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Holomorphic and Singular Solutions of Non Linear Singular Partial Differential Equations

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In this note, I will report some results on holomorphic and singular solutions of singular partial differential equations of the following three cases:

1. linear case;
2. non linear first order case;
3. non linear higher order case.

1 Linear case

First of all, let us survey my result in the case of linear Fuchsian case. Let $(t,x)=(t,x_1,\cdots,x_n)\in \boldsymbol{C}_t\times\boldsymbol{C}_x^n$ and let us consider

$$(\mathrm{E}_1)\qquad (t\frac{\partial}{\partial t})^m u=\sum_{\substack{j+|\alpha|\leq m\\ j<m}} a_{j,\alpha}(t,x)(t\frac{\partial}{\partial t})^j(\frac{\partial}{\partial x})^\alpha u+f(t,x),$$

where $m\in \boldsymbol{N}^*(=\{1,2,\cdots\})$, $\alpha=(\alpha_1,\cdots,\alpha_n)\in\boldsymbol{N}^n(=\{0,1,2,\cdots\}^n)$, $|\alpha|=|\alpha_1|+\cdots+|\alpha_n|$ and

$$(\frac{\partial}{\partial x})^\alpha=(\frac{\partial}{\partial x_1})^{\alpha_1}\cdots(\frac{\partial}{\partial x_n})^{\alpha_n}.$$

Assume the following conditions:

A$_1$) $a_{j,\alpha}(t,x)$ and $f(t,x)$ are holomorphic near the origin;

A$_2$) $a_{j,\alpha}(0,x) \equiv 0$, if $|\alpha| > 0$.

Then, (E$_1$) is called a Fuchsian type equation with respect to t. The indicial polynomial $C(\rho,x)$ is defined by

$$C(\rho,x) = \rho^m - \sum_{j<m} a_{j,0}(0,x)\rho^j$$

and the characteristic exponents $\rho_1(x),\cdots,\rho_m(x)$ are defined by the roots of $C(\rho,x)=0$.

Definition of $\widetilde{\mathcal{O}}$. $\widetilde{\mathcal{O}}$ is the set of all functions $u(t,x)$ satisfying the following : there are $\varepsilon>0$ and $r>0$ such that $u(t,x)$ is holomorphic in $\{(t,x)\in\mathcal{R}(\boldsymbol{C}\backslash\{0\})\times\boldsymbol{C}^n\ ;\ 0<|t|<\varepsilon \text{ and } |x|\leq r\}$, where $\mathcal{R}(\boldsymbol{C}\backslash\{0\})$ is the universal covering space of $\boldsymbol{C}\setminus\{0\}$.

THEOREM 1 (Tahara [1]). *Denote by $\mathcal{S}$ the set of all $\widetilde{\mathcal{O}}$–solutions of* (E$_1$). *Then, if $\rho_i(0)\notin\boldsymbol{N}$ $(1\leq i\leq m)$ and $\rho_i(0)-\rho_j(0)\notin\boldsymbol{Z}$ $(1\leq i\neq j\leq m)$ hold, we have*

$$\mathcal{S}=\{U(\varphi_1,\cdots,\varphi_m)\ ;\ (\varphi_1,\cdots,\varphi_m)\in(\boldsymbol{C}\{x\})^m\},$$

where $U(\varphi_1,\cdots,\varphi_m)$ is an $\widetilde{\mathcal{O}}$– solution of (E$_1$) *depending on $(\varphi_1,\cdots,\varphi_m)\in(\boldsymbol{C}\{x\})^m$ which can be taken arbitrarily and having an expansion of the following form:*

$$\begin{aligned} U(\varphi_1,\cdots,\varphi_m) &= \sum_{i=0}^{\infty} u_i(x)t^i \\ &\quad + \sum_{i=1}^{m}\sum_{j=0}^{\infty}\sum_{k=0}^{mj} \phi_{i,j,k}(x)t^{\rho_i(x)+j}(\log t)^k \end{aligned}$$

with $\phi_{i,0,0}(x)=\varphi_i(x)$ $(i=1,\cdots,m)$.

2 Non linear first order case

Next, I will report a result for non linear first order equation of the following form:

$$(\mathrm{E}_2) \qquad t\frac{\partial u}{\partial t} = F(t, x, u, \frac{\partial u}{\partial x}),$$

where $(t, x) \in \boldsymbol{C}_t \times \boldsymbol{C}_x^n$ and $\frac{\partial u}{\partial x} = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n})$.

Put $v = (v_1, \cdots, v_n)$ and assume the following:

$\mathrm{B}_1)$ $F(t, x, u, v)$ is holomorphic near the origin ;

$\mathrm{B}_2)$ $F(0, x, 0, 0) \equiv 0$ near $x = 0$;

$\mathrm{B}_3)$ $\frac{\partial F}{\partial v_i}(0, x, 0, 0) \equiv 0$ for $i = 1, \cdots, n$.

Then, (E_2) is called an equation of Briot-Bouquet type with respect to t (in [3]). Put

$$\rho(x) = \frac{\partial F}{\partial u}(0, x, 0, 0).$$

Definition of $\widetilde{\mathcal{O}}_+$. We denote by $\widetilde{\mathcal{O}}_+$ the set of all $u(t, x)$ satisfying the following i) and ii):

i) There are $r > 0$ and a positive-valued continuous function $\varepsilon(s)$ on $\boldsymbol{R}_s$ such that $u(t, x)$ is a holomorphic function on

$$\{(t, x) \in \mathcal{R}(\boldsymbol{C} \setminus \{0\}) \times \boldsymbol{C}^n \ ; \ 0 < | t | < \varepsilon(\arg t), | x | \leq r\};$$

ii) There is an $a > 0$ such that for any $\theta > 0$ we have

$$\max_{|x| \leq r} | u(t, x) | = O(| t |^a)$$

as $t \longrightarrow 0$ under the condition $| \arg t | < \theta$.

THEOREM 2 (Gérard-Tahara [4]). *Denote by* $\mathcal{S}_+$ *the set of all* $\widetilde{\mathcal{O}}_+$*-solutions of* ($E_2$). *Then, if* $\rho(0) \notin \boldsymbol{N}^*$ *holds, we have:*

$$\mathcal{S}_+ = \begin{cases} \{u_0\}, & \text{when } \mathrm{Re}\rho(0) \leq 0, \\ \{u_0\} \cup \{U(\varphi);\ 0 \neq \varphi(x) \in \boldsymbol{C}\{x\}\}, & \text{when } \mathrm{Re}\rho(0) > 0, \end{cases}$$

where u_0 *is the unique holomorphic solution of* (E_2) *and* $U(\varphi)$ *is an* $\widetilde{\mathcal{O}}_+$*-solution of* ($E_2$) *having an expansion of the following form:*

$$U(\varphi) = \sum_{i \geq 1} u_i(x) t^i + \sum_{\substack{i+2j \geq k+2 \\ j \geq 1}} \varphi_{i,j,k}(x) t^{i+j\rho(x)} (\log t)$$

with $\varphi_{0,1,0}(x) = \varphi(x)$ *which can be taken arbitrarily.*

3 Non linear higher order case

Lastly, I will report a generalization of the result in section 2 to higher order case.

Let us consider

$$(E_3) \qquad (t\frac{\partial}{\partial t})^m u = F(t, x, \{(t\frac{\partial}{\partial t})^j (\frac{\partial}{\partial x})^\alpha u\}_{\substack{j+|\alpha| \leq m \\ j<m}}),$$

where $(t, x) \in \boldsymbol{C}_t \times \boldsymbol{C}_x^n$ and $m \in \boldsymbol{N}^*$. Put

$$z = \{z_{j,\alpha}\}_{\substack{j+|\alpha| \leq m \\ j<m}}$$

and assume the following conditions:

C1) $F(t, x, z)$ is holomorphic near the origin ;

C2) $F(0, x, 0) \equiv 0$ near $x = 0$;

C3) $\dfrac{\partial F}{\partial z_{j,\alpha}}(0, x, 0) \equiv 0$ near $x = 0$, if $|\alpha| > 0$.

Note the following: 1) if $m = 1$, (E_3) is nothing but (E_2); 2) if (E_3) is linear, (E_3) is nothing but (E_1). Thus, (E_3) includes both cases (E_1) and (E_2).

Put

$$C(\rho, x) = \rho^m - \sum_{j<m} \frac{\partial F}{\partial z_{j,0}}(0, x, 0)\rho^j$$

and denote by $\rho_1(x), \cdots, \rho_m(x)$ the roots of $C(\rho, x) = 0$ in ρ. Set

$$\mu = \text{the cardinal of } \{\, i\,; \mathrm{Re}\rho_i(0) > 0\}.$$

If $\mu = 0$, this implies that $\mathrm{Re}\rho_i(0) \leq 0$ for all $i = 1, \cdots, m$. When $\mu \geq 1$, by a renumeration we may assume

$$\begin{cases} \mathrm{Re}\rho_i(0) > 0, & \text{for } 1 \leq i \leq \mu, \\ \mathrm{Re}\rho_i(0) \leq 0, & \text{for } \mu + 1 \leq i \leq m. \end{cases}$$

Then we have:

THEOREM 3 (Gérard-Tahara [5]). *Denote by $\mathcal{S}_+$ the set of all $\widetilde{\mathcal{O}}_+$ -solutions of* (E_3). *Then we have:*

(I) *If $\mu = 0$, we have*

$$\mathcal{S}_+ = \{u_0\},$$

where u_0 is the unique holomorphic solution of (E_3).

(II) *If $\mu \geq 1$, under the additional conditions:*

1) $\rho_i(0) \neq \rho_j(0)$ *for* $1 \leq i \neq j \leq \mu$,
2) $C(1, 0) \neq 0$,
3) $C(i + j_1\rho_1(0) + \cdots + j_\mu\rho_\mu(0), 0) \neq 0$ *for any* $(i, j) \in \boldsymbol{N} \times \boldsymbol{N}^\mu$ *satisfying* $i + |j| \geq 2$,

we have

$$\mathcal{S}_+ = \{U(\varphi_1, \cdots, \varphi_\mu)\,;\, (\varphi_1, \cdots, \varphi_\mu) \in (\boldsymbol{C}\{x\})^\mu\},$$

where $U(\varphi_1, \cdots, \varphi_\mu)$ *is an* $\widetilde{\mathcal{O}}_+$ *-solution of* (E_3) *depending on* $(\varphi_1, \cdots, \varphi_\mu) \in (\boldsymbol{C}\{x\})^\mu$ *which can be taken arbitrarily and having an expansion of the following form:*

$$U(\varphi_1, \cdots, \varphi_\mu) \;=\; \sum_{i \geq 1} u_i(x) t^i$$

$$+ \sum_{\substack{i+2m|j| \geq k+2m \\ |j| \geq 1}} \phi_{i,j,k}(x) t^{i+j_1\rho_1(x)+\cdots+j_\mu\rho_\mu(x)} (\log t)^k$$

with $\phi_{0,e_p,0}(x) = \varphi_p(x)$ $(p = 1, \cdots, \mu)$ *where* $e_1 = (1, 0, \cdots, 0), \cdots, e_\mu = (0, \cdots, 0, 1) \in \boldsymbol{N}^\mu$.

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