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Determinantal Ideals and Their Betti Numbers— A Survey

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Abstract

This note is an introduction to the ring-theoretical approach to the study of determinantal varieties, especially to the study of minimal free resolutions of determinantal ideals.

1 Determinantal Rings as ASL's

Let A be a noetherian ring, I an ideal of A, and M a finitely generated A-module. We define the *I-depth* of M to be $\min\{i \mid \operatorname{Ext}^i_A(A/I, M) \neq 0\}$ and denote it by $\operatorname{depth}(I, M)$.

If A is a local ring with the maximal ideal $\mathfrak{m}$, then $\operatorname{depth}(\mathfrak{m}, M)$ is sometimes denoted by $\operatorname{depth} M$. In this case, we have $\operatorname{depth} M \leq \dim M$ for $M \neq 0$, where $\dim M$ is the Kurll dimension of $A/\operatorname{ann}_A M$. We say that M is Cohen-Macaulay when the equality holds, or $M = 0$. We say that the local ring A is Cohen-Macaulay when so is A as an A-module. A noetherian ring (which may not be local) A is said to be Cohen-Macaulay when its localization at any maximal ideal is Cohen-Macaulay local.

Cohen-Macaulay property is one of the most important notion in the modern commutative ring theory.

Lemma 1.1 *Let A be a d-dimensional graded K-algebra (K a field) generated by finite degree one elements. Then, the following hold.*

1 *A is Cohen-Macaulay if and only if* $\operatorname{depth}(A_+, A) = d$, *where A_+ is the ideal of A consisting of all degree positive elements.*

2 *(K is assumed to be infinite) Let $\theta_1, \ldots, \theta_d$ be degree one elements such that A is a finite module over $K[\underline{\theta}] = K[\theta_1, \ldots, \theta_d] \subset A$ (such $\theta_1, \ldots, \theta_d$ do exist). Then, A is Cohen-Macaulay if and only if A is a* free *$K[\underline{\theta}]$-module (hence, this condition does not depend on the choice of $\theta_1, \ldots, \theta_d$).*

3 *Let $a_1, \ldots, a_r$ be the degree one generator of A as a K-algebra so that the map*

$$S = K[x_1, \ldots, x_r] \to K[a_1, \ldots, a_r] = A \quad (x_i \mapsto a_i)$$

is a surjective map of graded K-algebras. Then, A is Cohen-Macaulay if and only if $\operatorname{pd}_S A = r - d$, where pd *denotes the projective dimension.*

Gorenstein property is also important homological property. A noetherian local ring A is said to be Gorenstein when its self-injective dimension is finite. A noetherian ring is said to be Gorenstein when its localization at any maximal ideal is Gorenstein. Any Gorenstein ring is Cohen-Macaulay, but the converse is not true in general.

Lemma 1.2 *Let A be a d-dimensional Cohen-Macaulay graded K-algebra (K a field) generated by finite degree one elements. Then, the following hold.*

1 *A is Gorenstein if and only if $\mathrm{Ext}_A^d(A/A_+, A) \cong K$.*

2 *Let $F_A(t) = \sum_{i\geq 0}(\dim_K A_i)t^i$. Then, $(1-t)^d F_A(t)$ is a polynomial in t, say, $h_0 + h_1 t + \cdots + h_s t^s$ ($h_s \neq 0$). If A is Gorenstein, then $h_s = 1$. The converse is true when A is an integral domain.*

3 *Let $a_1, \ldots, a_r$ be degree-one generators of A, and consider A as a module over $S = K[x_1, \ldots, x_r]$. Then, the following are equivalent.*

  a *A is Gorenstein.*

  b *$\mathrm{Ext}_S^{r-d}(A, S)$ is cyclic as an S-module.*

  b' *$\mathrm{Ext}_S^{r-d}(A, S) \cong A$ as an S-module.*

For a graded K-algebra A, a graded A-module M is said to be free when M is a direct sum of modules of the form $A(i)$, where $A(i)$ is simply A as an A-module, and the grading is given by $A(i)_j = A_{i+j}$. Clearly, a free module is projective in the category of graded A-modules. Assume that A is generated by finite elements of positive degree. For a finitely generated graded A-module M and its subset $S = \{m_1, \ldots, m_r\}$, S generates M if and only if the image of S generates M/A_+M (an analogue of Nakayama's lemma). So, S is a set of minimal generators if and only if its image in M/A_+M is a K-basis.

Let R be a commutative ring with unity. For a matrix $(a_{i,j}) \in \mathrm{Mat}_{m,n}(R)$ with coefficients in R and a positive integer t, we define the *determinantal ideal* $I_t((a_{i,j}))$ of the matrix $(a_{i,j})$ to be the ideal of R generated by all t-minors of $(a_{i,j})$.

We are interested in the *generic case* here. Let $S = R[x_{ij}]_{1\leq i\leq m, 1\leq j\leq n}$ be a polynomial ring over R in mn variables. We set $X = (x_{ij}) \in \mathrm{Mat}_{m,n}(S)$. The ideal $I_t = I_t(X) \subset S$ is considered to be a generic determinantal ideal. When we consider S as a coordinate ring of the affine space $\mathrm{Mat}_{m,n}(R)$, the ideal I_t defines the closed subscheme Y_t, the space of $m \times n$ matrices whose rank is smaller than t (because the rank of a matrix is smaller than t if and only if its all t-minors vanish). The following is a fundamental theorem on determinantal ideals.

Theorem 1.3 (Hochster-Eagon [HE]) *Let R be noetherian. The following hold.*

1. $\dim S/I_t = \dim R + mn - (m-t+1)(n-t+1)$.

2. *The ideal I_t is perfect (of codimension $(m-t+1)(n-t+1)$). Namely, we have*

$$\mathrm{depth}_S(I_t, S) = \mathrm{pd}_S\, S/I_t = (m-t+1)(n-t+1).$$

3. S/I_t *is* R*-flat.*

4. If R *is a domain, then so is* S/I_t.

5. If R *is normal, then so is* S/I_t.

Where pd denotes the projective dimension, and $\operatorname{depth}(I_t, S) = \min\{i \mid \operatorname{Ext}^i_S(S/I_t, S) \neq 0\}$. There are some different proof of this theorem. In this section, we give a (sketch of a) purely algebraic (or combinatorial) proof of the theorem which uses the theory of ASL's. The lecture note [BV] gives a systematic account on this treatment.

The general theory tells us that it suffices to prove the following provided we have proved that S/I_t is R-flat.

Corollary 1.4 *Assume that* R *is a field. Then, we have* S/I_t *is a Cohen-Macaulay normal domain of dimension* $mn - (m - t + 1)(n - t + 1)$.

Definition 1.5 Let R be a commutative ring, and P a finite poset (= partially ordered set). We say that A is a (graded) ASL (algebra with straightening lows) on P over R if the followings hold.

ASL-0 An injective map $P \hookrightarrow A$ is given, A a graded R-algebra generated by P, and each element of P is homogeneous of positive degree. We call a product of elements of P a monomial in P. Formally, a monomial M is a map $P \to \mathbb{N}_0$, and we denote $M = \prod_{x \in P} x^{M(x)}$ so that it also stands for an element of A. A monomial in P of the form

$$x_{i_1} \cdots x_{i_l}$$

with $x_{i_1} \leq \cdots \leq x_{i_l}$ is called *standard*.

ASL-1 The set of standard monomials in P is an R-free basis of A.

ASL-2 For $x, y \in P$ such that $x \not\leq y$ and $y \not\leq x$, there is an expression of the form

$$xy = \sum_M c_M^{xy} M \quad (c_M^{xy} \in R) \tag{1.6}$$

where the sum is taken over all standard monomials $M = x_1 \cdots x_{r_M}$ $(x_1 \leq \cdots \leq x_{r_M})$ with $x_1 < x, y$ and $\deg M = \deg(xy)$.

The expression (1.6) in (ASL-2) condition is called the straightening relations of A. The most simple example of an ASL on P over R is the Stanley-Reisner ring $R[P] = R[x \mid x \in P]/(xy \mid x \not\leq y, y \not\leq x)$. The (ASL-2) condition is satisfied with letting the right-hand side zero. The Stanley-Reisner rings play central rôle in the theory of ASL.

Theorem 1.7 ([DEP]) *Let* R *be a commutative ring,* P *a finite poset, and* A *an ASL on* P *over* R*. Then, there is a sequence of ASL's on* P *over* R $A = A_0, A_1, \ldots, A_m = R[P]$ *and an ideal* I_i *of* A_i *for each* $i < m$ *such that* $A_{i+1} = G_{I_i} A_i$ *for* $i < m$.

Here, for a ring A and its ideal I, $G_I(A)$ denotes the associated graded ring $A[t^{-1}I, t]/(t)$. Usually, the associated graded ring $G_I(A)$ is worse than A. Hence, by the theorem, if $R[P]$ enjoys good property, then so does any ASL on P over R.

Corollary 1.8 *If $R[P]$ is an integral domain (resp. Cohen-Macaulay, normal, Gorenstein), then any ASL on P over R enjoys the same property.*

As is clear, ASL's on P over a field have the same Hilbert function provided we give the same degree to each element in P. So it is completely determined only by the combinatorial information on P (because $H_R(n, R[P]) = \#\{\text{standard monomials in } P \text{ of degree } n\}$). The corollary is not a good criterion of integrality, normality or Gorenstein property, because $R[P]$ rarely satisfies these conditions. However, the corollary gives a good criterion of Cohen-Macaulay property.

Proposition 1.9 *If R is Cohen-Macaulay and if P is a distributive lattice, then $R[P]$ is Cohen-Macaulay.*

For the proof of these results, see [DEP].

As a result, the determinantal ring S/I_t has a structure of an ASL on a distributive lattice over R, where $S = R[x_{ij}]_{1\le i\le m, 1\le j\le n}$. This shows that S/I_t is R-flat (by ASL-1) and that S/I_t is Cohen-Macaulay when so is R.

First, we introduce an ASL structure into S.

We set

$$\Omega_s = \{[i_1, \ldots, i_s; j_1, \ldots, j_s] \mid 1 \le i_1 < \cdots < i_s \le m,\ 1 \le j_1 < \cdots < j_s \le n\}$$

and $\Omega = \bigcup_{s=1}^{\min(m,n)} \Omega_s$. We introduce an order structure into Ω. For elements $d = [i_1, \ldots, i_s; j_1, \ldots, j_s]$ and $d' = [i'_1, \ldots, i'_{s'}; j'_1, \ldots, j'_{s'}]$ of Ω, we say that $d \le d'$ if $s \ge s'$ and if $i_l \le i'_l$, $j_l \le j'_l$ for $1 \le l \le s'$. It is easy to see that Ω is a distributive lattice with this order structure.

We have a map $\Omega \to S$ given by

$$[i_1, \ldots, i_s; j_1, \ldots, j_s] \mapsto \det(x_{i_\alpha, j_\beta})_{1\le\alpha,\beta\le s}.$$

Lemma 1.10 *With the structure above, S is an ASL on Ω over R with the straightening relation of the form*

$$ab = (a \wedge b)(a \vee b) + \sum_{c,d} u^{ab}_{cd} cd (+ve) \quad (u^{ab}_{cd}, v \in R)$$

for each $a = [a_1, \ldots, a_s; a'_1, \ldots, a'_s]$ and $b = [b_1, \ldots, b_{s'}; b'_1, \ldots, b'_{s'}]$, where $a \wedge b = \inf(a, b)$, $a \vee b = \sup(a, b)$, the sum is taken over all $c \in \Omega_l$ and $d \in \Omega_{l'}$ such that $c < a \wedge b$ and that $l + l' = s + s'$. If $a_1, \ldots, a_s, b_1, \ldots, b_{s'}$ are all distinct with its rearrangement in the increasing order is $c_1, \ldots, c_{s+s'}$, and if $a'_1, \ldots, a'_s, b'_1, \ldots, b'_{s'}$ are all distinct with its rearrangement in the increasing order is $c'_1, \ldots, c_{s+s'}$, then the term ve ($v \in R$) may appear in the right-hand side, where $e = [c_1, \ldots, c_{s+s'}; c'_1, \ldots, c'_{s+s'}]$. The grading of S is the usual grading (i.e., each x_{ij} is degree one).

This is proved using the Laplace expansion rule. See for example, [ABW2]. The ASL structure above is good with the determinantal ideals I_t.

For a poset P and its subset Q, we say that Q is a poset ideal of P when for any $x \in P$ and $y \in Q$, $x \leq y$ implies $x \in Q$.

Lemma 1.11 *Let A be an ASL on P over R with the straightening relation (1.6). If Q is a poset ideal of P, then A/I_Q is an ASL on $P - Q$ over R with the straightening relation*

$$xy = \sum_M c_M^{xy} M \quad (c_M^{xy} \in R)$$

where I_Q is the ideal $(x \mid x \in Q)$ in A generated by Q and the sum is taken over all M that appears in (1.6) such that no element in Q appears in M.

The proof is straightforward. Applying this lemma to the ASL S on Ω and the poset ideal $\Omega_{\geq t} = \bigcup_{s \geq t} \Omega_s$ of Ω, we conclude that S/I_t is an ASL on $\Omega_{<t} = \Omega - \Omega_{\geq t}$. Thus, S/I_t is R-flat for any R by (ASL-1). Moreover, it is easy to see that $\Omega_{<t}$ is a sublattice of Ω, and hence is a distributive lattice. This shows that S/I_t is Cohen-Macaulay when so is R.

It remains to show that S/I_t is a normal domain when R is a field. There is a good criterion of normality for ASL's on distributive lattices due to Ito.

Theorem 1.12 ([Ito, Corollary]) *Assume that R is a Cohen-Macaulay normal domain. Let A be an ASL over a distributive lattice L with the straightening relation*

$$xy = (x \wedge y)(x \vee y) + \sum_M c_M^{xy} M \quad (c_M^{xy} \in R),$$

where the sum is taken over standard monomials $M = x_1 \cdots x_{r_M}$ which have the same degree as xy with $x_1 < x \wedge y$. Then, A is a Cohen-Macaulay normal domain.

The determinantal ring S/I_t satisfies the assumption of this criterion, so it is a normal domain. It is straightforward to see that $\operatorname{rank} \Omega_{<t} = mn - (m-t+1)(n-t+1) - 1$ so that $\dim S/I_t = \dim R + mn - (m-t+1)(n-t+1)$, and the proof of Theorem 1.3 is completed.

Hibi [Hib] defined the algebra $\mathcal{R}_R[L] = R[x \in L]/(xy - (x \wedge y)(x \vee y))$ for distributive lattices, and showed that this algebra is a Cohen-Macaulay normal domain. The algebra $\mathcal{R}_R[L]$ is called the *Hibi ring* of L over R. It follows that any distributive lattice is integral. He posed a question that an ASL on L with some good straightening relation is a normal domain [Hib, p.103]. Ito's criterion is a good answer to this question.

For the Gorenstein property, Hibi completely determined when Hibi ring is Gorenstein.

Theorem 1.13 ([Hib, p.107]) *Let A be as in Theorem 1.12. Then, A is Gorenstein if and only if R is Gorenstein, and P is pure, where P is the set of join-irreducible elements in L. That is,*

$$P = \{x \in L \mid \#\{y \in L \mid \#\{z \in L \mid y \leq z \leq x\} = 2\} = 1\}.$$

Note that if the theorem is true for Hibi rings, then the theorem is true in general by **2** of Lemma 1.2. It is obvious that $x = [a_1, \ldots, a_s; b_1, \ldots, b_s] \in \Omega_{<t}$ is join-irreducible if and only if one of the following is satisfied (we assume $t \leq \min(m, n)$).

1. $x = [1, \ldots, s; 1, \ldots, s]$ $(1 \leq s \leq t-2)$
2. $x = [1, \ldots, i, m-s+i+1, \ldots, m; 1, \ldots, s]$ $(1 \leq s \leq t-1,\ 0 \leq i \leq s-1)$
3. $x = [1, \ldots, s; 1, \ldots, i, n-s+i+1, \ldots, n]$ $(1 \leq s \leq t-1,\ 0 \leq i \leq s-1)$

From this, it is not so difficult to show that $\Omega_{<t}$ is pure if and only if $t = 1$ $(\Omega_{<t} = \emptyset)$ or $m = n$.

Corollary 1.14 *S/I_t is Gorenstein if and only if R is Gorenstein, and $t = 1$ or $m = n$.*

2 A Mininal Free Resolution

There has been much interest in determinantal ideals from the viewpoint of homological algebra. Among them, the following is an interesting problem.

Problem 2.1 1. Construct a minimal free resolution of S/I_t as a graded S-module.

2. Assume that R is a field. Calculate the graded Betti numbers

$$\beta_{ij}^R = \dim_R [\operatorname{Tor}_i^S(S/S_+, S/I_t)]_j,$$

where $S_+ = I_1 = (x_{ij})$, and $[\ ]_j$ denotes the degree j component of a graded S-module.

Here, a graded S-complex (i.e., a chain complex in the category of graded S-modules)

$$\mathsf{F} : \cdots \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

is said to be a free resolution of a graded S-module M when each F_i is free, $H_i(\mathsf{F}) = 0$ $(i > 0)$ and $H_0(\mathsf{F}) = M$. It is called minimal when the boundary maps of $S/S_+ \otimes \mathsf{F}$ are all zero. A graded minimal free resolution is unique up to isomorphism. It exists when the base ring R is a field.

Since S/I_t is free as an R-module, we have $\operatorname{Tor}_i^R(M, S/I_t) = 0$ for $i > 0$ and any R-module M. Hence, if F is a projective resolution of S/I_t over the base ring R, and if R' is an R-algebra, then $R' \otimes_R \mathsf{F}$ is a projective resolution of $R' \otimes_R S/I_t$. If F is graded minimal free, then so is $R' \otimes_R S/I_t$. So, if 1 of the problem is solved for the ring of integers $\mathbf{Z}$, then 1 is solved for any R, because we can get the resolution by base change $R \otimes_{\mathbf{Z}} ?$.

Let F be a graded minimal free resolution of S/I_t. Then, $H_i(S/S_+ \otimes_S \mathsf{F}) = S/S_+ \otimes F_i$ is an R-free module, and we have

$$\infty > \operatorname{rank}_R \operatorname{Tor}_i^S(S/S_+, S/I_t) = \operatorname{rank}_S F_i.$$

Note that the right hand side is invariant under the base change. In particular, for any R-algebra K which is a field, we have $\beta_i^K = \operatorname{rank}_S F_i$. Thus, the problem 2 is easier than 1 (for example, if 1 is solved for any field, then 2 is completely solved).

Assume that R is a field. Since S/I_t is Cohen-Macaulay of dimension $\dim S - (m-t+1)(n-t+1)$, we have $\operatorname{pd}_S S/I_t = (m-t+1)(n-t+1)$. We set $h = (m-t+1)(n-t+1)$. Then, we have $\beta_h^R \neq 0$ and $\beta_i^R = 0$ for $i > h$. The ring S/I_t is Gorenstein if and only if $\beta_h = 1$ by Lemma 1.2. Let F be a graded minimal free resolution of S/I_t. Then, we have

$$H_i(\operatorname{Hom}_S(\mathsf{F}, S)) = \operatorname{Ext}_S^{-i}(S/I_t, S) = 0$$

unless $i = -h$ by Lemma 1.1, since S/I_t is Cohen-Macaulay of codimension h. So the complex $\operatorname{Hom}_S(\mathsf{F}, S)[-h]$ ([ ] denotes the shift of the degree as a chain complex) is a minimal free resolution of the S-module $\operatorname{Ext}_S^h(S/I_t, S)$. When S/I_t is Gorenstein, we have $\operatorname{Ext}_S^h(S/I_t, S) \cong S/I_t$. This shows that $\operatorname{Hom}_S(\mathsf{F}, S)[-h]$ is a graded minimal free resolution of S/I_t (the grading as a graded S-module may be different, so we should say $\operatorname{Hom}_S(\mathsf{F}, S)(a)[h]$ is a graded minimal free resolution of S/I_t for some $a \in \mathbf{Z}$). This shows that

$$F_i \cong \operatorname{Hom}_S(\mathsf{F}, S)[h]_i(a) = \operatorname{Hom}_S(F_{h-i}, S)(a),$$

and we have $\beta_i = \beta_{h-i}$.

Why is the problem a problem? First, constructing a graded minimal free resolution of S/I as an S-module (for a homogeneous polynomial ring $S = K[x_1, \ldots, x_r]$ over a field K and its homogeneous ideal I) has been considered as an ultimate aim of homological study of the algebra S/I—knowing a minimal free resolution yields ample information on the ring in question. For example, S/I is Cohen-Macaulay if and only if $\beta_i(S/I) = 0$ for $i > \dim S - \dim S/I$. It is Gorenstein if and only if it is Cohen-Macaulay and $\beta_{\dim S - \dim S/I}(S/I) = 1$. So the Betti numbers β_i of an algebra contain a lot of information of the algebra (however, nowadays the progress of the theory of commutative algebra provides us a lot of tools for studying important homological properties (such as Cohen-Macaulay property) of commutative algebras without constructing a resolution).

Secondly, the theory of the resolution of determinantal ideals is an interaction between the theory of commutative algebra, combinatorics and the representation theory of algebraic groups, and is interesting itself.

The number β_i^K depends only on the characteristic p of K, so we also write β_i^p.

When there exists a graded minimal free resolution F of S/I_t over the ring of integers so that the resolution is obtained by base change for an arbitrary ring? Clearly, if such a resolution exists over $\mathbf{Z}$, then β_i^p is independent of p. The converse is true.

Lemma 2.2 ([Rob, Chapter 4, Proposition 2], [HK, Proposition II.3.4]) *Assume that R is a noetherian reduced ring such that any finitely generated projective R module is free. Let $A = R[x_1, \ldots, x_n]$ be a homogeneous polynomial ring over R, and M a finitely generated graded A-module which is flat as an R-module. Then, the following are equivalent for any $i \geq 0$.*

1 *There exists a graded minimal free complex*

$$0 \to \mathsf{F}_{i+1} \xrightarrow{\partial_{i+1}} \mathsf{F}_i \xrightarrow{\partial_i} \cdots \to \mathsf{F}_1 \xrightarrow{\partial_1} \mathsf{F}_0 \to 0$$

such that $H_0\mathsf{F} = M$ *and* $H_k\mathsf{F} = 0$ *for* $1 \le k \le i$.

2 *For any* $0 \le k \le i$ *and* $j \in \mathbb{N}_0$, *the numbers*

$$\beta_{kj}^K(M) \stackrel{def}{=} \dim_{R/\mathfrak{M}}[\mathrm{Tor}_k^{R/\mathfrak{M}\otimes_R A}(R/\mathfrak{M}\otimes_R A/A_+, R/\mathfrak{M}\otimes_R M)]_j$$

is independent of the maximal ideal $\mathfrak{M}$ *of* R, *where* $[\,]_j$ *denotes the degree* j *component of a graded* A*-module.*

3 *For any* $0 \le k \le i$, *the Betti numbers* $\beta_k^K(M) = \beta_k(R/\mathfrak{M}\otimes_R M)$ *(over the field* $K = R/\mathfrak{M}$*) is independent of the maximal ideal* $\mathfrak{M}$ *of* R.

4 *For any* $0 \le k \le i$, $\mathrm{Tor}_k^A(A/A_+, M)$ *is a free* R*-module.*

Thus, there exists a graded minimal free resolution of S/I_t over $\mathbf{Z}$ if and only if $\beta_i^p(S/I_t)$ is independent of p for any i.

Problem 2.3 Are the Betti numbers $\beta_i^p(S/I_t)$ independent of the characteristic?

Known approaches to the problem of the resolutions of determinantal ideals more or less depend on representation theory of GL. Let $V = R^n$ and $W = R^m$. Then, the polynomial ring $S = R[x_{ij}]_{1\le i\le m, 1\le j\le n}$ is identified with the symmetric algebra $S(V\otimes W)$, on which $G = \mathrm{GL}(V)\times\mathrm{GL}(W)$ acts. It is clear that I_t is invariant under the action of G.

Among various tools in the representation theory of GL, Schur modules and Schur complexes are very important.

Let R be a commutative ring which contains the field of rationals $\mathbf{Q}$, and C a finite free R-complex (i.e., bounded R-complex with each term finite free). For $n > 0$, the symmetric group $\mathfrak{S}_n$ acts on $C^{\times n}$ by

$$\sigma(a_1\otimes\cdots\otimes a_n) = (-1)^{\sum_{i<j,\,\sigma i>\sigma j}\deg(a_i)\deg(a_j)} a_{\sigma^{-1}1}\otimes\cdots\otimes a_{\sigma^{-1}n}$$

for $\sigma\in\mathfrak{S}_n$.

For a partition (i.e., a weakly decreasing sequence of non-negative integers) $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_l)$ with $\sum_i\lambda_i = n$, we set $L_\lambda C := \mathrm{Hom}_{\mathfrak{S}_n}(s_{\tilde\lambda}, C^{\otimes n})$, where $s_{\tilde\lambda}$ is the Specht module (see e.g., [Gr]) of $\tilde\lambda$ ($\tilde\lambda = (\tilde\lambda_1,\tilde\lambda_2,\ldots)$ is the transpose of λ, namely, the partition given by $\tilde\lambda_i = \#\{j \mid \lambda_j \ge i\}$). This complex was used effectively, and the resolution of determinantal ideals over the field of characteristic zero was constructed [Las], [Nls], [PW1].

It seems to be difficult to extend the definition of Schur complex $L_\lambda C$ (with good property) of a finite free complex to the general base ring R. However, there is a good extension of the notion of Schur complex of a map (i.e., a complex of length at most one) to the general base ring [ABW2]. For a map of finite free R-modules $\varphi : F\to E$, the Schur complex $L_\lambda\varphi$ is defined. The definition is compatible with the base extension. Namely, for any map of commutative rings $R\to R'$, there is a canonical isomorphism of R'-complexes

$$R'\otimes_R L_\lambda\varphi \cong L_\lambda(R'\otimes_R\varphi).$$

Moreover, the definition of $L_\lambda\varphi$ agrees with that of Nielsen's when the base ring R contatins $\mathbf{Q}$.

The characteristic-free Schur complex is used to construct the minimal free resolution of S/I_t^r $(r \geq 1)$ for $t = \min(m,n)$. Using this, Akin, Buchsbaum and Weyman constructed the minimal free resolution of S/I_t for the case $t = \min(m,n) - 1$ [ABW2].

Using characteristic-free representation theory developed by Akin, Buchsbaum and Weyman, Kurano [Kur] obtained the following result.

Theorem 2.4 *The second Betti number β_2^K of the determinantal ring S/I_t is independent of the base field K.*

In the proof of the theorem, the characteristic-free Cauchy's formula [ABW2] played the central rôle. Cauchy's formula for the characteristic zero case is stated as follows. Let $R \supset \mathbf{Q}$, and V and W be finite free R-modules. Then, for $r \geq 0$, we have an isomorphism of $G = \mathrm{GL}(V) \times \mathrm{GL}(W)$-modules

$$S_r(V \otimes W) \cong \bigoplus_\lambda L_\lambda V \otimes L_\lambda W,$$

where the sum is taken over all partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\sum_i \lambda_i = r$. Note that each summand of the right-hand side is irreducible as a G-module or 0.

The characteristic-free version is stated using the characteristic-free Schur modules. After that, Kurano and the author extended the characteristic-free Cauchy's formula to the chain complex version [HK], and proved that Problem 2.3 is true for the case $m = n = t + 2$. After that, Problem 2.3 was solved negatively.

Theorem 2.5 ([Has1]) *We have $\beta_3^{\mathbf{Z}/3\mathbf{Z}} > \beta_3^{\mathbf{Q}}$ when $2 \leq t \leq \min(m,n) - 3$.*

After that, the author proved that there exists a graded minimal free resolution of S/I_t over $\mathbf{Z}$ when $t = \min(m,n) - 2$ [Has2]. Thus, we have

Theorem 2.6 *There exists a graded minimal free resolution of S/I_t if and only if $t = 1$ or $t \geq \min(m,n) - 2$.*

In the proof of the theorem, a certain class of subcomplexes of the Schur complex of the identity map $L_\lambda \mathrm{id}_F$, called the t-Schur complexes, was studied.

The t-Schur complexes are used to calculate the Betti numbers of other class of determinantal ideals [Has3], [Has4].

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