

## Algebraic integer valued holomorphic functions of exponential type

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### 1. Introduction

In this section we give a brief history of our subject.

In 1915, G.Polya proved following theorem.

**Theorem 1.** ([9]) Suppose that entire function  $f$  satisfies the following conditions:

- (1)  $f$  is an entire function of exponential type  $A$ ,
- (2)  $f(\mathbb{N}) \subset \mathbb{Z}$

If  $A$  is less than  $\log 2$ , then  $f$  is a polynomial with rational integer coefficients.

In 1946, C. Pisot generalized Polya's theorem using Laplace transform and transfinite diameter. ([8])

In 1974, V. Avanisian and R. Gay generalized Pisot's result to entire functions of exponential type in  $\mathbb{C}^m$  using the theory of analytic functionals. ([2])

In 1977, F.Gramain obtained several results about entire function  $f$  of exponential type in  $\mathbb{C}^1$  satisfying  $f(\mathbb{N}) \subset O_K$ .

Here  $O_K$  denotes the ring of algebraic integers contained in algebraic number field  $K$  over  $\mathbb{Q}$ . ([4])

In 1988, A.Bazylewicz generalized F.Gramain's result to entire functions of exponential type in  $\mathbb{C}^m$ . ([3])

We will generalize Bazylewicz's result to non entire functions of

exponential type in  $\mathbb{C}^m$  using the theory of analytic functionals with non-compact carrier (a kind of Fourier ultra hyperfunction). To close this section, we notice the special relation between the theory of arithmetic holomorphic function and the Ising model in statistical physics. ([7]).

## 2. Notations

Let  $K$  be a number field over  $\mathbb{Q}$  with degree  $[K:\mathbb{Q}] = d = r+2s$ .  $K^{(i)}$  ( $1 \leq i \leq r$ ) and  $\overline{K}^{(r+j)} = K^{(r+s+j)}$  ( $1 \leq j \leq s$ ) are its conjugate field.

$\delta$  is defined as follows:

$$\delta = \begin{cases} d & \text{if } K \subset \mathbb{R} \\ d/2 & \text{if } K \not\subset \mathbb{R} \end{cases}$$

For algebraic integer  $a$ , we put

$$\overline{|a|} = \max_{1 \leq i \leq s} |a_i| \quad (a_i \text{'s are conjugates over } \mathbb{Q} )$$

Then following inequality valids :

$$\log |a| \geq -(\delta-1) \log \overline{|a|}$$

(inegalite de la taille)

$O_A$  denotes the ring of algebraic integers.

$O_K^{(j)}$  denotes the ring of algebraic integers in  $K^{(j)}$ .

For  $p(z) = \sum a_n z^n \in K[z]$ , we put  $p^{(j)}(z) = \sum a_n^{(j)} z^n \in K^{(j)}[z]$ ,

where  $a_n^{(j)}$  is conjugate of  $a_n$ .

$\tau(A)$  denotes the transfinite diameter of compact set  $A$ .

For the details of transfinite diameter we refer the reader to [1].

Supporting function  $H_A(z)$  of set  $A$  is defined by

$$H_A(z) = \sup_{\zeta \in A} \operatorname{Re} \langle z, \zeta \rangle$$

$$|z| = |z_1| + |z_2| + \dots + |z_m|, \quad (z_1, z_2, \dots, z_m) \in \mathbb{C}^m$$

### 3. Some results on $O_K$ valued entire function of exponential type

In 1976-1977, F. Gramain obtained following theorems.

**Theorem 2.** ([4]) Let  $K_1$  be a compact convex set in  $\mathbb{C}$  contained in  $\{\zeta \in \mathbb{C} : |\operatorname{Im} \zeta| < \pi\}$ . Suppose that  $f(z)$  satisfies following conditions (i), (ii), (iii).

- (i)  $f(z)$  is entire function of exponential type  $H_{K_1}$ .
- (ii)  $f(\mathbb{N}) \subset O_K$
- (iii)  $\limsup_{n \rightarrow \infty} 1/n \log |f(n)| \leq c$

If  $\log \tau(\exp(K_1)) < -c(\delta-1)$ , then  $f(z) = \sum P_b(z) b^z$ .

Where  $P_b(z) \in K(\vartheta)[z]$  and  $\vartheta = \{a : a \text{ algebraic integer, } |a| \leq e^c, a \text{ is contained in } \exp(K_1) \text{ together with its conjugates over } K.\}$ .

**Theorem 3.** (A generalization of Polya 's theorem [5] )

- (i)  $f(z)$  is entire function of exponential type  $\alpha$
- (ii)  $f(\mathbb{N}^m) \subset O_K$
- (iii)  $\limsup_{n \rightarrow \infty} 1/n \log |f(n)| \leq c$

If  $\log(e^\alpha - 1) < -(\delta-1)\log(1+e^c)$ , then  $f(z) \in K[z]$ .

In 1988, Bazylewicz generalized theorem 2 to entire functions of exponential type in  $\mathbb{C}^n$ . Namely, he obtained following theorem 4.

**Theorem 4.** ([3]) Suppose that  $f(z)$  is entire function of exponential type satisfying following conditions:

(i) For any  $\varepsilon > 0$ , there exists  $C_\varepsilon \geq 0$  such that

$$|f(z)| \leq C_\varepsilon \exp\left(\sum_{i=1}^m H_{T_i}(z_i) + \varepsilon |z|\right), \quad z = (z_1, z_2, \dots, z_m) \in \mathbb{C}^m$$

where  $T_i$ 's are compact convex sets in  $\mathbb{C}$ .

(ii)  $f(\mathbb{N}^m) \subset O_K$

(iii) there exists a positive number  $c$  such that

$$\limsup_{n \rightarrow \infty} 1/n \log \overline{|f(n)|} \leq c$$

If  $\log \tau(\exp(T_i)) < -c(\delta-1)$  ( $1 \leq i \leq m$ ), then

$$f(z) = \sum P_b(z) b^z,$$

where  $P_b(z) \in K(\mathfrak{A})[z]$  and  $\mathfrak{A}$  is a finite set of  $O_A^m$ .

#### 4. Main result

Using the theory of analytic functional with non-compact carrier we can generalize theorem 4 to holomorphic functions of exponential type defined in the product of half plane. Namely, we can generalize theorem 4 as follows.

**Theorem 5. (Main result)** Suppose that  $0 \leq k' < 1$  and function  $f(z)$  satisfies following conditions :

(i)  $f(z)$  is holomorphic in  $\prod_{i=1}^m \{z_i : \operatorname{Re} z_i < -k'\}$

(ii) For any  $\varepsilon > 0$  and  $\varepsilon' > 0$ , there exists  $C_{\varepsilon, \varepsilon'} \geq 0$  such that

$$|f(z)| \leq C_{\varepsilon, \varepsilon'} \exp(H_L(z) + \varepsilon|z|) \quad (\operatorname{Re} z_i \leq -k' - \varepsilon')$$

where  $L$  is a convex set contained in  $\prod_{i=1}^m \{ \zeta_i \in \mathbb{C} : |\operatorname{Im} \zeta_i| \leq b_i < \pi, \operatorname{Re} \zeta_i \geq a_i \}$ .

$$(iii) f((-N)^m) \subset O_K$$

$$(iv) \limsup_{n \rightarrow \infty} 1/n \log |f(-n)| \leq c$$

If  $\tau(\exp(-T_i)) < -c(\delta-1)$  ( $1 \leq i \leq m$ ), then  $f(z) = \sum P_b(z) b^z$  where  $P_b \in K(\mathfrak{F})[z]$ , and  $\mathfrak{F}$  is a finite set of  $O_A^m$ .

### 5. Transformations of analytic functionals with non-compact carriers

In this section we recall Avanissian-Gay transform and Laplace transform of analytic functionals with non-compact carrier.

Let  $L$  be a convex closed set in  $\mathbb{C}^m$  bounded in imaginary direction. we define holomorphic test function space  $Q(L; k')$  as follows:

$$Q(L; k') = \varprojlim_{\varepsilon > 0, \varepsilon' > 0} \operatorname{ind}_{\varepsilon, \varepsilon'} Q_b(L_\varepsilon; k' + \varepsilon')$$

$$Q_b(L_\varepsilon; k' + \varepsilon') = \{ f \in \mathcal{O}(L_\varepsilon) \cap C(\overline{L_\varepsilon}) : \sup_{\zeta \in L_\varepsilon} |f(\zeta) \exp((k' + \varepsilon')\zeta)| \leq \infty \}$$

$Q'(L; k)$  denotes the dual space of  $Q(L; k')$ . The element of dual space is called analytic functional with carrier  $L$ . Now we define Laplace transform  $\tilde{T}(z)$  of  $T \in Q'(L; k')$  as follows:

$$\tilde{T}(z) = \langle T_\zeta, \exp(z\zeta) \rangle$$

$$\text{where } z\zeta = z_1\zeta_1 + \dots + z_m\zeta_m$$

Following Paley-Wiener type theorem characterizes Laplace transform of  $Q'(L; k')$ .

**Theorem 6.** ([10])

Let  $T$  belong to  $Q'(L:k')$ , then  $\tilde{T}(z)$  is holomorphic function in  $D = \prod_{i=1}^m \{z_i \in \mathbb{C} : \operatorname{Re} z_i < -k'\}$  and satisfies following estimate:

for any  $\varepsilon > 0$ ,  $\varepsilon' > 0$ , there exists  $C_{\varepsilon, \varepsilon'} \geq 0$  such that

$$(*) \quad |\tilde{T}(z)| \leq C_{\varepsilon, \varepsilon'} \exp(H_L(z) + \varepsilon |z|) \quad (\operatorname{Re} z_i \leq -k' - \varepsilon')$$

Conversely, if holomorphic function  $f(z)$  in  $D$  satisfies  $(*)$ , then  $f(z)$  is a Laplace transform of some  $T \in Q'(L:k')$ .

For the details of Laplace transform of analytic functionals, we refer the reader to [10]

To define Avannissian-Gay transform we put following assumptions :

(i)  $0 \leq k' < 1$

(ii)  $\operatorname{pr}_i(L) \subset \{\zeta_i \in \mathbb{C} : |\operatorname{Im} \zeta_i| \leq b_i < \pi, \operatorname{Re} \zeta_i \geq a_i\}$

where  $\operatorname{pr}_i(L)$  denotes the  $i$ -th projection of  $L$ .

Avannissian-Gay transform  $G_T(w)$  of  $T \in Q(L:k)$  is defined as follows :

$$G_T(w) = \langle T_\zeta, \prod_{i=1}^m (1 - w_i \zeta_i)^{-1} \rangle$$

Avannissian-Gay transform has following properties:

**Proposition 1.** ([7],[10])

(i)  $G_T(w)$  is holomorphic in  $\prod_{i=1}^m (\mathbb{C}/\exp(-L_i))$ .

(ii)  $G_T(w) = (-1)^m \sum T(-n_1, -n_2, \dots, -n_m) w_1^{-n_1} w_2^{-n_2} \dots w_m^{-n_m}$

(iii) for any  $\varepsilon > 0$ , and  $\varepsilon' > 0$ , there exists  $C_{\varepsilon, \varepsilon'} \geq 0$  such that

$$|G_T(w)| \leq C_{\varepsilon, \varepsilon'} |w_1|^{-k'-\varepsilon'} |w_2|^{-k'-\varepsilon'} \dots |w_m|^{-k'-\varepsilon'}$$

(iv) (inversion formula )

$$T(z) = (2\pi i)^{-m} \int_{\Gamma} G_T(w_1, w_2, \dots, w_m) w_1^{-z-1} w_2^{-z-1} \dots w_m^{-z-1} dw_1 dw_2 \dots dw_m$$

where  $\Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_m$ ,  $\Gamma_i (1 \leq i \leq m)$  is the contour surrounding  $\exp(-L_i)$ .

### 6. Proof of main result.

In this section we will prove theorem 5. Following proposition is essential in our discussion.

#### Proposition 2(Bazylewicz[3])

$S_{i,j} (1 \leq i \leq m)$  are compact sets in  $\mathbb{C}$  and  $\tau_{i,j} = \tau(S_{i,j})$  are their transfinite diameters. Suppose that  $S_{i,j}$  satisfies following conditions:

$$(i) \quad S_{i,j} = \overline{S_{i,j}} \quad (1 \leq j \leq r) \quad (\text{i.e. } S_{i,j} \subset \mathbb{R})$$

$$S_{i,j+r} = \overline{S_{i,r+j+s}} \quad (1 \leq j \leq s)$$

$$(ii) \quad \prod_{j=1}^d \tau_{i,j} < 1 \quad (1 \leq i \leq m)$$

We suppose that

$$g^{(j)}(z) = \sum_{n \in \mathbb{N}^m} a_n^{(j)} z^{-1-n} \quad \text{is holomorphic in } \prod_{i=1}^m (\mathbb{C} / S_{i,j})$$

where  $a_n^{(j)}$ 's are algebraic integers in  $K$ . Then there exists

polynomials  $P(z_1, z_2, \dots, z_m)$ ,  $Q_i(z_i)$  satisfying following conditions :

(1)  $Q_i(z_i)$  are monic (coefficient of highest degree term is unit.)

$$(2) \quad \deg_{z_i} P(z_1, z_2, \dots, z_m) \leq \deg Q_i(z_i)$$

$$(3) \quad g^{(j)}(z) = P^{(j)}(z_1, z_2, \dots, z_m) / \prod_{i=1}^m Q_i^{(j)}(z_i)$$

Now we give the proof of theorem 5. By Theorem 4, there exists  $T \in Q(L: k)$  such that  $f(z) = T(z)$ . We put  $g(w) = G_T(w)$ . By (ii) in proposition 1, we have

$$g(w) = (-1)^m \sum_{n \in \mathbb{N}^m} f(-n) w^{-1-n}.$$

Now we put

$$g^{(j)}(w) = (-1)^m \sum_{n \in \mathbb{N}^m} f(-n)^{(j)} w^{-1-n} \quad (1 \leq j \leq d).$$

Each  $g^{(j)}$  is holomorphic in  $\prod_{i=1}^m \{|w_i| \geq e^c\}$ .

Since  $g(w)$  is Avanissian-Gay transform of  $T$ ,  $g(w)$  and  $g^{(j)}(w)$  is holomorphic in

$$\prod_{i=1}^m (\mathbb{C}/\overline{\exp(-L_i)}) , \quad \prod_{i=1}^m (\mathbb{C}/\exp(-L_i)).$$

Here  $\overline{\exp(-L_i)}$  is a closure of  $\exp(-L_i)$ , and  $\mathbb{C}/\overline{\exp(-L_i)}$  is complex conjugate of  $\mathbb{C}/\exp(-L_i)$ .

We put  $S_i = \overline{\exp(-L_i)}$  and

$$S_i^{(j)} = \begin{cases} S_i & \text{if } K = K^{(j)} \\ \overline{S_i} & \text{if } \overline{K} = K^{(j)} \\ |w_i| \leq e^c, & \text{other case.} \end{cases}$$

Then  $g^{(j)}$  and  $S_i^{(j)}$  satisfy all assumptions in proposition 2. So we have

$$G_T(w) = g(w) = P(w_1, w_2, \dots, w_m) / \prod_{i=1}^m Q_i(w_i)$$

where  $P$  and  $Q_i$ 's are polynomials satisfying the conditions in proposition 2.

By inversion formula in proposition 1 and residue theorem, we obtain

$$\begin{aligned}
f(z) &= (2\pi i)^{-1} \int G_T(w) w_1^{-z-1} w_2^{-z-1} \dots w_m^{-z-1} dw_1 dw_2 \dots dw_m \\
&= (2\pi i)^{-1} \int P(w_1, \dots, w_m) / \prod_{i=1}^m Q_i(w_i) w_1^{-z-1} \dots w_m^{-z-1} dw_1 \dots dw_m \\
&= \sum_{\substack{\gamma \\ \text{integer}}} \text{algebraic } P_\gamma(z) \gamma^z.
\end{aligned}$$

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