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Toeplitz Operators and Models for Pseudo-differential Algebras

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In these notes we show examples of operator algebras which have the same symbolic calculus as the algebra of pseudo-differential operators. Some of these examples are comparatively older (Toeplitz operators, operators "of polynomial type" on $\mathbf{R}^n$). The last example, dealing with functions of exponential type, is more recent and was studied by D. Nguon in his thesis.

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1. The symbolic calculus.

In the first paragraph we describe what is expected from the symbolic calculus of our pseudo-differential models.

a. Symplectic cones.

The first ingredient is a symplectic cone Σ, ie. Σ is a smooth (paracompact) manifold equipped with a free action of the multiplicative group $\mathbf{R}_+^\times$, and with a symplectic form ω, homogeneous of degree 1.

Thus Σ is isomorphic (as a $\mathbf{R}_+^\times$-manifold) to a product $S\times\mathbf{R}_+^\times$, where $S=\Sigma/\mathbf{R}_+^\times$ is the basis. Since ω is homogeneous, there is a unique horizontal, homogeneous 1-form λ such that $d\lambda = \omega$: the Liouville form $\lambda = \rho \llcorner \omega$ where ρ is the ifinitesimal generator of the action of $\mathbf{R}_+^\times$.

In the examples described below the basis $S=\Sigma/\mathbf{R}_+^\times$ will always be compact.

For the standard model of pseudodifferential operators on a manifold X, we have $\Sigma = T^*X-\{0\}$, the cotangent bundle minus its zero section, and if x_j are local coordinates on X and ξ_j the dual coordinates on the fibers, we have (locally) $\lambda = \Sigma\, \xi_j\, dx_j$, $\omega = \Sigma\, d\xi_j\, dx_j$. In the general case Σ is locally (ie. above small sets of its basis S) isomorphic to this canonical model.

Let us recall that the Hamiltonian field H_f and the Poisson bracket $\{f,g\}$ (for smooth functions f and g) are defined by

$$H_f \llcorner \omega = - df$$
$$\{f, g\} = H_f\, g$$

(for the standard model $\{f,g\} = \Sigma\, \partial f/\partial\xi_j\, \partial g/\partial x_j - \partial f/\partial x_j\, \partial g/\partial\xi_j$)

b. Scale of Sobolev spaces and algebra of operators.

The operators will act on a scale H^s of Hilbert spaces, mimicking the scale $H^s(X)$ on a compact manifold. The base space H^0 mimicks L^2 and H^s is the domain of A^s where A is a suitable self adjoint positive unbounded operator (resp. the range of A^{-s} if $s<0$; we can choose $A=(1-\Delta)^{1/2}$ for the usual Sobolev spaces).

An algebra of operators of the type we consider is a filtered algebra $A=\cup A_n$ ($n\in \mathbf{Z}$), together with an isomorphism (symbol map) of A on the graded algebra $O(\Sigma)$ of smooth homogeneous functions of integral degree on Σ [1]. If $P\in A_m$ we will usually denote $\sigma(P)$ or $\sigma_m(P)\in O_m(\Sigma)$ (the symbol of P) the image in $O(\Sigma)$ of the class of P in grA. The fact that we have an isomorphism means that

$\sigma_m(P)$ is of degree m if $P\in A_m$

$\sigma_m(P) = 0$ means that P is of degree $\leq m-1$ ($P\in A_{m-1}$)

$\sigma_{p+q}(PQ) = \sigma_p(P)\ \sigma_q(Q)$ if P and Q are of respective degree p and q.

We further require the following law for commutators:

$$\sigma_{p+q-1}([P,Q])= -i\ \{\sigma_p(P), \sigma_q(Q)\} \tag{1.1}$$

One also makes some completeness assumptions on A: the weakest is that any elliptic operator P (ie. such that the symbol σ_p is invertible) is invertible mod. operators of degree $-\infty$. A stronger condition, which will be true for all the models below (but would have to be suitably modified if one wanted to describe analytic operators) is that A is complete for the topology defined by its filtration (ie. if P_k is any sequence such that deg $P_k \rightarrow -\infty$, there exists $P\in A$ such that $\deg(P-\Sigma_{j\leq k} P_k \rightarrow -\infty)$.

We may consider such a data (the scale H^s and the algebra A) as a "quantization" of the symplectic cone Σ (or the contact manifold S). A "quantized canonical transformation" would then be a lifting of an isomorphism of symplectic cones. Such liftings play the same role as elliptic Fourier integral operators. It has been shown in Boutet de Monvel-Guillemin (see also Boutet de Monvel [3]) that for any real symplectic cone Σ there exists such a scale of Hilbert spaces and an algebra of operators giving rise to symbolic calculus as described above (in fact the operators areFourier integral operators belonging to a self reproducing complex canonical relation suitably related to the contact structure of the basis S, the scale of Hilbert spaces is a subscale of the Sobolev scale of S, and the whole construction is unique up to isomorphism and mod. operators of finite rank; it can further be made equivariant under a compact group action).

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[1] For the symbolic calculus it is sometimes convenient to replace Σ by the pure imaginary part $i\Sigma$ of its "complexification" $\Sigma \times_{\mathbf{R}_+^\times} \mathbf{C}^\times$.

2. The model of standard pseudo-differential operators.

For the standard model of pseudodifferential operators, we have a smooth compact manifold X, and $\Sigma = T^*X-\{0\}$ is the cotangent bundle minus its zero section. H^s is the scale of Sobolev spaces, and $A = \cup A_k$ is the algebra of pseudodifferential operators (we limit ourselves to those which are regular, ie> whose total symbol in any set of local coordinates has an asymptotic expansion

(2.1) $a(x,\xi) \sim \Sigma\, a_k(x,\xi)$ where a_k is homogeneous of degree k w.r.t. to ξ and k ranges among integers $\leq m$ = degree of a.

The microsupport SSf of a distribution f can be naively defined as follows: f will be said to be smooth at a cotangent vector $\xi \in T^*X$ if there exists a pseudodifferential oprator A elliptic at ξ (ie. whose symbol does not vanish there) such that $Af = 0$; SSf is the closed conic set of points at which f is not smooth.

From there one can "microlocalize" pseudodifferential operators and distributions: microdifferential operators and microfunctions form sheaves on $T^*X-\{0\}$, whose set of germs at a cotangent vector ξ are the set of pseudodifferential operators, resp. distributions, whose total symbol is of degree $-\infty$, (for the study of analyticity these definitions should be changed). In fact it is possible to microlocalize in this manner all models, and all models are microlocally (not globally) isomorphic to the standard one.

3. The model of polynomial pseudo-differential operators.

We may, as did I.N.Bernstein, count the degrees of pseudodifferential operators on $\mathbf{R}^n$ differently. We will note x, ξ the points of $T^*\mathbf{R}^n = \mathbf{R}^{2n}$. The canonical symplectic form is again $\omega = \Sigma\, d\xi_j\, dx_j$. But now we decide that x and ξ have both the same degree 1/2 (so that ω is still of degree 1). Σ is the cone $\mathbf{R}^{2n}$-{the origin}; its basis is the sphere S^{2n-1}; the Liouville form is now $1/2\,(\Sigma\, \xi_j\, dx_j - x_j\, d\xi_j)$.

In this model pseudodifferential operators are defined by the usual formula:

(3.1) $$a(x,D)f = \int e^{ix\xi}\, a(x,\xi)\, d\xi$$

where the total symbol $a(x,\xi)$ is a smooth function on $\mathbf{R}^{2n}$ admitting, for $(x,\xi) \to \infty$, an asymptotic expansion:

(3.2) $$a \sim \Sigma\, a_k(x, \xi)$$

with a_k homogeneous of degree k when x and ξ are both attributed the degree 1/2 (ie. a_k is homogeneous of degree 2k in the usual sense), and k ranges among integers $\leq m$ = degree of a. For example an even differential operator with polynomial coefficients is a "polynomial pseudodifferential operator" in the sense above. Eg. the harmonic oscillator

$$A = 1/2(-\Delta + \|x\|^2) \tag{3.3}$$

is in this context an elliptic operator of degree 1, and in fact its eigenvalues have the same order of magnitude as those of elliptic operators of degree 1 on n-dimensional manifolds. (It would seem reasonable to modify the definition and allow half-integers in the expansion (3.2) above - but in fact mixing even and odd operators would take us away from the standard pseudodifferential model, just as would the adjunction of operators of half-integral degree; for example the set of equations

$$(\partial_j - a_j)\, f = 0\,, \quad j = 1\ldots n \quad (a_j = \text{constants}) \tag{3.4}$$

always define a holonomic module, because the fact of being holonomic does not depend on the choice of degrees or filtrations; but the fact of being regular holonomic does and in the present context the D-module corresponding to the equations above is regular if and only if $a_j = 0$ for all $j = 1\ldots n$.

The analogue of the Sobolev space is the domain H^s of the power A^s of the harmonic oscillator A (the dual for s<0). Thus for s a positive integer, H^s is the space of functions f such that $x^\alpha f \in L^2$ and $\partial^\alpha f \in L^2$ for $|\alpha| \leq 2s$. The intersection H^∞ is the Schwartz space of rapidly decreasing functions, and the union $H^{-\infty}$ is the space of tempered distributions.

The definition of the microsupport SSf is copied from the standard one in this situation; it describes a compromise between the regularity of f and its growth at infinity (and is not much related to the standard one where x is of degree 0 and ξ of degree 1).

In this model, the creation and annihilation operators constitute useful tools. We recall that they are defined by

$$Z_j = \frac{x_j - \partial/\partial x_j}{\sqrt{2}} \quad \text{(creation operators)} \tag{3.5}$$
$$Z_j^* = \frac{x_j - \partial/\partial x_j}{\sqrt{2}} \quad \text{(annihilation operators)}$$

The Hermite functions are eigenfunctions of the harmonic operator $(A=\sum Z_j Z_j^* + n/2)$, and form a useful orthogonal basis of $H^0 = L^2$. Up to constant factors they are given by

(3.6) $h_\alpha = Z^\alpha e^{-x^2/2}$ (with $Z^a = \prod Z_j^{\alpha_j}$)

4. Toeplitz operators.

Let H be the Hilbert space of square integrable functions $f = \sum_0^\infty a_n z^n$ on the unit circle $S^1 \subset \mathbf{C}$ which extend holomorphically to the disc, and let a be a continuous function on S^1. Classically the Toeplitz operator of symbol a is defined is the operator $T_a \in L(H)$ is defined by

(4.1) $T_a f = S\, af$

where S is the orthogonal projector (Szegö projector) $L^2(S^1) \rightarrow H$. It is easy to prove that the C*-algebra $\mathcal{T}$ generated by Toeplitz operators contains the ideal $\mathcal{K}$ of all compact operators, and that $a \rightarrow T_a$ mod. compact operators defines an isomorphism of $C(S^1)$ on $\mathcal{T}/\mathcal{K}$ (symbolic calculus).

In the definition of several variable Toeplitz operators we make stronger smoothness assumptions on the symbols a to get a more precise symbolic calculus: let Ω be a relatively compact strictly pseudoconvex domain with smooth boundary in $\mathbf{C}^n$ [2]. We denote $\mathcal{O}^s(\partial\Omega)$ the space of boundary values of holomorphic functions on Ω which belong to the Sobolev space $H^s(\partial\Omega)$. If Q is pseudo-differential operator of degree m on $\partial\Omega$, let T_Q denote the operator $f \rightarrow S(Qf)$, which is continuous $\mathcal{O}^s \rightarrow \mathcal{O}^{s-n}$ for all s (if Q is the multiplication by a smooth function a on $\partial\Omega$, this is the straightforward n-variable generalization of the one variable Toeplitz operators above) [3].

We may define $\Sigma \subset T^*\partial\Omega$ as the cone of positive multiples of $-i\partial u$, where $u<0$ is a defining inequation for Ω as in footnote 1: Σ is the microsupport of singularities of

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[2] This means that Ω can be defined by an inequation $u<0$, where u is a smooth real function, $du \neq 0$ on $\partial\Omega$, and the Levi matrix $\partial^2 u/\partial z_j\, \partial \bar{z}_k$ is hermitian positive. More generally Ω can be a relatively compact, strictly pseudoconvex domain with smooth boundary in a Stein space, which allows a finite number of singularities

[3] It is also natural to consider Toeplitz operators such as T_a on the space $\mathcal{O}^0(\partial\Omega)$ of sqare integrable, boundary values of holomorphic functions, using the Szegö projector S, or similarily on the space $\mathcal{O}^0(\Omega)$ of square integrable holomorphic functions on Ω, using the Bergman projector. In this context it is natural to assume that Ω is pseudoconvex, but one does not get in general a "nice" symbolic calculs: commutators are not necessarily compact (e.g. they are not if Ω is a plolydisc - in fact compacity requires something like a subelliptic estimate for the $\bar\partial$-Neuman problem); for a full symbolic calculus including th Poisson bracket rule for commutators, strict pseudo-convexity is obviously required.

boundary values of holomorphic functions; the strict pseudoconvexity assumption implies that it is symplectic. The scale of "Sobolev spaces" is the scale $\mathcal{O}^s$ above. It was shown in [Boutet de Monvel 3] that Toeplitz operators form an algebra and give rise to symbolic calculus: if Q is a pseudo-differential operator on Σ, the symbol of T_Q is $\sigma(Q) \mid \Sigma$, and we have the following rules, where A,A' are Toeplitz operators of degree m,m':

(4.2) $$\sigma_{m+m'}(AA') = \sigma_m(A)\ \sigma_{m'}(A')$$

$$\sigma_{m+m'-1}([A,A'] = -i\ \{\sigma_m(A)\ ,\ \sigma_{m'}(A')\}$$

$$\sigma_m(A) = 0 \quad \text{iff} \quad A \text{ is in fact of degree} \leq m-1$$

which mean that the algebra of Toeplitz operators is a model for pseudo-differential operators in the sense above. Let us emphasize that in this context several operations of pseudo-differential operator theory take a simple geometric form; eg. the analogue of Fourier integral transformations are operators of the form $f \to A(f_o\psi)$ where A is a Toeplitz operator and ψ a contact transformation, up to a positive factor, for the contact form $-i\partial u \mid \partial\Omega$.

5 Bargman space and spaces of functions of exponential type..

We finally to describe two models which give rise to the same symbolic calculus. The first arises in the context of the Bargman space (cf. Boutet de Monvel [2]). Here the analogue of Sobolev space H^s is the space B^s of holomorphic functions f on C^n such that $\int \|z\|^{2s} |f(z)|^2\, e^{-|z|^2}\, dz < \infty$. Let B be the orthogonal projector of the Hilbert space $L^2(\mathbf{C}^n, e^{-|z|^2}dz)$ to its subspace B^0 of holomorphic functions. If a is a symbol of degree m on $\mathbf{C}^n$, we define again T_a by $T_a f = B(af)$: this is an operator of degree m, ie. continuous $B^s \to B^{s-m}$ for any s. The operators T_a form, mod. operators of degree $-\infty$, an algebra, which gives rise to the same symbolic calculus and is an analogue of the algebra of pseudo-differential operators or of Toeplitz operators are described above. Here the symplectic cone Σ is $\mathbf{C}^n$-{0} equipped with its canonical symplectic form.

There is simple correspondance with "polynomial operators" given by the "Fourier integral operator"

(5.1) $$f(x) \to g(z) = \int \exp -\tfrac{1}{2}(x^2+z^2)+\sqrt{2}x.z)\ f(x)\ dx$$

(acting from functions of x on $\mathbf{R}^n$ to functions of z on $\mathbf{C}^n$). In this correspondance the operators T_{z_j} and $T_{\bar{z}_j} = \partial/\partial\bar{z}_j$ correspond to the creation and annilihation operators

$\frac{1}{\sqrt{2}}(x_j-\partial/\partial x_j)$, resp. $\frac{1}{\sqrt{2}}(x_j+\partial/\partial x_j)$, and up to normalizing constants the Hermite functions h_α correspond to the monomials z^α.

The last model uses functions of exponential type, and was described by D. Nguon in his thesis. Let U be a bounded strictly convex domain with smooth boundary in $\mathbf{C}^n$. If f is a bounded holomorphic function on U its Borel-Laplace transform g(z) is defined by

(5.2) $$\mathcal{B}f = g(z) = \int_{\partial U} e^{z.x}\, d\sigma(x)$$

For the description of such Fourier-Borel transforms, the adequate scale of Sobolev spaces is the scale $\mathcal{E}^s$ of holomorphic functions f on $\mathbf{C}^n$ such that $|z|^s \exp{-h(z)}f$ is square integrable, where h is the supporting function of U: $h(z) = \sup_{x\in U} \mathrm{Re}\langle z|x\rangle$.

Here the analogue of Toeplitz operators are as follows: let Π be the orthonormal projector from $L^2(\mathbf{C}^n, e^{-2h}dz)$ to its subspace $\mathcal{E}^0$ of holomorphic functions. If a is a symbol of degree m on $\mathbf{C}^n=\mathbf{R}^{2n}$, denote T_a the operator $f\to\Pi(af)$. Then again T_a is an operator of degree m, ie. it is continuous $\mathcal{E}^s\to\mathcal{E}^{s-m}$ for any s, and, mod. operators of degree $-\infty$, the operators T_a for, an algebra, which gives rise to the same symbolic calculus. The essential point, which links these operators to Toeplitz oprators on ∂U, is the fact that $\mathcal{B}^*\mathcal{B}$ is an invertible Toeplitz operator on ∂U (in fact it is positive elliptic, of degree -1/2).

Here the symplectic cone is $\Sigma=\mathbf{C}^n-\{0\}$, equipped with the symplectic form

(5.3) $$-2i\, \partial\bar{\partial}h.$$

This follows from the fact that the operators $\partial_k=\partial/\partial z_k$ (k=1,...n) coincide with the operators $2T_{\partial_k h}$ [4]; so in this case the Liouville form is $\lambda=i\Sigma\sigma(\partial_k)\, dz_k = 2\partial h$, and the symplectic form is $\omega=d\lambda=2i\, \bar{\partial}\partial h$ as announced. Note that the operator ∂_k corresponds through $\mathcal{B}$ to the Toeplitz operator $T_{\bar{w}_k}$ on ∂U. A more complete set of such formulas can be found in the paper of Nguon.

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[4] because for any $f,g\in\mathcal{E}^0$ we have $\int e^{-2h}\, \partial_k f\, \bar{g} = -\int \partial_k(e^{-2h}\bar{g})\, f = 2\int(\partial_k h)e^{-2h}\, \bar{g}\, f$ since $\partial_k\bar{g}=0$.

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