if $(g^{\otimes n})^*F = F$ for all $g \in O(E; H)$. During the proof of Theorem 7.1 we have established the following

LEMMA 7.4. *Let $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$. Then $\Xi_{l,m}(\kappa)$ is rotation-invariant if and only if κ is rotation-invariant.*

Thus the proofs of Theorems 7.2 and 7.3 are essentially reduced to listing up the rotation-invariant distributions. The full list is, in fact, described satisfactorily as below, though the long combinatorial proof is omitted, see [**19**].

PROPOSITION 7.5. *Assume that* $F \in (E_{\mathbf{C}}^{\otimes n})^*$ *is rotation-invariant. If* n *is odd, then* $F = 0$. *If* n *is even, say* $n = 2m$, *then* F *is a linear combination of* $(\tau^{\otimes m})^\sigma$, $\sigma \in \mathfrak{S}_n$. *Moreover, the dimension of rotation-invariant distributions in* $(E_{\mathbf{C}}^{\otimes n})^*$ *is* $(n-1)!!$.

Here is notation. For $F \in (E_{\mathbf{C}}^{\otimes n})^*$ and $\sigma \in \mathfrak{S}_n$ we define F^σ by

$$\langle F^\sigma, \xi_1 \otimes \cdots \otimes \xi_n \rangle = \left\langle F, \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)} \right\rangle, \qquad \xi_1, \cdots, \xi_n \in E_{\mathbf{C}}.$$

PROOF OF THEOREM 7.2. Let $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ and suppose that $\Xi_{l,m}(\kappa)$ is rotation-invariant. Then, κ is rotation-invariant by Lemma 7.4. If $l+m$ is odd, it follows from Proposition 7.5 that $\kappa = 0$ and hence $\Xi_{l,m}(\kappa) = 0$.

We next consider the case when $l+m$ is even. It follows again from Proposition 7.5 that κ is a linear combination of $(\tau^{\otimes(l+m)/2})^\sigma$, $\sigma \in \mathfrak{S}_{l+m}$. For each $\sigma \in \mathfrak{S}_{l+m}$ we may find $\sigma' \in \mathfrak{S}_l \times \mathfrak{S}_m$ such that

$$\begin{aligned}
&(\tau^{\otimes(l+m)/2})^{\sigma\sigma'} \\
&= \sum e_{i_1}^{\otimes 2} \otimes \cdots \otimes e_{i_\alpha}^{\otimes 2} \otimes e_{j_1} \otimes \cdots \otimes e_{j_\beta} \otimes e_{j_1} \otimes \cdots \otimes e_{j_\beta} \otimes e_{k_1}^{\otimes 2} \otimes \cdots \otimes e_{k_\gamma}^{\otimes 2} \\
&= \tau^{\otimes \alpha} \otimes \lambda_\beta \otimes \tau^{\otimes \gamma}
\end{aligned}$$

for some non-negative integers α, β, γ with $2\alpha + \beta = l$ and $2\gamma + \beta = m$, where

$$\lambda_\beta = \sum_{j_1, \cdots, j_\beta = 0}^{\infty} e_{j_1} \otimes \cdots \otimes e_{j_\beta} \otimes e_{j_1} \otimes \cdots \otimes e_{j_\beta}.$$

In view of (6-1) we have

$$\Delta_G = \Xi_{0,2}(\tau) = \sum_{j=0}^{\infty} \Xi_{0,2}(e_j \otimes e_j), \qquad \Delta_G^* = \Xi_{2,0}(\tau) = \sum_{j=0}^{\infty} \Xi_{2,0}(e_j \otimes e_j).$$

Then a straightforward computation implies that

$$\Xi_{l,m}((\tau^{\otimes(l+m)/2})^\sigma) = \Xi_{l,m}((\tau^{\otimes(l+m)/2})^{\sigma\sigma'}) = (\Delta_G^*)^\alpha \Xi_{\beta,\beta}(\lambda_\beta) \Delta_G^\gamma.$$

Note that $\Xi_{\beta,\beta}(\lambda_\beta)$ is a polynomial of the number operator N of degree β, in fact,

$$\Xi_{\beta,\beta}(\lambda_\beta) = N(N-1)\cdots(N-(\beta-1)).$$

Hence $\Xi_{l,m}((\tau^{\otimes(l+m)/2})^\sigma)$ is a linear combination of $(\Delta_G^*)^\alpha N^\beta \Delta_G^\gamma$ with $\alpha + \beta + \gamma \leq (l+m)/2$ and therefore, so is $\Xi_{l,m}(\kappa)$. QED

For the proof of Theorem 7.3, taking Theorem 7.2 into account, we only need to show that Δ_G^* does not appear if $\Xi \in \mathcal{L}((E),(E))$. But this is an easy consequence of the fact (see (6-4)) that $\Delta_G^*\phi \notin (E)$ for any $\phi \in (E)$ with $\phi \neq 0$.

By Theorem 3.2 any bounded operator on (L^2) commuting with all $\Gamma(g)$, $g \in O(E;H)$, is a function of the number operator N. However, it is clear that the Gross Laplacian is not a function of N, see e.g., (6-4). In other words, the Gross Laplacian can not be grasped whenever we restrict ourselves to operators on (L^2). This is also illustrated by the fact that $\Delta_G^* = 0$ on its proper L^2-domain, i.e., on the space of all $\phi \in (E)$ with $\Delta_G^*\phi \in (L^2)$.

We here recall Umemura's heuristic argument of deriving the number operator from finite dimensional Laplacians. First the finite dimensional Laplacian $\sum_{j=1}^{D} \partial^2/\partial x_j^2$ should be modified using the mapping $\phi \mapsto (2\pi)^{D/2}e^{|x|^2/4}\phi$ which is a unitary isomorphism from $L^2(\mathbf{R}^D, dx)$ onto the L^2-space over $\mathbf{R}^D$ with Gaussian measure. The resultant expression is;

$$\text{(7-7)} \qquad \sum_{j=1}^{D}\left(\frac{\partial^2}{\partial x_j^2} - x_j\frac{\partial}{\partial x_j} + \frac{x_j^2}{4} - \frac{1}{2}\right).$$

(E^* is a projective limit of $\mathbf{R}^D$ with *Gaussian measure.*) Then, taking the "convergent terms," Umemura defined an infinite dimensional Laplacian by

$$\text{(7-8)} \qquad \Delta = \sum_{j=1}^{\infty}\left(\frac{\partial^2}{\partial x_j^2} - x_j\frac{\partial}{\partial x_j}\right).$$

This operator acts on cylindrical functions of the form:

$$\phi(x) = f(\langle x, e_1\rangle, \cdots, \langle x, e_n\rangle), \qquad x \in E^*,$$

where $x_j = \langle x, e_j\rangle$. It is then easily verified that $\Delta = -N$. In fact, (7-8) is comparable to (6-9).

Umemura [**27**] showed that N (or equivalently Δ) is the essentially unique rotation-invariant operator. However, as was shown above, within white noise calculus N is decomposed into two rotation-invariant operators. Furthermore, we shall see that the "divergent terms" in (7-7) involves another rotation-invariant operator. Consider white noise analogue of the Euclidean norm:

$$R(x) = \left\langle :x^{\otimes 2}:, \tau\right\rangle.$$

This is a generalized white noise functional (see §4) and admits another expression:

$$\begin{aligned} R(x) &= \sum_{j=0}^{\infty} \left\langle :x^{\otimes 2}:, \, e_j \otimes e_j \right\rangle \\ &= \sum_{j=0}^{\infty} \left(\langle x \otimes x, \, e_j \otimes e_j \rangle - \langle \tau, \, e_j \otimes e_j \rangle \right) \\ &= \sum_{j=0}^{\infty} \left(\langle x, \, e_j \rangle^2 - 1 \right). \end{aligned}$$

Then, it is apparent that R is involved in the "divergent terms" of (7-7). Moreover, as multiplication operator, R is related to the Laplacians:

$$R = 2N + \Delta_G + \Delta_G^*.$$

We have thus observed an interesting contrast between rotation-invariant operators on white noise functionals and those on a finite dimensional Euclidean space.

8. Regular One-parameter Subgroups

We begin with general notion. Let $\mathfrak{X}$ be a nuclear Fréchet space with defining Hilbertian seminorms $\{\|\cdot\|_\alpha\}_{\alpha \in \mathcal{A}}$, taking $\mathfrak{X} = E$ or $\mathfrak{X} = (E)$ into consideration. Let $GL(\mathfrak{X})$ be the group of all linear homeomorphisms from $\mathfrak{X}$ onto itself. A one-parameter subgroup $\{g_\theta\}_{\theta \in \mathbf{R}} \subset GL(\mathfrak{X})$ is called *differentiable* if

$$X\xi = \lim_{\theta \to 0} \frac{g_\theta \xi - \xi}{\theta} \tag{8-1}$$

converges in $\mathfrak{X}$ for any $\xi \in \mathfrak{X}$. In that case X becomes a linear operator from $\mathfrak{X}$ into itself and, as usual, is called the *infinitesimal generator* of $\{g_\theta\}_{\theta \in \mathbf{R}} \subset GL(\mathfrak{X})$.

It is known that a subset of a nuclear space is compact if and only if it is closed and bounded. Then simple application of the Banach-Steinhaus theorem leads us to the following

LEMMA 8.1. *Let $\{g_\theta\}_{\theta \in \mathbf{R}} \subset GL(\mathfrak{X})$ be a differentiable one-parameter subgroup. Then its infinitesimal generator X is always continuous, i.e., $X \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$. Moreover, the convergence (8-1) is uniform on every compact (or equivalently, bounded) subset of $\mathfrak{X}$, namely,*

$$\lim_{\theta \to 0} \sup_{\xi \in K} \left\| \frac{g_\theta \xi - \xi}{\theta} - X\xi \right\|_\alpha = 0 \tag{8-2}$$

for any $\alpha \in \mathcal{A}$ *and any compact (or bounded) subset* $K \subset \mathfrak{X}$.

By a standard argument one may prove the uniqueness of an infinitesimal generator of a differentiable one-parameter subgroup. However, in general, not every $X \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ can be an infinitesimal generator of a differentiable one-parameter subgroup of $GL(\mathfrak{X})$. We give here a sufficient condition.

PROPOSITION 8.2. *Let* $X \in \mathcal{L}(\mathfrak{X}, \mathfrak{X})$ *and assume that there exists* $r > 0$ *such that* $\{(rX)^n/n!\}_{n=0}^{\infty}$ *is equicontinuous, namely, for every* $\alpha \in \mathcal{A}$ *there exist* $C = C(\alpha) \geq 0$ *and* $\beta = \beta(\alpha) \in \mathcal{A}$ *such that*

$$\sup_{n\geq 0} \frac{1}{n!} \| (rX)^n \xi \|_{\alpha} \leq C \| \xi \|_{\beta}, \qquad \xi \in \mathfrak{X}.$$

Then there exists a differentiable one-parameter subgroup $\{g_\theta\}_{\theta\in\mathbf{R}} \subset GL(\mathfrak{X})$ *with infinitesimal generator* X.

In that case we observe a stronger property than stated in (8-2): for any $\alpha \in \mathcal{A}$ there exists $\beta \in \mathcal{A}$ such that

$$\lim_{\theta\to 0} \sup_{\| \xi \|_{\beta}\leq 1} \left\| \frac{g_\theta \xi - \xi}{\theta} - X\xi \right\|_{\alpha} = 0.$$

Such a differentiable one-parameter subgroup $\{g_\theta\}_{\theta\in\mathbf{R}} \subset GL(\mathfrak{X})$ is called *regular*. Although it is not yet clear whether the notion of a regular one-parameter subgroup plays an essential role in white noise calculus, we feel it practically useful.

Here are simple examples in case of $\mathfrak{X} = (E)$.

EXAMPLE 8.3 (TRANSLATION OPERATOR). For $y \in E^*$ we put

$$T_y\phi(x) = \phi(x+y), \qquad x \in E^*, \quad \phi \in (E).$$

It is known that $T_y \in \mathcal{L}((E),(E))$. Moreover, $\{T_{\theta y}\}_{\theta\in\mathbf{R}}$ is a regular one-parameter subgroup of $GL((E))$ with infinitesimal generator D_y. Incidentally, the Fock expansion of T_y is given as

$$T_y = \sum_{n=0}^{\infty} \frac{1}{n!} \Xi_{0,n}(y^{\otimes n}).$$

Since $\Xi_{0,n}(y^{\otimes n}) = D_y^n$, it follows from Theorem 5.2 that

$$\phi(x+y) = \sum_{n=0}^{\infty} \frac{1}{n!} (D_y \phi)(x),$$

where the series converges in (E) and therefore pointwisely as well. This is the Taylor expansion of $\phi \in (E)$.

EXAMPLE 8.4 (WEYL FORM OF CCR). For $\xi \in E$ we define

$$\begin{cases} P_\xi\phi(x) = \exp\left(-\frac{1}{2}\langle x, \xi\rangle - \frac{1}{4}\langle \xi, \xi\rangle\right)\phi(x+\xi), \\ Q_\xi\phi(x) = e^{i\langle x, \xi\rangle}\phi(x). \end{cases} \tag{8-4}$$

It can be checked that both belong to $GL((E))$ and give rise to unitary representations of the additive group E. Moreover, put

$$\begin{cases} p_\xi = \frac{1}{2}(D_\xi - D_\xi^*), \\ q_\xi = i(D_\xi + D_\xi^*). \end{cases}$$

Then, $\{P_{\theta\xi}\}_{\theta\in\mathbf{R}}$ and $\{Q_{\theta\xi}\}_{\theta\in\mathbf{R}}$ are regular one-parameter subgroups of $GL((E))$ with infinitesimal generators p_ξ and q_ξ, respectively.

We have introduced white noise coordinate system $\{x(t)\}$ in §4. By a similar argument as in the proof of Lemma 6.2 one can prove easily that

$$x(t) = \partial_t + \partial_t^*, \qquad t \in T, \tag{8-5}$$

where $x(t)$ is regarded as multiplication operator. Hence a white noise analogy of an infinitesimal generator of finite dimensional rotations is given as

$$x(s)\partial_t - x(t)\partial_s = (\partial_s^* + \partial_s)\partial_t - (\partial_t^* + \partial_t)\partial_s = \partial_s^*\partial_t - \partial_t^*\partial_s. \tag{8-6}$$

This is, in fact, an operator in $\mathcal{L}((E),(E)^*)$ and we shall investigate its definite meaning in Theorem 8.6 below.

For $X \in \mathcal{L}(E_{\mathbf{C}}, E_{\mathbf{C}})$ we define an operator $d\Gamma(X)$ as follows. Suppose that $\phi \in (E)$ is given as

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f_n\right\rangle, \qquad x \in E^*.$$

Then we put

$$d\Gamma(X)\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \gamma_n(X)f_n\right\rangle,$$

where

$$\begin{cases} \gamma_n(X) = \sum_{k=0}^{n-1} I^{\otimes k} \otimes X \otimes I^{\otimes(n-1-k)}, & n \ge 1, \\ \gamma_0(X) = 0. \end{cases}$$

It is checked easily that $d\Gamma(X) \in \mathcal{L}((E),(E))$. Formally, $d\Gamma(X)$ is an infinitesimal generator of $\{\Gamma(g_\theta)\}_{\theta\in\mathbf{R}}$, where $\{g_\theta\}_{\theta\in\mathbf{R}}$ is a one-parameter subgroup with X being the infinitesimal generator. However, it is not clear whether or not $\{\Gamma(g_\theta)\}_{\theta\in\mathbf{R}}$ becomes a differentiable one-parameter subgroup of $GL((E))$ for any differentiable one-parameter subgroup $\{g_\theta\}_{\theta\in\mathbf{R}} \subset GL(E)$. In this connection regularity introduced above seems useful. In fact, we have the following result.

LEMMA 8.5. *If $\{g_\theta\}_{\theta\in\mathbf{R}}$ is a regular one-parameter subgroup of $GL(E)$ with infinitesimal generator X, then $\{\Gamma(g_\theta)\}_{\theta\in\mathbf{R}}$ is a regular one-parameter subgroup of $GL((E))$ with infinitesimal generator $d\Gamma(X)$.*

For the proof we need a long calculation, see [8]. As is easily seen, the infinitesimal generator X of $\{g_\theta\}_{\theta\in\mathbf{R}} \subset O(E;H)$ is skew-symmetric in the sense that

$$\langle X\xi, \eta\rangle = -\langle \xi, X\eta\rangle, \qquad \xi,\eta \in E.$$

Hence by a simple argument one comes to the following result including the meaning of $x(s)\partial_t - x(t)\partial_s$ introduced in (8-6).

THEOREM 8.6 ([8]). *Let $\{g_\theta\}_{\theta\in\mathbf{R}}$ be a regular one-parameter subgroup of $O(E;H)$ with infinitesimal generator X. Then, $\{\Gamma(g_\theta)\}_{\theta\in\mathbf{R}}$ is a regular one-parameter subgroup of $GL((E))$ with infinitesimal generator $d\Gamma(X)$. Moreover, there exists a skew-symmetric distribution $\kappa \in E \otimes E^*$ such that*

$$d\Gamma(X) = \int_{T\times T} \kappa(s,t)(\partial_s^*\partial_t - \partial_t^*\partial_s)\, dsdt.$$

9. Further Topics

Group of Diffeomorphisms

The proof of characterizing the rotation-invariant operators (see §7) owes essentially to Proposition 7.5. Although we omitted the proof, it requires only a subgeroup of $O(E;H)$ consisting of rotations g such that $ge_j = e_j$ except finitely many e_j, namely, which act identically on the subspace generated by $\{e_j, e_{j+1}, \cdots\}$ for some j. Hence it is interesting to investigate operators which are invariant under another subgroups of $O(E;H)$.

One of the most interesting would be the case of T being a (Riemannian) manifold with smooth (Riemannian) volume as measure ν. A diffeomorphism γ of T is called *admissible* to the Gelfand triple $E \subset L^2(T,\nu) \subset E^*$ or to the operator A on $L^2(T,\nu)$ if

$$g_\gamma\xi(t) = \left(\frac{d\nu(\gamma^{-1}t)}{d\nu(t)}\right)^{1/2} \xi(\gamma^{-1}t), \qquad \xi \in E,$$

gives rise to an infinite dimensional rotation $g_\gamma \in O(E;H)$. In other words, γ is admissible if E is stable under g_γ. We denote by $\mathrm{Diff}_A(T)$ the group of admissible diffeomorphisms of T. Then it would be very interesting to investigate $\mathrm{Diff}_A(T)$-invariant operators in $\mathcal{L}((E),(E)^*)$. The study of $\mathrm{Diff}_A(T)$ as subgroup of $O(E;H)$ is also deeply connected with unitary representation theory of a diffeomorphism group, see e.g., [**2**].

In the special case of $T = \mathbf{R}$ with Lebesgue measure and $A = 1 + t^2 - d^2/dt^2$, we see that

$$\int_{\mathbf{R}} \partial_t \, dt.$$

is invariant under $\mathrm{Diff}_A(T)$ as well as N and Δ_G. We conjecture that the converse is also true.

Kuo's Fourier Transform

As in the case of finite dimension "Fourier transform" should be important in harmonic analysis on Gaussian space. One might think that T-transform introduced in §4 would be one of the candidates of Fourier transform on Gaussian space. However, T-transform is not a mapping from $(E)^*$ into itself and therefore we can not discuss the relation with differential operators, multiplication opertors, Laplacians and rotations.

Answering a question posed by Hida [**3**], [**4**], about a decade ago Kuo invented a Fourier transform by formal calculus and proved that it intertwines differential operators and multiplication operators as usual Fourier transform:

$$\mathfrak{F}\partial_t = ix(t)\mathfrak{F}, \qquad \mathfrak{F}x(t) = i\partial_t\mathfrak{F},$$

in a slightly formal form, see e.g., [**6**] for a precise statement. There is now a firm ground for Kuo's Fourier transform (see [**13**]) and $\mathfrak{F} = T^{-1}S$ is one of the equivalent definitions of Kuo's Fourier transform, for T- and S-transforms see §4. Finally we note that Kuo's Fourier transform is the unique (up to a constant factor) continuous linear operator on $(E)^*$ which possesses the intertwing property mentioned above. The constant is determined, for example by $\mathfrak{F}1 = \delta_0$, see [**6**] for details.

Volterra Laplacian and Lévy Laplacian

In the eary years of this century Volterra, Gâteaux and Lévy discussed "Laplacians" acting on functions of infinitely (or rather continuously) many variables, see the book of Lévy [**15**]. Later on various attempts have been made to reformulate their works with modern language, namely, within the framework of Hilbert spaces or Banach spaces. It seems also interesting to discuss those operators whithin our setup.

Let F be a $\mathbf{C}$-valued function on E of C^2-class in the sense of Fréchet. Since E is nuclear, for each $\xi \in E$ there exists $F''(\xi) \in (E \otimes E)^*$ such that

$$\frac{d^2}{d\theta^2}\bigg|_{\theta=0} F(\xi + \theta\eta) = \left\langle F''(\xi), \eta \otimes \eta \right\rangle, \qquad \xi, \eta \in E.$$

Then F is called an *LV-functional* if F'' has a special form:

$$\left\langle F''(\xi), \eta \otimes \zeta \right\rangle = \int_T F''_{\mathrm{sing}}(\xi; t)\eta(t)\zeta(t)dt + \int_{T\times T} F''_{\mathrm{reg}}(\xi; s, t)\eta(s)\zeta(t)dsdt,$$

where $F''_{\mathrm{sing}}(\xi; \cdot) \in L^1_{\mathrm{loc}}(T)$ and $F''_{\mathrm{reg}}(\xi; \cdot, \cdot) \in L^1_{\mathrm{loc}}(T \times T)$. We call F''_{sing} the ***singular part*** and F''_{reg} the ***regular part*** of F''.

Let F be an LV-functional. If the regular part of F'' defines a trace class operator on H, we define

$$\Delta_V F(\xi) = \mathrm{Trace}\, F''_{\mathrm{reg}}(\xi) = \int_T F''_{\mathrm{reg}}(\xi; t, t)\, dt,$$

where the integral expression is valid under certain regularity condition. While, if $F''_{\mathrm{sing}}(\xi; \cdot) \in L^1(T)$, we put

$$\Delta_L F(\xi) = \int_T F''_{\mathrm{sing}}(\xi; t)\, dt.$$

The operators Δ_V and Δ_L are called *Volterra Laplacian* and *Lévy Laplacian*, respectively.

Recall that the S-transform of $\Phi \in (E)^*$, denoted by $S\Phi$, is a $\mathbf{C}$-valued function on $E_{\mathbf{C}}$ and therefore on E by restriction. Thus we may discuss the actions of Δ_V and Δ_L on white noise functionals and obtain

$$\Delta_V S\phi(\xi) = S\Delta_G \phi(\xi), \qquad \Delta_L S\phi(\xi) = 0, \qquad \xi \in E, \quad \phi \in (E).$$

Thus we understand that the Volterra Laplacian is an extension of Δ_G. While, it is further proved that the Lévy Laplacian acts as zero operator on (L^2). However, it is known that Δ_L acts effectively on a space of generalized white noise functionals.

The Lévy Laplacian is also connected with "asymptotic spherical mean" on Hilbert space and this justifies the name of Laplacian, see [**17**]. In their quite recent paper Accardi, Gibilisco and Volovich [**1**] investigate a relation between the Lévy Laplacian and Yang-Mills equations. These works suggest that the Lévy Laplacian plays a more interesting role in infinite dimensional harmonic analysis than we have expected so far.

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For more complete information on white noise calculus and related topics, see [**7**], [**21**], [**22**] and references cited therein.

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