数理解析研究所講究録
第 856 巻 1994 年 132-135

The behavior of solutions with singularities of linear partial differential equations in C^{n+1}

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Let $P(z,\partial)$ be a linear partial differential operator with order $m \geq 1$. Its coefficients are holomorphic in a neighbourhood of the origin $z=0$ in C^{n+1}. K is a nonsingular complex hypersurface through $z=0$, which is characteristic for $P(z,\partial)$. We choose the coordinate so that $K=\{z_0=0\}$. In the present talk we consider

$$P(z,\partial)u(z)=f(z), \tag{1}$$

where $u(z)$ and $f(z)$ are holomorphic except on K. In order to state the results we give notations and definitions: $z=(z_0,z_1,\cdots,z_n)=(z_0,z')$, $\partial_i=\partial/\partial_i$, $\partial=(\partial_0,\partial_1,\cdots,\partial_n)=(\partial_0,\partial')$ and $|z|=\max\{|z_i|;\ 0\leq i\leq n\}$. We write $P(z,\partial)$ in the form

$$\begin{cases} P(z,\partial)=\sum_{k=0}^{m} P_k(z,\partial) \\ P_k(z,\partial)=\sum_{l=s_k}^{k} A_{k,l}(z,\partial')\partial_0^{k-l} \\ A_{k,l}(z,\partial')=(z_0)^{j(k,l)}a_{k,l}(z,\partial'), \end{cases} \tag{2}$$

where $P_k(z,\partial)$ is the homogeneous part of order k. If $P_k(z,\partial)\not\equiv 0$, $A_{k,s_k}(z,\partial')\not\equiv 0$, and if $A_{k,l}(z,\partial')\not\equiv 0$, $a_{k,l}(0,z',\partial')\not\equiv 0$. If $P_k(z,\partial)\equiv 0$, put $s_k=+\infty$. Let us define the irregularities of K, which are closely related to the characteristic indices introduced in [1] and others. Put $d_{k,l}=l+j(k,l)$, $d_k=\min\{d_{k,l};\ s_k\leq l\leq k\}$ and $e_k=d_k-k$. Put $\Sigma^*=$ *the convex hull of* $\cup_{k=0}^{m}\Pi(k,e_k)$, where $\Pi(a,b)=\{(x,y)\in \boldsymbol{R}^2; x\leq a,\ y\geq b\}$. The boundary of Σ^* consists of a vertical half line Σ_0^*, a holizontal half line Σ_p^* and segments Σ_i^* $(1\leq i\leq p-1)$. The set of vertices of Σ^* consists of p points (k_i,e_{k_i}), $0\leq k_{p-1}<k_{p-2}<\cdots<k_1<k_0=m$ (see Figure 1). Let γ_i be the slope of Σ_i^*, $0=\gamma_p<\gamma_{p-1}<\cdots<\gamma_1<\gamma_0=+\infty$.

Definition 1. *We call γ_i, $(0\leq i\leq p)$ the irregularities of K for $P(z,\partial)$. In particular γ_{p-1} is called the minimal irregularity and denote by $\gamma_{\min,P}$.*

Let us define some functions spaces. Let $\Omega=\Omega^0\times\Omega'$ be a polydisk: $\Omega^0=\{z_0\in C^1;\ |z_0|\leq R\}$, $\Omega'=\{z'\in C^n; |z'|\leq R\}$. Put $\Omega_\theta^0=\{z_0\in\Omega^0-\{0\};\ |arg z_0|<\theta\}$ and $\Omega_\theta=\Omega_\theta^0\times\Omega'$. $\mathcal{O}(\Omega)$ $(\mathcal{O}(\Omega'))$ is the set of all holomorphic functions on Ω (*resp.* Ω'). $\mathcal{O}(\Omega_\theta)$ is the set of all holomorphic functions on Ω_θ, which contains multi-valued functions on $\Omega-K$, if $\theta>\pi$. Now we introduce

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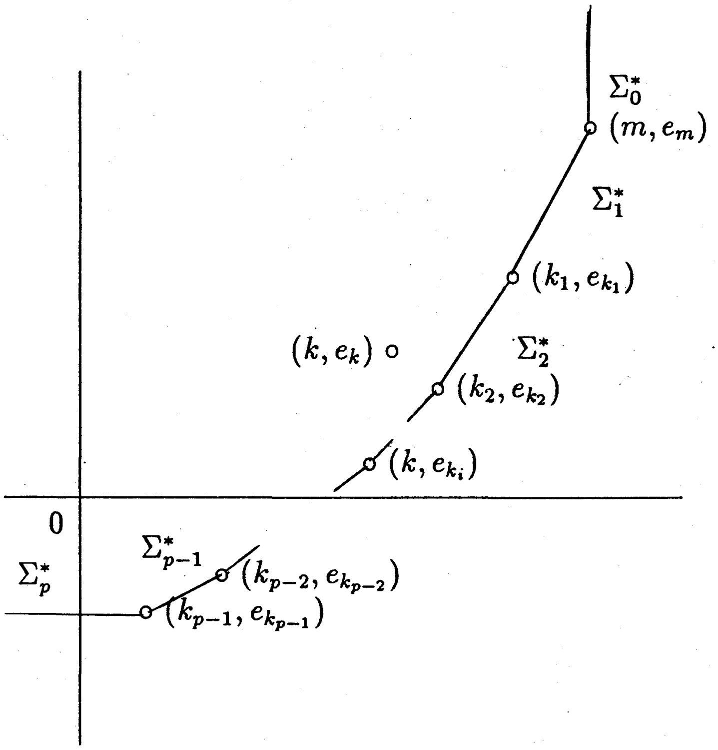

FIGURE 1. Characteristic polygon

Definition 2. *$Asy_{\{\kappa\}}(\Omega_\theta)$ $(0 < \kappa \leq +\infty)$ is the set of all $f(z) \in \mathcal{O}(\Omega_\theta)$ with the following asymptotic expansion: for any θ' with $0 < \theta' < \theta$ and any N*

$$|f(z) - (\sum_{k=0}^{N-1} a_k(z')z_0^k)| \leq A_{\theta'} B_{\theta'}^N \Gamma(N/\kappa + 1)|z_0|^N \quad \text{in } \Omega_{\theta'}, \tag{3}$$

where $a_k(z') \in \mathcal{O}(\Omega')$.

Definition 3. *$\tilde{\mathcal{M}} - Asy_{\{\kappa\}}(\Omega_\theta)$ $(0 < \kappa \leq +\infty)$ is the set of all $f(z) \in \mathcal{O}(\Omega_\theta)$ with the following asymptotic expansion: for any θ' with $0 < \theta' < \theta$ and any N*

$$\begin{aligned} &|f(z) - (\sum_{k=0}^{N-1} a_k(z')z_0^k) \log z_0 - (\sum_{k=-H}^{N-1} b_k(z')z_0^k))| \\ &\leq A_{\theta'} B_{\theta'}^N \Gamma(N/\kappa + 1)|z_0|^N |\log z_0| \quad \text{in } \Omega_{\theta'}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} &|f(z) - (\sum_{k=0}^{N} a_k(z')z_0^k) \log z_0 - (\sum_{k=-H}^{N-1} b_k(z')z_0^k)| \\ &\leq A_{\theta'} B_{\theta'}^N \Gamma(N/\kappa + 1)|z_0|^N \quad \text{in } \Omega_{\theta'}, \end{aligned} \tag{5}$$

where $H \in N$ *and* $a_k(z'), b_k(z') \in \mathcal{O}(\Omega')$.

Definition 4. $\mathcal{M}(\Omega)$ *is the set of all* $f(z) \in \mathcal{O}(\Omega_{+\infty})$ *with the form* $f(z) = a(z)\log z_0 + b(z)z_0^{-H}$, *where* $H \in N$ *and* $a(z),\ b(z) \in \mathcal{O}(\Omega)$.

Definition 5. $\mathcal{O}_{(\kappa)}(\Omega_\theta)$ $(\kappa > 0)$ *is the set of all* $f(z) \in \mathcal{O}(\Omega_\theta)$ *with the following bound: for any* θ' *with* $0 < \theta' < \theta$ *and any* $\epsilon > 0$

$$|f(z)| \leq C_\epsilon \exp(\epsilon |z_0|^{-\kappa}) \quad in\ \Omega_{\theta'}. \tag{6}$$

Now we suppose that $P(z, \partial)$ satisfies the following condition:

Condition

$$\begin{cases} (a)\ \gamma_{\min,P} \neq +\infty \quad (b)\ d_{k_{p-1}} = 0 \\ (c)\ d_{k_i} = s_{k_i} \quad for \quad 0 \leq i \leq p-1. \end{cases} \tag{7}$$

Put $\gamma = \gamma_{\min,P}$. Then the main results are the following.

Theorem 6. *If* $u(z) \in \mathcal{O}_{(\gamma)}(\Omega_\theta)$ *is a solution of*

$$P(z,\partial)u(z) = f(z) \in Asy_{\{\kappa\}}(\Omega_\theta) \quad (0 < \kappa \leq \gamma), \tag{8}$$

then $u(z) \in Asy_{\{\kappa\}}(\Omega_\theta)$.

Corollary 7. *Suppose that* $f(z) \in \mathcal{O}(\Omega)$ *and* $\theta > (\pi/2\gamma) + \pi$ *in Theorem* 6. *Then* $u(z) \in \mathcal{O}(\Omega)$.

Theorem 8. *If* $u(z) \in \mathcal{O}_{(\gamma)}(\Omega_\theta)$ *is a solution of*

$$P(z,\partial)u(z) = f(z) \in \tilde{\mathcal{M}} - Asy_{\{\kappa\}}(\Omega_\theta) \quad (0 < \kappa \leq \gamma), \tag{9}$$

then $u(z) \in \tilde{\mathcal{M}} - Asy_{\{\kappa\}}(\Omega_\theta)$.

Corollary 9. *Suppose that* $f(z) \in \tilde{\mathcal{M}}(\Omega)$ *and* $\theta > (\pi/2\gamma) + 2\pi$ *in Theorem* 8. *Then* $u(z) \in \tilde{\mathcal{M}}(\Omega)$.

Condition (a) means that K is an irregular characteristic surface in the sense in [1] and it is equivalent to $p > 1$. Condition (b) means that the $(p-1)$-th localization on K of $P(z, \partial)$, which is defined in [1], is a function. Condition (c) is an assumption imposed on the verticies of the characteristic polygon Σ^*. Theorems 6 and 8 are shown by the detailed analysis of the integral representation of solutions singular on K ([2, 3]) and for this purpose we assume (c).

A simple example satisfying the conditions in Theorems is

$$P(z,\partial) = a(z)\partial_1^m + b(z)\partial_1^{l'}\partial_0^{k'-l'} + \partial_0^k, \quad z = (z_0, z_1) \quad m > k' > k, \tag{10}$$

where we assume $a(0)b(0) \neq 0$ and $l' > k' - k$. For this $P(z, \partial)$, we have $\gamma = \min\{k/(m-k), (l'-k'+k)/(k'-k)\}$.

We can also obtain results similar to Theorems 6 and 8 of the following type for other $\mathcal{F}(\Omega_\theta) \subset \mathcal{O}(\Omega_\theta)$:

$$\begin{cases} u(z) \in \mathcal{O}_{(\gamma)}(\Omega_\theta) \\ P(z, \partial)u(z) = f(z) \in \mathcal{F}(\Omega_\theta) \end{cases} \Longrightarrow \quad u(z) \in \mathcal{F}(\Omega_\theta).$$

References

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