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A proof of the Gauss-Bonnet-Chern Theorem by the symbol calculus of pseudo-differential operators

Chisato IWASAKI (岩崎　千里)
Department of mathematics, Himeji Institute of Technology

§1. Introduction. The aim of this paper is to give an analytic proof of the Gauss-Bonnet-Chern theorem for a smooth orientable Riemannian manifold with boundary by means of symbol calculus of pseudo-differential operators. The similar attempts for a smooth Riemannian manifold without boundary are found in E.Getzler [6], H.L.Cycon-R.G.Froese-W.Kirsch-B.Simon[5] and N.Berline-E.Getzler-M.Vergne[2] .

Let M be a Riemannian manifold and let $\chi(M)$ be the Euler characteristic of M.Let dv and $d\sigma$ be a volume element of M and one of its boundary ∂M respectively.

Analytical proofs are based of the following formula

$$\chi(M)=\int_M\sum_{p=0}^{n}(-1)^p\, tre_p(t,x,x)dv,$$

where $e_p(t,x,y)$ is the kernel of the fundamental solution $E_p(t)$ for the Cauchy problem for the heat equation for Δ_p on differental p-forms $A^p(M)=\Gamma(\wedge^pT^*(M))$, that is

$$E_p(t)f(x)=\int_M e_p(t,x,y)\varphi(y)dv_y,\quad \varphi\in A^p(M)$$

satisfies

$$\text{(1.1)}\qquad \begin{cases}(\dfrac{\partial}{\partial t}+\Delta_p)E_p(t)=0 & \text{in } (0,T)\times M,\\ E_p(0)=I & \text{in } M.\end{cases}$$

If M has boundary ∂M, then $E_p(t)$ satisfies $(1.2)_p$ instean of (1.1).

$$(1.2)_p\qquad \begin{cases}(\dfrac{\partial}{\partial t}+\Delta_p)E_p(t)=0 & \text{in } (0,T)\times M,\\ B_pE_p(t)=0 & \text{on } (0,T)\times\partial M,\\ E_p(0)=I & \text{in } M,\end{cases}$$

with some boundary condition B_p (See §6).

(Δ_p,B_p) is an elliptic boundary value problem. So it is well-known that $e_p(t,x,y)$ has singularity only at $x=y$ as follows.

$$tre_p(t,x,x)\sim c_0(x)t^{-\frac{n}{2}}+c_1(x)t^{-\frac{n}{2}+\frac{1}{2}}+\cdots+\cdots\qquad t\to 0.$$

The vanishing of the singularity of super trace at a point of M defined by

$$stre(t,x,x) = \sum_{p=0}^{n}(-1)^p tre_p(t,x,x)$$

is due to algebraic theorem in §3 stated in [5] which is owing to V.K.Patodi [15]. The point of this paper is that according to this theorem and the method of construction of the fundamental solution for the mixed problem in C.Iwasaki[11], even if M has boundary , one can prove the Gauss-Bonnet-Chern theorem only by symbol calcuclus of the top term of the asymptotic of the fundamental solution, considering operators acting on $A^*(M) = \sum_{p=0}^{n} A^p(M)$

Main theorem . *We get the Gauss-Bonnet-Chern theorem. Moreover we have that*
(1)

$$\lim_{t\to 0}\int_M \sum_{p=0}^{n}(-1)^p\, tre_p(t,x,x)\psi(x)dv = \int_M C_n(x,M)\psi(x)dv + \int_{\partial M} D_{n-1}(x)\psi(x)d\sigma$$

for any $\psi(x) \in C^\infty(M)$.

(2)For M *without boundary or for* x *contained in* $M\backslash\partial M$

$$\sum_{p=0}^{n}(-1)^p\, tre_p(t,x,x) = C_n(x,M) + 0(\sqrt{t}) \qquad \text{as } t\to 0.$$

(3) If M *has boundary,*

$$\sum_{p=0}^{n}(-1)^p\, tre_p(t,x,x) = 2D_{n-1}(x)\frac{1}{\sqrt{t}} + 0(1) \qquad \text{as } t\to 0$$

for $x \in \partial M$ *,where*

$$C_n(x,M)dv = \begin{cases} the\ Euler\ form, & \text{if } n \text{ is even (See (5.2) for the pricise definition);} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

and

$$D_{n-1}(x) = \begin{cases} \frac{1}{2}C_{n-1}(x,\partial M), & \text{if } n \text{ is odd;} \\ \text{See Definition 7}, & \text{if } n \text{ is even.} \end{cases}$$

There are many studies to prove the Gauss- Bonnet-Chern theorem analytically. McKean-Singer [14] proved

$$stre(t,x,x) = C_n(x) + 0(t)$$

when M is a closed manifold of dimention 2. V.K.Patodi[15] extended this equation for a manifold of any dimention. Moreover P.B.Gilkey[7],[8] proved the Gauss-Bonnet-Chern theorem by invariant theory in case M has boundary. There are probabilitistic proofs in N.Ikeda-S.Watanabe[9] and I.Shigekawa-N.Ueki-S.Watanabe[16].

§2. The representation of Δ. Let M be a smooth Riemannian manifold with a Riemannian metric g. Set $X_1, X_2, \cdots, X_n$ be a local orthonormal frame of $T(M)$ in a local patch of chart U. And let $\omega^1, \omega^2, \cdots, \omega^n$ be its dual.

The differential d and its dual ϑ acting on $A^*(M)$ are given as follow, using the Levi-Civita connection ∇ .

$$d = \sum_{j=1}^{n} e(\omega^j)\nabla_{X_j}, \qquad \vartheta = -\sum_{j=1}^{n} \imath(X_j)\nabla_{X_j},$$

where we use the following notations.

Notations.

$$e(\omega^j)\omega = \omega^j \wedge \omega, \ \ \imath(X_j)\omega(Y_1, \cdots, Y_{p-1}) = \omega(X_j, Y_1, \cdots, Y_{p-1}).$$

Let $c_{i,j}^k$ be the folloing function and let $R(X,Y)$ be the curvature transformation, that is

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]},$$

$$\nabla_{X_i} X_j = \sum_{\ell=1}^{n} c_{i,j}^{\ell} X_{\ell}.$$

From the fact that our connection is the Riemannian connection we have

Proposition 1. *The coefficients $c_{i,j}^k$ of connection form satisfy*

$$c_{i,j}^k = -c_{i,k}^j, \ \ [X_i, X_j] = \sum_{k=1}^{n} (c_{i,j}^k - c_{j,i}^k) X_k.$$

We have the following representation for $\Delta = d\vartheta + \vartheta d$ which is known as Weitzenböck's formula.

Lemma 1. *The Laplacian Δ on $A^*(M)$ is given by*

$$\Delta = -\{\sum_{j=1}^{n} \nabla_{X_j}\nabla_{X_j} - \sum_{i,j=1}^{n} c_{i,i}^{j}\nabla_{X_j} + \sum_{i,j=1}^{n} e(\omega^i)\iota(X_j)R(X_i,X_j)\}.$$

We use the following notations in the rest of this paper.

$$e(\omega^j) = a_j^*, \qquad \iota(X_m) = a_m,$$

$$a_I = a_{i_1}a_{i_2}\cdots a_{i_p}, \qquad a_I^* = a_{i_p}^*\cdots a_{i_1}^* \qquad \text{for } I = \{i_1 < i_2 < \cdots < i_p\},$$

$$\omega^I = \omega^{i_1}\wedge\omega^{i_2}\wedge\cdots\wedge\omega^{i_p} \qquad \text{for } I = \{i_1 < i_2 < \cdots < i_p\},$$

$$R(X_i,X_j)X_k = \sum_{\ell=1}^{n} R_{kij}^{\ell}X_\ell \quad 1 \le i,j,k, \le n.$$

Then we have

$$\Delta = -\{\sum_{j=1}^{n}(X_jI - G_j)^2 - \sum_{i,j=1}^{n} c_{i,i}^{j}(X_jI - G_j) - \sum_{i,j,\ell,m=1}^{n} R^{m}{}_{\ell ij}a_i^*a_ja_\ell^*a_m\}$$

on $A^*(M)$. Here

$$G_j = \sum_{\ell,m=1}^{n} c_{j,\ell}^{m}a_\ell^*a_m$$

and I is an identity operator on $\wedge^*(T^*(M))$.

Take a local coordinate $\{x_1,\cdots,x_n\}$ of U. Let $\{\xi_1,\cdots,\xi_n\}$ be its dual. By the above Lemma 1 we have

Lemma 2. *The symbol of Δ is given by*

$$\sigma(\Delta) = -\{\sum_{j=1}^{n}(\alpha_jI - G_j)^2 - \sum_{i,j,\ell,m=1}^{n} R^{m}{}_{\ell ij}a_i^*a_ja_\ell^*a_m\} + r_1,$$

where

$$r_1 = \sum_{k,j=1}^{n} i\{(\frac{\partial}{\partial\xi_k})\alpha_j(\frac{\partial}{\partial x_k})\alpha_j - (\frac{\partial}{\partial\xi_k})\alpha_j(\frac{\partial}{\partial x_k})G_j\}I + \sum_{j,k=1}^{n} c_{k,k}^{j}(\alpha_jI - G_j)$$

and

$$\sigma(X_j) = \alpha_j.$$

The following proposition is fundamental for a_i, a_j^*.

Proposition 2.

$$a_i a_j + a_j a_i = 0,$$
$$a_i^* a_j^* + a_j^* a_i^* = 0,$$
$$a_i a_j^* + a_j^* a_i = \delta_{ij}.$$

§3. Berezin-Patodi formula. Let V be a vector space of dimention n with inner product and let $\wedge^p(V)$ be its anti-symmetric p tensors. Set $\wedge^*(V) = \sum_{p=0}^n \wedge^p(V)$. Let $\{v_1, \cdots, v_n\}$ be an orthnormal basis for V. Set a_i^* be a linear transformation on $\wedge^*(V)$ defined by $a_i^* v = v_i \wedge v$ and set a_i be an adjoint operator of a_i^* on $\wedge^*(V)$. Then $\{a_i^*, a_j\}$ satisfy Proposition 2. The following Theorem 1 was shown in [5] under the above assumptions.

Theorem 1(Berezin-Patodi[5]). *For any linear operator A on $\wedge^*(V)$, we can write uniquely in the form $A = \sum_{I,J} \alpha_{I,J} a_I^* a_J$ and*

$$\sum_{p=0}^{n} \mathrm{tr}[(-1)^p A_p] = (-1)^n \alpha_{\{1,2,\cdots,n\}\{1,2,\cdots,n\}},$$

where $A_p = A|_{\wedge^p(V)}$.

§4. Construction of the asymptotics of the fundamental solution for the Cauchy problem.

Now let us cosider the Cauchy problem on $\mathbf{R}^n$.

$$\text{(4.1)} \qquad \begin{cases} (\frac{\partial}{\partial t} + R(x,D))U(t) = 0 & \text{in } (0,T) \times \mathbf{R}^n, \\ U(0) = I & \text{in } \mathbf{R}^n, \end{cases}$$

where $R(x,D)$ is a differential operator of which symbol $r(x,\xi) = p_2(x,\xi)I + p_1(x,\xi)$ satisfies $p_j \in S^j_{1,0}$ and $p_2 \geq \delta|\xi|^2$.

Definition 1. (1)Let $(\mathcal{A})_{ij} = a_i^* a_j \quad 1 \leq i,j \leq n$.
(2)Let K^m be a subset of $S^m_{1,0}$ as follows.
$K^m = \{p(x,\xi : \mathcal{A})$;polynomial with respect to ξ and $\mathcal{A}$ of order m with coefficients $\mathcal{B}(\mathbf{R}^n)\}$.
(3)We define a pseudo-differential operator action on $A^*(M)$ by $P = p(x,D : \mathcal{A})$ of a symbol $\sigma(P) = p(x,\xi : \mathcal{A}) = \sum_{I,J} p_{I,J}(x,\xi) a_I^* a_J \in K^m$ as follows.

$$p(x,D : \mathcal{A})(\varphi_K \omega^K) = \sum_{I,J} p_{I,J}(x,D)\varphi_K a_I^* a_J(\omega^K)$$

Definition 2. (1) For a real munber m, K_m is the set of all polynomials with respect to t of degree d with cefficient of K^{m+2d}.
(2)For a real number ℓ, R_ℓ is the subset of $\cup_m B_t(S^m_{1,0})$ which satisfies the following enequality for nonnegative constants $C_{\alpha,\beta}, C$ and $\ell_{\alpha,\beta}$

$$\|(\frac{\partial}{\partial t})^k(\frac{\partial}{\partial \xi})^\alpha(\frac{\partial}{\partial x})^\beta q(t,x,\xi)\| \leq C_{\alpha,\beta}e^{-p_2t+C<\xi>t}(t<\xi>^2+1)^{\ell_{\alpha,\beta}}(\frac{1}{\sqrt{t}}+<\xi>)^{\ell+2k-|\alpha|}.$$

Now assume (4.2) for the symbol $r(x,\xi)$ of $R(x,D)$ in (4.1)

$$r(x,\xi:\mathcal{A}) = r_2(x,\xi:\mathcal{A}) + r_1(x,\xi:\mathcal{A}), \qquad r_j \in K^j\ (j=1,2), \tag{4.2}$$

$$r_2(x,\xi:\mathcal{A}) - p_2(x,\xi)I \in S^1_{1,0}.$$

Let

$$u_0 = e^{-tr_2(x,\xi:\mathcal{A})}.$$

Theorem 2. *For any non negative integer N we have the asymptotics u^N of the fundamental solution for (4.1) of the form $u^N = \sum_{j=0}^N u_j$, $\quad u_j = v_j u_0$ with $v_j \in K_{-j}$ in the sense*

$$\begin{cases} (\frac{\partial}{\partial t}+r)\circ u^N = 0 & \mod R_{-N+1}, \\ u^N(0) = I. \end{cases}$$

§5. The proof of the Gauss-Bonnet-Chern theorem without boundary.

We will construct the asymptotics of the fundamental solution for the Cauchy problem on M , that is,

$$\begin{cases} (\frac{\partial}{\partial t}+\Delta)U(t) = 0 & \text{in } (0,T)\times M, \\ U(0) = I & \text{in } M, \end{cases}$$

where the operator $U(t)$ is cosidered acting on $A^*(M) = \sum_{p=0}^n A^p(M)$. Owing to the fact that the fundamental solution has the pseudo-local property, it is sufficient to consider the fundamental solution in a local chart. We have

$$\operatorname{str} e(t,x,x) = \operatorname{str} \tilde{u}^N(t,x,x) + 0(t^{-\frac{n}{2}+\frac{N}{2}}),$$

where

$$\tilde{u}^N(t,x,y) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i(x-y)\cdot\xi} u^N(t,x,\xi)d\xi.$$

In our case $r = r_2 + r_1$, where r_1 is given in Lemma 2 and

$$r_2 = -\sum_{j=1}^{n}(\alpha_j I - G_j)^2 + R.$$

Here

$$R = \sum_{i,j,\ell,m=1}^{n} R^m_{\ell ij} a_i^* a_j a_\ell^* a_m. \tag{5.1}$$

The principal symbol of Δ is equal to $p_2 = -\sum_{j=1}^{n}(\alpha_j)^2 I$.

By Theorem 1 , we have

$$\operatorname{str}\tilde{u}_0(t,x,x) = \begin{cases} (\frac{1}{2\sqrt{\pi t}})^n \sqrt{\det g}\operatorname{str}\{\frac{(-1)^m}{m!}R^m t^m\} + 0(t), & \text{if } n = 2m\ ; \\ 0(\sqrt{t}), & \text{if } n \text{ is odd.} \end{cases}$$

$$\operatorname{str}\tilde{u}_j(t,x,x) = 0(\sqrt{t}^j).$$

Noting (5.1), we have

Lemma 3. *If $n = 2m$,*

$$(\frac{1}{2\sqrt{\pi}})^n \operatorname{str}\{\frac{(-1)^m}{m!}R^m\} = C_n(x,M),$$

where

$$C_n(x,M) = (\frac{1}{2\sqrt{\pi}})^n \frac{1}{m!} \sum_{\pi,\sigma\in S_n} (\frac{1}{2})^m sign(\pi) sign(\sigma) \tag{5.2}$$

$$\times R_{\pi(1)\pi(2)\sigma(1)\sigma(2)} \cdots R_{\pi(n-1)\pi(n)\sigma(n-1)\sigma(n)}.$$

§6. Asymptotics of the fundamental solution for intial-boundary value problems. The study in [11] is applicable for the construction of the fundamental solution for our intial-boundary value problem . But as we have studied in §5 , the lower parts of the asymptotics of the fundamental solution play the important part for the proof of Gauss-Bonnet-Chern theorem . So in this case we introduce new class $\mathcal{J}_s$ instead of

$\mathcal{H}_s$ in [11], as we used K^m instead of $S^m_{1,0}$ in §4. The main part of the construction of the fundamental solution or its asymptotics is how to construct these ones in a local chart (cf.[11]).

We will write down the boundary operator B_p in a local coordinate. Take a local patch Ω near ∂M such that ∂M is defined by $\{\rho = 0\}$ in Ω and $M \cap \Omega \subset \{\rho \geq 0\}$. Assume that $\omega^n = cd\rho$ with some function c on M.

Choose a local coordinate $\{x_1, \cdots, x_n\}$ in Ω such that $M \cap \Omega = \{(x', x_n); x' \in \mathcal{U}, x_n \geq 0\}, \Gamma = \partial M \cap \Omega = \{(x', 0); x' \in \mathcal{U}\}$ and

$$X_n|_\Gamma = \frac{\partial}{\partial x_n}.$$

The boundary operator B_p is as follows.

$$\varphi \in Dom(\vartheta), \qquad d\varphi \in Dom(\vartheta),$$

where $Dom(\vartheta) = \{\varphi = \sum_J \varphi_J \omega^J, \varphi_J|_\Gamma = 0 \ for \ n \in J\}$. So we obtain the eqation for the boundary condition

$$\frac{\partial}{\partial x_n}\varphi|_\Gamma = 0$$

for $\varphi \in A^0(M)$ and for $\sum_J \varphi_J \omega^J \in A^p(M), \ p \geq 1$

$$\begin{cases} \varphi_J|_\Gamma = 0 \text{ if } n \in J, \\ \{(\frac{\partial}{\partial x_n} - \gamma + b) \sum_{n \notin J} \varphi_J \omega^J\}|_\Gamma = 0, \end{cases}$$

where γ and b are given in the following (6.1).

Definition 3. (1) We define $h^\star = h^\star(t, x', \xi) = h(t, x', 0, \xi)$ for a function $h(t, x, \xi)$ given in $\mathbf{R}^{2n+1}$.
(2)Set

$$\gamma = \gamma(x' : \mathcal{A}) = \sum_{1 \leq j,k \leq n} (c^j_{n,k})^\star a^*_k a_j, \tag{6.1}$$

$$b = b(x' : \mathcal{A}) = -\sum_{1 \leq j,k \leq n-1} (c^n_{j,k})^\star a^*_j a_k + \sum_{j=n \text{ or } k=n} (c^j_{n,k})^\star a^*_k a_j.$$

(3) $\mathcal{P} = a^*_n a_n$, $\mathcal{Q} = a_n a^*_n = I - \mathcal{Q}$, $B = \frac{\partial}{\partial x_n} - \gamma + b$.

As the argument in [11], it is enough to construct the fundamental solution in $\mathbf{R}^n_+$. Suppose that the fundamental solution is in the form $U_B(t) = U(t) + V(t)$, where $U(t)$ is the fumdamental solution for the Cauchy problem. Then we consider the following problem in $\mathbf{R}^n_+$.

$$\begin{cases} (\frac{d}{dt} + R(x,D))V(t) = 0 \quad \text{in } I \times \mathbf{R}^n_+, \\ \qquad \mathcal{P}V(t) = -\mathcal{P}U(t) \quad \text{on } I \times \mathbf{R}^{n-1} \times \{x_n = 0\}, \\ \qquad BQV(t) = -BQU(t) \quad \text{on } I \times \mathbf{R}^{n-1} \times \{x_n = 0\}, \\ \qquad \lim_{t\to 0} V(t) = 0 \quad \text{in } \mathbf{R}^n_+. \end{cases}$$

Definition 4. Let $\{q_j\}_{j\leq 2}$ be defined as

$$q_2 = r_2(x', 0, \xi', \xi_n : \mathcal{A}) = r_2^\star,$$

$$q_{2-j} = \sum_{\ell+k=j, 0\leq k\leq 2} ((\frac{\partial}{\partial x_n})^\ell r_{2-k})^\star \frac{x_n^\ell}{\ell!}, \quad j \geq 1.$$

For any fixed N we set

$$\hat{q} = \sum_{j=2}^{-N+1} q_j.$$

We have by Definition 3 and 4

$$q_2 = (\xi_n + i\gamma)^2 + \beta(x', \xi' : \mathcal{A}),$$

where

$$\beta = -\sum_{j=1}^{n-1}((\alpha_j)^\star I - (G_j)^\star)^2 + R^\star \in K^2.$$

Let $\{\tilde{w}_{j,k}\}$ be symbols defined in Definition 7 of [11] and let $\{W_{j,k}\}$ be operators defined by $\{\tilde{w}_{j,k}\}$.

Definition 5. For a pair (j,k) of integer j and nonpositive integer k we define a function

$$\{\tilde{v}_{j,k}(t, x_n, y_n; b, \gamma)\}_{j,k} = e^{\gamma(x_n - y_n)}\tilde{w}_{j,k}(t, x_n + y_n; b).$$

An operator $V_{j,k}(t; b, \gamma)$ corresponding to $\tilde{v}_{j,k}$ is defined as follows for a function $\varphi(y_n)$ defined on $\mathbf{R}^1_+$.

$$(V_{j,k}(t; b, \gamma)\varphi)(x_n) = \int_0^\infty \tilde{v}_{j,k}(t, x_n, y_n; b, \gamma)\varphi(y_n)dy_n.$$

Here

$$w_{0,0}(t, \xi_n) = \exp(-t\xi_n^2)$$

$$w_{j,0}(t,\xi_n) = (i\xi_n)^j w_{0,0}(t,\xi_n), j \geq 0,$$

$$\tilde{w}_{j,0}(t,\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\omega\cdot\xi_n} \tilde{w}_{j,0}(t,\xi_n) d\xi_n,\ j \geq 0,$$

$$\tilde{w}_{j,0}(t,\omega;b) = -\frac{1}{\sqrt{\pi}}(\frac{1}{2\sqrt{t}})^{j+1} \int_0^{\infty} e^{-(\sigma+\frac{\omega}{2\sqrt{t}})^2} \frac{(-\sigma)^{-j-1}}{(-j-1)!} d\sigma,\ j \leq -1,$$

for $k \leq -1$ $\quad \tilde{w}_{j,k}(t,\omega;b) =$

$$\begin{cases} -\frac{1}{\sqrt{\pi}}(\frac{1}{2\sqrt{t}})^{j+k+1} \int_0^{\infty} e^{-(\sigma+\frac{\omega}{2\sqrt{t}})^2+2b\sqrt{t}\sigma} \frac{(-\sigma)^{-k-1}}{(-k-1)!} h_j(\sigma+\frac{\omega}{2\sqrt{t}}) d\sigma, & \text{if} j \geq 0; \\ \frac{1}{\sqrt{\pi}}(\frac{1}{2\sqrt{t}})^{j+k+1} \int_0^{\infty} \frac{(-\tau)^{-j-1}}{(-j-1)!} d\tau \int_0^{\infty} e^{-(\sigma+\tau+\frac{\omega}{2\sqrt{t}})^2+2b\sqrt{t}\sigma} \frac{(-\sigma)^{-k-1}}{(-k-1)!} d\sigma, & \text{if} j \leq -1, \end{cases}$$

where $h_j(\sigma) = \{(\frac{\partial}{\partial\sigma})^j e^{-\sigma^2}\} e^{\sigma^2}$.

Definition 6.
(1) $\mathcal{J}_s$ is the set of all finite sum of the following functions

$$\{g(t,x_n,y_n,x',\xi':\mathcal{A}) = t^d (x_n)^\ell q(x',\xi':\mathcal{A}) \tilde{v}_{j,k}(t,x_n,y_n;b(x':\mathcal{A}),\gamma(x':\mathcal{A})) e^{-\beta(x',\xi':\mathcal{A})t};$$
$$q \in K^m(\mathbf{R}^{n-1}), d \geq 0, \ell \geq 0, k \leq 0, m = s+2d+\ell-j-k\}.$$

(2)$\tilde{R}_\ell$ is the set of all matrices which belong to $B([0,T]\times[0,\infty)\times[0,\infty); S^m_{1,0}(\mathbf{R}^{n-1}))$ and satify for any α,β,a,b,k

$$\|(\frac{\partial}{\partial\xi'})^\alpha(\frac{\partial}{\partial x'})^\beta(\frac{\partial}{\partial x_n})^a(\frac{\partial}{\partial y_n})^b(\frac{\partial}{\partial t})^k g\|$$

$$\leq C_{\alpha,\beta} \min(|\xi'|^{-|\alpha|}, \sqrt{t}^{|\alpha|})(\frac{1}{\sqrt{t}})^{\ell+1+2k+a+b} \exp(-\delta\frac{(x_n+y_n)^2}{4t} - c_0|\xi'|^2 t)$$

for any $\delta < 1$ and some $c_0 > 0$.
(3) For a symbol $g(t,x_n,y_n,x',\xi',\mathcal{A}) \in \mathcal{J}_s$ we define a integral-pseudodifferential operator as follows.

$$(G\varphi)(t,x',x_n:\mathcal{A}) = \int_0^{\infty} g(t,x_n,y_n,x',D':\mathcal{A})\varphi(\cdot,y_n)dy_n.$$

Theorem 3. *(1)For any $g(t) \in \mathcal{J}_s$ and $h(t) \in \mathcal{J}_{s-1}$ there exists $v(t) \in \mathcal{J}_{s-2}$ such that*

$$\begin{cases} (\frac{\partial}{\partial t}+\hat{q})\circ v(t) = g(t) \ \ mod\ \mathcal{J}_{s-1}+\tilde{R}_{-N} & \text{in } I\times\mathbf{R}^n_+, \\ Bv(t)|_{x_n=0} = h(t) \ \ mod\ \mathcal{J}_{s-2}+\tilde{R}_{-N-1} & \text{in } I\times\mathbf{R}^{n-1}. \end{cases}$$

(2)For any $g(t) \in \mathcal{J}_s$ and $h(t) \in \mathcal{J}_{s-2}$ there exists $v(t) \in \mathcal{J}_{s-2}$ such that

$$\begin{cases} (\frac{\partial}{\partial t} + \hat{q}) \circ v(t) = g(t) \ \ mod\ \mathcal{J}_{s-1} + \tilde{R}_{-N} & \text{in } I \times \mathbf{R}^n_+, \\ v(t)|_{x_n=0} = h(t) \ \ mod\ \mathcal{J}_{s-3} + \tilde{R}_{-N-2} & \text{in } I \times \mathbf{R}^{n-1}. \end{cases}$$

Now we discuss our boundary value problem .

For a function $h(x)$ defined in $\bar{\mathbf{R}}^n_+$, we set a function $h^+(x)$ defined in $\mathbf{R}^n$ as follows.

$$h^+(x) = \begin{cases} h(x', x_n), & \text{if } x_n \geq 0; \\ 0, & \text{if } x_n < 0. \end{cases}$$

Also we set

$$\varphi^+ = \sum_J \varphi^+ \omega^J \text{ for } \varphi = \sum_J \varphi_J \omega^J.$$

Theorem 4. *For any N the asymptotics of the fundamental solution $U_B(t)$ for the boundary problem (7.2) in the sense*

$$\begin{cases} (\frac{\partial}{\partial t} + \hat{q}) \circ v(t) = -(\frac{\partial}{\partial t} + \hat{q}) \circ u(t) \ \ \text{mod } \tilde{R}_{-N+2} & \text{in } I \times \mathbf{R}^n_+, \\ \mathcal{P}(u(t) + v(t))|_{x_n=0} = 0 \ \ \text{mod } \tilde{R}_{-N} & \text{in } I \times \mathbf{R}^{n-1}, \\ B\mathcal{Q}(u(t) + v(t))|_{x_n=0} = 0 \ \ \text{mod } \tilde{R}_{-N+1} & \text{in } I \times \mathbf{R}^{n-1}. \end{cases}$$

is obtained in the form $U_B(t)\varphi = U(t)\varphi^+ + V(t)\varphi$, where $V(t)$ is the operator defined by a symbol $v(t) \in \mathcal{J}_1$ such that $v(t) = \sum_{j=0}^N v_{1-j}(t),\ v_j(t) \in \mathcal{J}_{-j},\ v_1(t) = 2Q\tilde{v}_{1,-1}e^{-t\beta}$.

For a function $h(x)$ defined in $\bar{\mathbf{R}}^n_+$, we set a function $h^+(x)$ defined in $\mathbf{R}^n$ as follows.

$$h^+(x) = \begin{cases} h(x', x_n), & \text{if } x_n \geq 0; \\ 0, & \text{if } x_n < 0. \end{cases}$$

Also we set

$$\varphi^+ = \sum_J \varphi^+ \omega^J \text{ for } \varphi = \sum_J \varphi_J \omega^J.$$

Theorem 4. *For any N the asymptotics of the fundamental solution $U_B(t)$ for the boundary problem (7.2) in the sense*

$$\begin{cases} (\frac{\partial}{\partial t}+\hat{q})\circ v(t) = -(\frac{\partial}{\partial t}+\hat{q})\circ u(t) \mod \tilde{R}_{-N+2} & \text{in } I\times\mathbf{R}^n_+, \\ \mathcal{P}(u(t)+v(t))|_{x_n=0} = 0 \mod \tilde{R}_{-N} & \text{in } I\times\mathbf{R}^{n-1}, \\ BQ(u(t)+v(t))|_{x_n=0} = 0 \mod \tilde{R}_{-N+1} & \text{in } I\times\mathbf{R}^{n-1}. \end{cases}$$

is obtained in the form $U_B(t)\varphi = U(t)\varphi^+ + V(t)\varphi$, where $V(t)$ is the operator defined by a symbol $v(t)\in\mathcal{J}_1$ such that $v(t)=\sum_{j=0}^N v_{1-j}(t)$, $v_j(t)\in\mathcal{J}_{-j}$, $v_1(t)=2Q\tilde{v}_{1,-1}e^{-t\beta}$.

§7. The proof of Gauss-Bonnet-Chern theorem with boundary. Let $\hat{R}(W,Z,X,Y)$ be the Riemannnian curvature tensors induced on Γ. From Equation of Gauss we have

$$R(X_i,X_j,X_k,X_\ell) = \hat{R}(X_i,X_j,X_k,X_\ell) + c^n_{k,j}c^n_{\ell,i} - c^n_{\ell,j}c^n_{k,i},$$

$$1\le i,j,k,\ell\le n-1 \ \text{ on } \Gamma.$$

Definition 7.

$$D_{n-1}(x) = (\frac{1}{2})(\frac{1}{2\sqrt{\pi}})^{n-1}\frac{1}{m!}\sum_{\pi,\sigma\in S_{n-1}}(\frac{1}{2})^m sign(\pi)sign(\sigma)\hat{R}_{\pi(1)\pi(2)\sigma(1)\sigma(2)}\cdots$$

$$\cdots\hat{R}_{\pi(n-2)\pi(n-1)\sigma(n-2)\sigma(n-1)}$$

if n is odd $(n-1=2m)$.

$$D_{n-1}(x) = \sum_{k=0}^{m-1}\frac{1}{2^{m+k}\pi^m k!1\cdot3\cdot5\cdots(2m-2k-1)}(\frac{1}{2})^k\sum_{\pi,\sigma\in S_{n-1}} sign(\pi)sign(\sigma)$$

$$\times R^\star_{\pi(1)\pi(2)\sigma(1)\sigma(2)}\cdots R^\star_{\pi(2k-1)\pi(2k)\sigma(2k-1)\sigma(2k)}$$

$$\times c^n_{\pi(2k+1),\sigma(2k+1)}c^n_{\pi(2k+2),\sigma(2k+2)}\cdots c^n_{\pi(n-1),\sigma(n-1)}$$

if n even $(n=2m)$.

By Theorem 4 asymptotic of the fundamental solution for the mixed problem is given by $U_0+U_1+\cdots+U_N+V_1+V_0+\cdots+V_{-N}$, $v_j\in\mathcal{J}_j$, $v_j=g_je^{-t\beta}$, $g_1=2Q\tilde{v}_{1,-1}$.

For the supertrace of kernel $\tilde{v}_j(t,x,y)$ of operator V_j, we have the following lemma.

Lemma 4. *For any integer N we have*

$$\operatorname{str}\tilde{v}_j(t,x,x) = \begin{cases} 0(\sqrt{t}^N), & \text{if } x_n \neq 0; \\ 0((\sqrt{t})^{-j}), & \text{if } x_n = 0. \end{cases}$$

$$\int_0^\varepsilon \operatorname{str}\tilde{v}_j(t,x,x)\psi(x)dx_n = 0((\sqrt{t})^{-j+1}).$$

Moreover we have

$$\operatorname{str}\tilde{v}_1(t,x',0,x',0) = \frac{2}{\sqrt{t}}D_{n-1}(x')\sqrt{detg} + 0(1).$$

$$\int_0^\varepsilon \operatorname{str}\tilde{v}_1(t,x,x)\psi(x)dx_n = \psi(x',0)D_{n-1}(x')\sqrt{det\hat{g}} + 0(\sqrt{t}),$$

where $\hat{g}$ is the Riemannain metric induced on ∂M.

Theorem 5. *For any N we have*

$$\operatorname{str}\tilde{v}(t,x,x) = \begin{cases} 0(\sqrt{t}^N) & \text{if } x_n > 0 \\ \frac{2}{\sqrt{t}}D_{n-1}(x')\sqrt{detg} + 0(1) & \text{if } x_n = 0. \end{cases}$$

$$\int_0^\varepsilon \operatorname{str}\tilde{v}(t,x,x)\psi(x)dx_n = D_{n-1}(x')\sqrt{det\hat{g}}\psi(x',0) + 0(\sqrt{t}).$$

Proof of Main Theorem. It is sufficient to consider the fundamental solution locally if we study the asymptotic behavior of the fundamental solution. In a local patch we have

$$e(t,x,x)dv = \tilde{u}(t,x,x)dx + \tilde{v}(t,x,x)dx.$$

Then for any N we get by Theorem 5

$$\operatorname{str} e(t,x,x) = \operatorname{str}\tilde{u}(t,x,x) + 0(t^N), \; x \in M\backslash\partial M$$

$$\operatorname{str} e(t,x,x) = C_n(x) + 0(\sqrt{t}), \; x \in M\backslash\partial M$$

$$\operatorname{str} e(t,x,x) = \frac{2}{\sqrt{t}}D_{n-1}(x) + 0(1), \; x \in \partial M.$$

We remakt that the induced volume element of ∂M is defined by $d\sigma = (-1)^n \sqrt{det\tilde{g}} dx_1 dx_2 \cdots dx_n$ in a local chart. In our case $D_{n-1}(x')d\sigma$ is independet of orientation of M. So we have

$$\int_M \mathrm{str}\, e(t,x,x)\psi(x)dv = \int_M C_n(x)\psi(x)dv + \int_{\partial M} D_{n-1}(x')\psi(x')d\sigma + 0(\sqrt{t}).$$

The proof is complete. q. e. d.

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