数理解析研究所講究録
第 857 巻 1994 年 81-98

A Survey of H-vectors and Local H-vectors

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Abstract

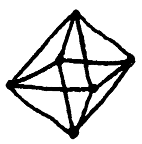
The h-vector of a simplicial complex is a well-known combinatorial invariant which has been studied from the points of view of algebraic topology, commutative algebra, and toric varieties. We present main results on h-vectors and generalized h-vectors (for polyhedral complexes). We also examine local h-vectors, which measure how h-vectors change under simplicial subdivision.

1 Introduction to H-vectors

The following is a brief introduction to the theory of h-vectors. For a more complete survey (and references), I highly recommend [Stan85].

We begin with the simplicial case. Let Δ be a $(d-1)$-dimensional simplicial complex (e.g., boundary of a simplicial convex polytope), and let $f(\Delta)$ denote its *face-vector* $(f_{-1}, f_0, \ldots, f_{d-1})$, where f_i denotes the number of i-dimensional faces of Δ, and by convention, $f_{-1} = 1$. What can we say about $f(\Delta)$?

If Δ is homeomorphic to the sphere S^{d-1}, then the well-known Euler Formula says that $\tilde{\mathcal{X}}(\Delta) = (-1)^{d-1}$, where $\tilde{\mathcal{X}}(\Delta) = \sum_{i=-1}^{d-1}(-1)^i f_i$.



If Δ is the boundary complex of an octahedron, then $\Delta \simeq S^2$ and $f(\Delta) = (1, 6, 12, 8)$.
So $\tilde{\chi}(\Delta) = -1 + 6 - 12 + 8 = 1$.

To find out more about $f(\Delta)$, we need to study the h-vector $h(\Delta)$, first defined by Stanley to be $h(\Delta) = (h_0, h_1, \ldots, h_d)$, where

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

For the boundary of the octahedron pictured above, we thus have $h = (1,3,3,1)$. It follows immediately from the definition that for any Δ, $h_0 = 1$, $h_1 = f_0 - d$, $h_d = (-1)^{d-1}\tilde{\mathcal{X}}(\Delta)$, and $\sum_i h_i = f_{d-1}$.

One can prove by induction on d that the h-vector is the bottom row of the difference table with the f-vector written down the right diagonal, as shown for the boundary of the octahedron:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 6 & & & = f(\Delta) \\
 & & 1 & & 5 & & 12 & & \\
 & 1 & & 4 & & 7 & & 8 & \\
1 & & 3 & & 3 & & 1 & & = h(\Delta)
\end{array}
$$

Note also that $\sum_i h_i x^i = \sum_{F \in \Delta} x^{\#F}(1-x)^{d-\#F}$ follows easily from the definition. This is a useful identity as we will see later.

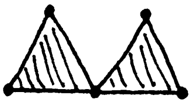
It turns out that the h-vector is in many senses more desirable than the f-vector, and provides the same information anyway, since if A is the matrix with entries $a_{ij} = (-1)^{i-j}\binom{d-j}{d-i}$ for $i,j = 0,1,\ldots,d-1$, then $A \cdot f(\Delta) = h(\Delta)$. *i.e.*, since A is an invertible linear transformation, finding linear relations among the f_i's is equivalent to finding linear relations among the h_i's.

Here are a couple of easy examples of the greater simplicity of the h-vector:

1. If $\Delta = 2^V$, the simplex with vertex set $V = \{1,2,\ldots,d\}$, then $f(\Delta) = (\binom{d}{0}, \binom{d}{1}, \ldots, \binom{d}{d})$ while $h(\Delta) = (1,0,\ldots,0)$

2. If Δ is homoemorphic to S^{d-1}, then Euler's Formula expressed in terms of $h(\Delta)$ is simply $h_d = h_0$

Unfortunately, the h-vector lacks combinatorial meaning in general. In some cases $h_i < 0$, in which case it clearly can not count anything.

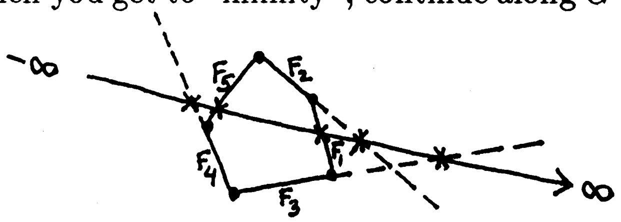


$f=(1,5,6,2) \Rightarrow h=(1,2,-1,0).$
$\quad \llcorner h_2 < 0.$

However, when Δ is the boundary of a simplicial convex polytope, and in many other cases, it *is* possible to prove that $h(\Delta) \geq 0$ by finding combinatorial meaning for each h_i. The following theorem is due to McMullen and Brugesser-Mani.

Theorem 1.1 *If Δ is the boundary of a simplicial convex polytope $\mathcal{P}$, then* $h(\Delta) \geq 0$.

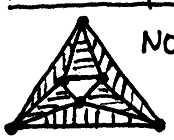
Proof: Choose a generic line G through the interior of $\mathcal{P}$ (*i.e.,* for any two maximal faces of Δ, G intersects their affine spans in distinct points). From the interior of $\mathcal{P}$, walk along G in one direction (it doesn't matter which), and label the maximal faces of Δ in the order in which G intersects their affine spans. (When you get to "infinity", continue along G from the opposite side.)



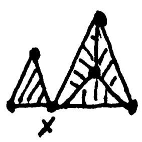
By [Br-Ma], the ordering $F_1, \ldots, F_r$ which results is a *shelling* of $\mathcal{P}$, *i.e.,* for each $i > 1$, the intersection $F_i \cap (\cup_{j<i} F_j)$ is homeomorphic to a $(d-2)$-dimensional ball or sphere. Thus by McMullen (see [Stan?]), $h_i = \#\{j : F_j \cap (\cup_{j<i} F_j)$ has exactly i faces of dimension $d-2\}$, so clearly $h_i \geq 0$. □

Note that the proof holds for any *shellable* Δ, *i.e.,* any simplicial complex Δ whose maximal faces have all the same dimension and can be ordered as a shelling.

<u>Non-examples:</u>

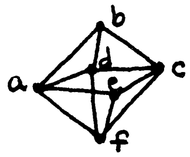


No ordering of the maximal faces is a shelling. For example, if the faces are ordered clockwise (starting with any face) then $F_5 \cap (F_1 \cup F_2 \cup F_3 \cup F_4) = \cdot\!/ \not\cong S^1$.



Clearly no ordering is a shelling, due to vertex x.
However, note that $h=(1,3,0,0)$ is nonnegative anyway.

Example:



The boundary of this octahedron has shelling
abe, abd, bdc, ecb, aef, adf, fdc, ecf.
0 1 1 2 1 2 2 3

The numbers shown are the number of edges in the intersection of each face with previous faces in the shelling. Thus by Thm 1.1, $h(\Delta) = (1,3,3,1)$.

Notice that $(1,3,3,1)$ is symmetric. This is true for all *Eulerian* complexes, which are defined as follows: For any $F \in \Delta$, the *link* of F in Δ is the subcomplex $lk_\Delta F = \{S \in \Delta : S \cap F = \emptyset, S \cup F \in \Delta\}$, where $S \cup F$ denotes the face of Δ with the vertices of S and F. If all the maximal faces of Δ are the same dimension and $\tilde{\mathcal{X}}(lk_\Delta F) = (-1)^{dim(lk_\Delta F)}$ for all $F \in \Delta$, then Δ is an Eulerian complex. For example, if Δ is homeomorphic to S^{d-1}, then Δ is Eulerian. All Eulerian complexes satisfy the Dehn-Sommerville Equations (proved in greater generality in [Stan87]):

Theorem 1.2 *If Δ is Eulerian, then $h_i = h_{d-i}$ for all i.*

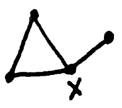
Proof:

$$\begin{aligned}
\sum_i h_i x^i &= \sum_{F\in\Delta} x^{\#F}(1-x)^{d-\#F} \\
&= \sum_{F\in\Delta}\sum_{S\subseteq F}(x-1)^{\#F-\#S}(1-x)^{d-\#F} \\
&= \sum_{S\in\Delta}(x-1)^{d-\#S}\sum_{F\in\Delta, S\subseteq F}(-1)^{d-\#F} \\
&= \sum_{S\in\Delta}(x-1)^{d-\#S}(-1)^{dim(lk_\Delta S)}\tilde{\mathcal{X}}(lk_\Delta S) \\
&= \sum_{S\in\Delta}(x-1)^{d-\#S} = \sum_i f_{i-1}(x-1)^{d-i} = \sum_i h_i x^{d-i}.
\end{aligned}$$

so $h_i = h_{d-i}$ for all i, as desired. □

The Dehn-Sommerville Equations represent the most general linear relations to hold among h-vectors (hence also f-vectors) of Eulerian complexes.

It is not true, however, that a symmetric h-vector must belong to an Eulerian complex:



$h(\Delta) = (1,2,1)$ symmetric, but

Δ not Eulerian (because of vertex x).

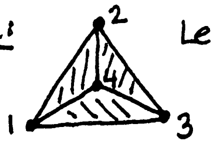
2 H-vectors in Commutative Algebra

Now let us move on to the connection to commutative algebra. (See [Stan83, Chapter 2] for background.)

Let K be any field, and let $\{1, 2, \ldots, n\}$ denote the vertices of a $(d-1)$-dimensional simplicial complex Δ. Form the polynomial ring $K[x_1, \ldots, x_n]$ and the ideal $I(\Delta) \subset K[x_1, \ldots, x_n]$ generated by monomials of the form $x^G = \Pi_{i \in G} x_i$ where $G \notin \Delta$. Then $K[\Delta] := \frac{K[x_1,\ldots,x_n]}{I(\Delta)}$ is the *Stanley-Reisner ring* of Δ over K, with the standard grading from $K[x_1, \ldots, x_n]$.

Let $\theta_1, \ldots, \theta_d$ be homogeneous elements of $K[\Delta]$, and let (θ) denote the ideal generated by the θ_i's. Then $\theta_1, \ldots, \theta_d$ is a *homogeneous system of parameters (h.s.o.p.)* for $K[\Delta]$ if $\frac{K[\Delta]}{(\theta)}$ is finite-dimensional as a K-vector space. From now on we assume that K is infinite, so that by the Noether Normalization Lemma, $K[\Delta]$ must have an h.s.o.p. of degree 1.

Example:



Let $\theta_1 = x_1 - x_4$, $\theta_2 = x_2 - x_4$, $\theta_3 = x_3 - x_4$.

$K[\Delta] = \frac{K[x_1, x_2, x_3, x_4]}{(x_1 x_2 x_3)}$ and $\frac{K[\Delta]}{(\theta)} = K + K \cdot x_4 + K \cdot x_4^2$.

$K[\Delta]$ is a *Cohen-Macaulay ring* if for some (hence every) h.s.o.p. $\theta_1, \ldots, \theta_d$, each θ_i is a non-zero-divisor on $\frac{K[\Delta]}{(\theta_1,\ldots,\theta_{i-1})}$. In this case we say that Δ is a Cohen-Macaulay complex. (*e.g.,* the example shown above is Cohen-Macaulay.)

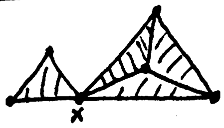
The following theorem of Reisner simplifies the question of when Δ is Cohen-Macaulay (see [Reis]):

Theorem 2.1 *Δ is Cohen-Macaulay if and only if $\tilde{H}_i(lk_\Delta F; K) = 0$ for all*

$i < dim(lk_\Delta F)$ *and all* $F \in \Delta$.

Corollary 2.2 *If* Δ *is homeomorphic to a sphere or ball, or if* Δ *is shellable, then* Δ *is Cohen-Macaulay.*

Non-example:



By Reisner's theorem, this simplicial complex is not Cohen-Macaulay, because of vertex x.

Since we already know that shellable Δ have $h(\Delta) \geq 0$, it is natural to ask if the same is true for Cohen-Macaulay Δ. The following theorem of Stanley answers our question.

Theorem 2.3 *If* Δ *is Cohen-Macaulay, then* $0 \leq h_i \leq \binom{f_0-d+i-1}{i}$ *for all* i.

Proof: Let $\theta_1, \ldots, \theta_d \in K[\Delta]$ be an h.s.o.p. of degree 1. Since Δ is Cohen-Macaulay, each θ_i is a non-zero-divisor in $\frac{K[\Delta]}{(\theta_1,\ldots,\theta_{i-1})}$, so the Poincare series $F(\frac{K[\Delta]}{(\theta)}, x) = (1-x)^d F(K[\Delta], x)$. Since $F(K[\Delta], x) = \sum_{F\in\Delta}(\frac{x}{1-x})\#F$, it follows that $F(\frac{K[\Delta]}{(\theta)}, x) = \sum_i h_i x^i$.

Thus $h_i \geq 0$ and $h_i \leq$ the number of distinct monomials of degree i in h_1 variables, so $h_i \leq \binom{f_0-d+i-1}{i}$, as desired. □

This theorem is sometimes called the "Upper Bound Conjecture" because (thanks to McMullen), it implies the following result for f-vectors of spheres:

Theorem 2.4 *If* Δ *is homeomorphic to* S^{d-1} *and* $\mathcal{P}$ *is a convex polytope with* $f_0(\Delta)$ *distinct vertices of the form* $(t_i, t_i^2, \ldots, t_i^d) \in R^d$, *then* $f_i(\Delta) \leq f_i(\mathcal{P})$ *for all* i.

3 Intersection Homology and Generalized H-vectors

Now let us consider an application of intersection homology theory to h-vectors. (See [Stan87] for background and references.)

If Δ is the boundary of a rational convex d-dimensional polytope $\mathcal{P}$ with 0 in its interior, then Δ defines a fan of rational cones in R^d which in turn defines a complex toric variety X (see [Dan]) such that $IH_{2i+1}(X;Q) = 0$ and $dim_Q IH_{2i}(X;Q) = h_i(\Delta)$ for all i. Thus, as noticed by Stanley, not only is $h(\Delta)$ symmetric (Dehn-Sommerville, or Poincare Duality), but $h(\Delta)$ is also unimodal, *i.e.*,

$$h_0 \leq h_1 \leq \ldots \leq h_{\lfloor \frac{d}{2} \rfloor} \geq \ldots \geq h_d,$$

by the Hard Lefschetz Theorem for Intersection Homology.

Since every simplicial convex polytope is combinatorially equivalent to a rational polytope, this means that the boundary of any simplicial convex polytope has unimodal h-vector.

It is now natural to ask whether or not the definition of h-vector generalizes to polyhedral complexes so that it still corresponds to the Intersection Homology betti numbers in the case of rational convex polytopes. The answer is yes. (Note that the old definition doesn't work, for example, for the boundary of a 3-dimensional cube, the old h-vector would be $(1,5,-1,1)$, which is not unimodal, symmetric, nor nonnegative!)

Let Γ be a $(d-1)$-dimensional polyhedral complex. If Γ is simplicial, then the old definition says that $h(\Gamma) = (h_0, \ldots, h_d)$ such that $\sum_i h_i x^{d-i} = \sum_{f \in \Gamma}(x-1)^{d-\#F}$. Let $\overline{h(\Gamma,x)}$ denote $\sum_i h_i x^{d-i}$. For general Γ, Stanley defined the generalized h-vector $h(\Gamma)$ as follows:

1. $\overline{h(\emptyset,x)} = g(\emptyset,x) = 1$

2. $\overline{h(\Gamma,x)} = \sum_{f\in\Gamma} g(\partial f,x)(x-1)^{d-r(f)}$, where $r(f) = 1 + dim(f)$, and $\partial f = \{f' \in \Gamma : f' \subset\neq f\}$

3. $g(\Gamma,x) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor}(k_i - k_{i-1})x^i$, where $\overline{h(\Gamma,x)} = \sum_i k_i x^i$

Proposition 3.1 *If $\Gamma = \partial 2^V$, then $g(\Gamma,x) = 1$. (Hence the generalized h-vector agrees with the old definition, in the simplicial case.)*

Proof: By induction on the dimension of Γ. Let $d = 1 + dim\Gamma$. (so # V = d+1.) Then $\overline{h(\Gamma,x)} = \sum_{f\in\Gamma} g(\partial f,x)(x-1)^{d-r(f)}$, but $g(\partial f,x) = 1$ for all $f \in \Gamma$, by inductive hypothesis. So $\overline{h(\Gamma,x)} = \sum_{f\in\Gamma}(x-1)^{d-\#f} = \frac{x^{d+1}-1}{x-1} = 1 + x + \ldots + x^d$, by an application of the binomial theorem. So $g(\Gamma,x) = 1$, as claimed. □

Let's compute the generalized h-vector of the square:

$$\overline{h(\square, x)} = g(\partial\phi, x)(x-1)^3 + 4g(\partial\bullet, x)(x-1)^2 + 4g(\partial I, x)(x-1) + g(\partial\square, x)$$
$$= (x-1)^3 + 4(x-1)^2 + 4(x-1) + (1+x) = x^3 + x^2$$

$$\left(\begin{array}{l} \text{Note that } g(\partial\square, x) = 1+x \text{ since} \\ \overline{h(\partial\square, x)} = \overline{h(\square, x)} = (x-1)^2 + 4(x-1) + 4 = x^2 + 2x + 1. \end{array} \right)$$

So $h(\square) = (1, 1, 0, 0)$.

The generalized h-vector has the following properties:

1. (Stanley) If Γ is the boundary of a rational convex polytope then $h(\Gamma)$ is unimodal (by the same argument as in the simplicial case). However, it is not true that every convex polytope is combinatorially equivalent to a rational one in the non-simplicial case. (An 8-dimensional example is due to Perles.)

2. (Stanley) If Γ is homeomorphic to S^{d-1} then $h(\Gamma)$ is symmetric. (the Dehn-Sommerville Equations for generalized h-vectors.) The proof is similar to that for the simplicial case, but uses Möbius inversion on the face poset of Γ.

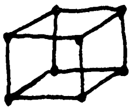
3. (Chan) If Γ is shellable and each face of Γ is combinatorially equivalent to a geometric cube, then $h(\Gamma) \geq 0$. In particular, if $F_1, \ldots, F_r$ is a shelling of Γ and $d = 1 + dim(\Gamma)$, let s_{ij} denote the number of F_k's such that $F_k \cap (\cup_{m<k} F_m)$ has exactly i unpaired and j antipodal pairs of $(d-2)$-dimensional faces, and define $f_d(i, j, x) = \sum_{k=0}^{d} c_d(i, j, k)^k$ as follows:

   1. if $j < d-1$, then $c_d(i, j, k)$ is the number of d-vertex plane-trees with exactly k nonforks which are not $1', \ldots, i'$ nor $1'', \ldots, j''$, where i' means i^{th} in preorder, with exactly one child; and j'' means $(d-j)^{th}$ in preorder, followed by a root, only, or inner child. (See [Chan91] for more detail.)

   2. $c_d(0, d-1, k)$ is the number of d-vertex plane-trees with exactly k forks

   Then $\sum_i h_i x^{d-i} = \sum_{i,j} s_{ij} f_d(i, j, x)$.

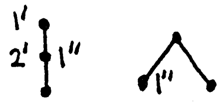
For example, let Γ be the boundary complex of a 3-dimensional cube:



shelling: front, top, right, left, bottom, back.

$(i,j) = (0,0), (1,0), (2,0), (2,0), (1,1), (0,2)$.

So $s_{00}=1$, $s_{10}=1$, $s_{20}=2$, $s_{11}=1$, $s_{02}=1$.



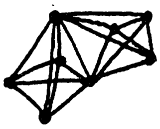
$f_3(0,0,x) = x^3+x^2$, $f_3(1,0,x) = 2x^2$, $f_3(2,0,x) = x+x^2$,
$f_3(1,1,x) = 2x$, $f_3(0,2,x) = 1+x$.

Thus $\overline{h(\Gamma,x)} = (x^3+x^2)+(2x^2)+2(x+x^2)+(2x)+(1+x) = x^3+5x^2+5x+1$.
So $h(\Gamma) = (1,5,5,1)$.

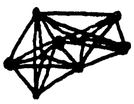
A natural open question is: If Γ is *any* shellable polyhedral complex then is $h(\Gamma) \geq 0$?

4 Subdivisions and Local H-vectors

The theory of local h-vectors (conceived by Stanley) was motivated by the question: If Δ' is a simplicial subdivision of a Cohen-Macaulay complex Δ, then is $h(\Delta') \geq h(\Delta)$? For non-Cohen-Macaulay complexes the answer can be no, for example:



Δ = two tetrahedra which share one edge, and which are each divided by an interior vertex.
$h(\Delta) = (1,4,1,2,0)$.



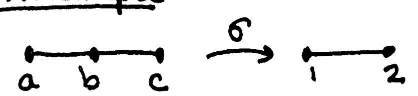
Δ' = the result of adding a new vertex to the interior of the shared edge, and dividing adjacent faces.
$h(\Delta') = (1,5,5,1,0)$.

Let us begin with a formal definition of *subdivision.* See [Stan92, Sections 1-3] for background and proofs of the results in this section.

A simplicial complex Γ with a simplicial map $\sigma : \Gamma \to 2^V$ (*i.e.*, $\sigma(F) \subseteq \sigma(G)$ if $F \subseteq G$) is called a *subdivision* of 2^V if for all $W \subseteq V$:
1. $\Gamma_W := \sigma^{-1}(2^W)$ is homeomorphic to 2^W; and
2. $\sigma^{-1}(W)$ is the set of interior faces of Γ_W.

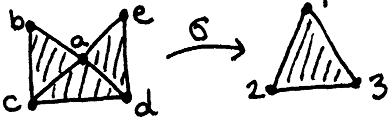
If $\sigma(F) \subseteq W$, we say *F lies on W*.

<u>Nonexample</u>:



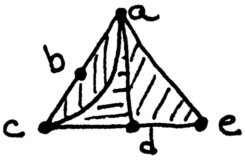
$\left.\begin{array}{l} \sigma(\phi) = \phi,\ \sigma(a) = 1,\ \sigma(b) = 2, \\ \sigma(c) = \sigma(ab) = \sigma(bc) = 12. \end{array}\right\}$ W=12 VIOLATES (ii).

Example:



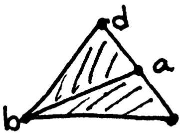
$\sigma(\phi)=\phi$, $\sigma(a)=1$, $\sigma(c)=2$, $\sigma(e)=3$, $\sigma(ae)=13$,
$\sigma(b)=\sigma(ab)=\sigma(bc)=12$,
$\sigma(d)=\sigma(cd)=\sigma(de)=23$,
$\sigma(F)=123$ for all other F.

We find it convenient to picture σ as shown:

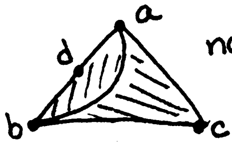


We are interested in three basic types of subdivisions:

1. *Quasigeometric:* No face $F \in \Gamma$ has all its vertices lying on a face of $W \in 2^V$ of dimension less than $\dim F$.

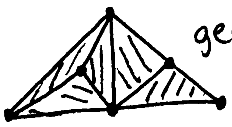


quasigeometric.



not quasigeometric, because of the face abd.

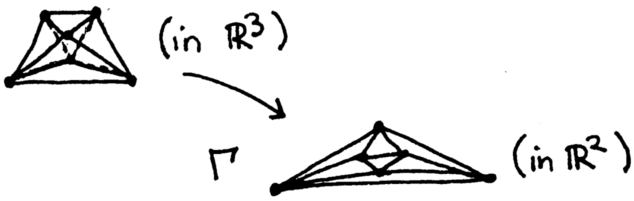
2. *Geometric:* Γ can be realized so that all of its faces are convex.



geometric.

(See [Chan92] for 3-dimensional nongeometric but quasigeometric Γ. (due to Jungreis))

3. *Regular:* Γ can be realized as the projection of a strictly convex polyhedral surface.



Clearly, regular implies geometric, but the converse is false ([Rud]).

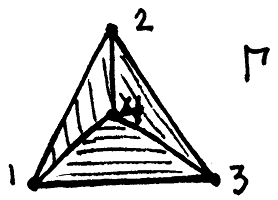
Now let us define the *local h-vector* of Γ with respect to V, denoted by $l_V(\Gamma) = (l_0, l_1, \ldots, l_d)$, where $d = \#V$. Let $h(\Gamma, x) = \sum_i h_i x^i$ if $h(\Gamma) = (h_0, \ldots, h_d)$, and define

$$l_V(\Gamma, x) = \sum_{i=0}^{d} l_i x^i = \sum_{W \subseteq V} (-1)^{\#V - \#W} h(\Gamma_W, x).$$

Example: If $V = \{1, 2, 3\}$ and Γ is as shown below, then

$$\begin{aligned} l_V(\Gamma, x) &= h(\Gamma, x) - h(\Gamma_{12}, x) - h(\Gamma_{13}, x) - h(\Gamma_{23}, x) \\ &\quad + h(\Gamma_1, x) + h(\Gamma_2, x) + h(\Gamma_3, x) - h(\Gamma_\emptyset, x) \\ &= (1 + x + x^2) - 1 - 1 - 1 + 1 + 1 + 1 - 1 \\ &= x + x^2 \end{aligned}$$

so $l_V(\Gamma) = (0, 1, 1, 0)$.



Alternatively, if $e(G) = \#\sigma(G) - \#G$ for all $G \in \Gamma$, then

$$l_V(\Gamma, x) = \sum_{G \in \Gamma} (-1)^{d - \#G} x^{d - e(G)} (x - 1)^{e(G)}$$

follows from the identity $h(\Delta, x) = \sum_{F \in \Delta} x^{\#F} (1 - x)^{d - \#F}$ (see [Stan92]).

Example: For V and Γ as above,

$$\begin{aligned} l_V(\Gamma, x) &= 3x^3(x-1)^0 && G = 124, 134, 234 \\ &\quad -3x^3(x-1)^0 && G = 12, 13, 23 \\ &\quad -3x^2(x-1)^1 && G = 14, 24, 34 \\ &\quad +3x^3(x-1)^0 && G = 1, 2, 3 \\ &\quad +x^1(x-1)^2 && G = 4 \\ &\quad -x^3(x-1)^0 && G = \emptyset \\ &= x^2 + x \end{aligned}$$

Notes:

1. If $\Gamma = 2^V$ and σ is the identity map, then $l_V(\Gamma, x) = 0$ unless $V = \emptyset$, in which case $l_V(\Gamma, x) = 1$

2. Using the above formula for $l_V(\Gamma, x)$ in terms of $G \in \Gamma$, it's easy to see that $l_0 = 0$, $l_1 =$ the number of interior vertices of Γ, and $l_d = \tilde{\mathcal{X}}(\Gamma) = 0$ (since Γ is homeomorphic to a ball)

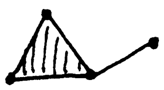
If Δ is any simplicial complex, then a subdivision of Δ is another simplicial complex Δ' with a simplicial map $\sigma : \Delta' \to \Delta$ such that for every $F \in \Delta$, the restriction of σ to Δ'_F is a subdivision of the simplex 2^F. The following theorem justifies the name "local" h-vector:

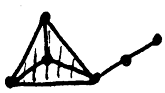
Theorem 4.1 *If Δ' is a subdivision of a pure simplicial complex Δ, then*

$$h(\Delta', x) = \sum_{F \in \Delta} l_F(\Delta'_F, x) h(lk_\Delta F, x).$$

This theorem is crucial in proving that $h(\Delta', x) \geq h(\Delta, x)$ in the case when Δ' is a quasigeometric subdivision of a Cohen-Macaulay complex Δ. Its proof relies on a technical lemma which follows from $h(\Delta, x) = \sum_{F \in \Delta} x^{\#F}(1 - x)^{d - \#F}$.

Note that if Δ is not pure (*i.e.*, not all maximal faces have the same dimension) then the theorem may not hold. For example:


$h(\Delta, x) = 1 + x - x^2$


$h(\Delta', x) = 1 + 3x - x^2$

$\Rightarrow \sum_{F \in \Delta} l_F(\Delta'_F, x) \cdot h(lk_\Delta F, x) = 1 + 3x \neq h(\Delta', x)$.

Another important result on local h-vectors is

Theorem 4.2 *For any subdivision Γ of the simplex 2^V, the local h-vector $l_V(\Gamma) = (l_0, l_1, \ldots, l_d)$ satisfies $l_i = l_{d-i}$ for all i.*

The proof depends on the fact that $h(Int(\Gamma), x) = x^d h(\Gamma, \frac{1}{x})$, which can be proved along the same lines as the proof given for the Dehn-Sommerville Equations.

5 Quasigeometric Subdivisions and Commutative Algebra

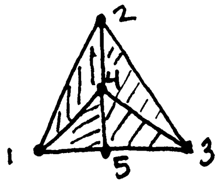
We now come to a main result of Stanley on local h-vectors in the quasigeometric case:

Theorem 5.1 *If* Γ *is a quasigeometric subdivision of the simplex* 2^V, *then* $l_V(\Gamma) \geq 0$.

The proof depends on a commutative algebra technique ([Stan92, Section 4]), which we summarize below.

Recall that $\theta_1, \theta_2, \ldots, \theta_d$ is an h.s.o.p. for $K[\Gamma]$ if it's a set of homogeneous elements such that $K[\Gamma]$ is finitely generated as a $K[\theta_1, \ldots, \theta_d]$-module. Moreover, since Γ is homeomorphic to a ball, it's Cohen-Macaulay, so $h(\Gamma, x) = F(\frac{K[\Gamma]}{(\theta)}, x)$.

Now let us consider a special class of h.s.o.p.'s for $K[\Gamma]$. By relabelling, we may assume that $x_1, \ldots, x_d$ correspond to the vertices of 2^V. An h.s.o.p. $\theta_1, \ldots, \theta_d$ for $K[\Gamma]$ is *special* if each θ_i is a linear combination of vertices of Γ which do not lie on the face $V - x_i$ of the simplex 2^V. For example:



$\theta_1 = x_1 - x_4 - x_5$
$\theta_2 = x_2 - x_4$
$\theta_3 = x_3 - x_4 - x_5$

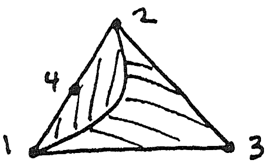
The following useful lemma is due to Kind & Kleinschmidt:

Lemma 5.2 *For any* $(d-1)$*-dimensional simplicial complex* Δ, $\theta_1, \ldots, \theta_d$ *is an h.s.o.p. of degree one in* $K[\Delta]$ *if and only if for all* $F \in \Delta$ *and all* $i \in F$, x_i *is a linear combination of* $\theta_1|_F, \ldots, \theta_d|_F$, *where* $\theta_i|_F$ *denotes* θ_i *with all vertices not in* F *set to 0.*

In the example shown above, if $F = \{1,2,4\}$, then $\theta_1|_F = x_1 - x_4$, $\theta_2|_F = x_2 - x_4$, and $\theta_3|_F = -x_4$, which clearly span x_1, x_2, x_4. The same holds for all $F \in \Gamma$ for this example, which verifies that $\theta_1, \theta_2, \theta_3$ is an h.s.o.p.

The lemma also shows that some subdivisions can not have special h.s.o.p.'s. For example if $\theta_1, \theta_2, \theta_3$ were a special h.s.o.p. for the Γ shown below,

$\theta_3|_F = 0$ for $F = \{1,2,4\}$, which violates the condition in the lemma.



It is not hard to show from the lemma that

Corollary 5.3 *If Γ is a subdivision of 2^V, then $K[\Gamma]$ has a special h.s.o.p. if and only if Γ is quasigeometric.*

From now on we assume Γ quasigeometric and $\theta_1, \ldots, \theta_d$ special. Then we can define the *local face module* $L_V(\Gamma)$ to be the image of the ideal $(Int\Gamma)$ in $\frac{K[\Gamma]}{(\theta)}$, with the standard grading. Let L_i denote the i^{th} graded piece of $L_V(\Gamma)$. For example:

$\theta_1 = x_1 - x_4$, $\theta_2 = x_2 - x_4$, $\theta_3 = x_3 - x_4$; $\frac{K[\Gamma]}{(\theta)} = K + K\cdot x_4 + K\cdot x_4^2$.

$(Int\,\Gamma) = (x_4) \Rightarrow L_V(\Gamma) = K\cdot x_4 + K\cdot x_4^2$

$\Rightarrow L_0 = 0$, $L_1 = K\cdot x_4$, $L_2 = K\cdot x_4^2$, $L_3 = 0$.

Recall that in this case, $l_V(\Gamma) = (0,1,1,0)$, so $l_i = dim_K L_i$ for each i.

Theorem 5.4 *For any quasigeometric Γ, we have $l_i = dim_K L_i$ for all i.*

Proof: The proof is based on a technical lemma, which says that if $\mathcal{K}$ is the complex

$$K[\Gamma] \to \oplus_i \frac{K[\Gamma]}{N_i} \to \oplus_{i<j} \frac{K[\Gamma]}{N_i + N_j} \to \ldots \to \frac{K[\Gamma]}{N_1 + \ldots + N_d} \to 0$$

where N_i is the ideal generated by monomials of the form x^F where $F \in \Gamma$ does not lie on $V - x_i$, and the maps are the usual coboundary maps, then $\frac{\mathcal{K}}{\theta \cdot \mathcal{K}}$ is exact.

Now note that $\frac{K[\Gamma]}{N_{i_1} + \ldots + N_{i_s}} = K[\Gamma_{V-\{i_1,\ldots,i_s\}}]$ and also that the kernel of the map $\frac{K[\Gamma]}{(\theta)} \to \oplus_i \frac{K[\Gamma]}{(\theta)+N_i}$ is simply $L_V(\Gamma)$. So since $\frac{\mathcal{K}}{\theta\cdot\mathcal{K}}$ is exact,

$$F(L_V(\Gamma), x) = F(\frac{K[\Gamma]}{(\theta)}, x) - F(\oplus_i \frac{K[\Gamma]}{(\theta) + N_i}, x) \ldots + (-1)^d F(\frac{K[\Gamma]}{(\theta) + N_1 + \ldots + N_d}, x)$$

$$= \sum_{W \subseteq V} (-1)^{\#V-\#W} F(\frac{K[\Gamma_W]}{(\theta)}, x)$$
$$= \sum_{W \subseteq V} (-1)^{\#V-\#W} h(\Gamma_W, x)$$

since the Γ_W's are Cohen-Macaulay and $\theta_1, \ldots, \theta_d$ is special. Thus $F(L_V(\Gamma), x) = l_v(\Gamma, x)$, as desired. □

Theorem 5.1 follows immediately. We also have the following corollary, which partially answers the motivating question:

Corollary 5.5 $h(\Delta') \geq h(\Delta)$ *for any quasigeometric subdivision* Δ' *of a Cohen-Macaulay complex* Δ.

Proof: $h(\Delta', x) = \sum_{F \in \Delta} l_F(\Delta'_F, x) h(lk_\Delta F, x) = h(\Delta, x) + \sum_{\emptyset \neq F \in \Delta} l_F(\Delta'_F, x) h(lk_\Delta F, x) \geq h(\Delta, x)$, since for all $F \in \Delta$, Δ' quasigeometric implies that $l_F(\Delta'_F, x) \geq 0$ and Δ Cohen-Macaulay implies that $lk_\Delta F$ Cohen-Macaulay, hence $h(lk_\Delta F, x) \geq 0$. □

Conjecture: If Δ is Cohen-Macaulay, then $h(\Delta') \geq h(\Delta)$ for *any* subdivision Δ' of Δ.

An interesting question to consider is when $l_V(\Gamma) = 0$.

Proposition 5.6 *If* $l_V(\Gamma) = 0$ *and* Γ' *is any quasigeometric subdivision of* 2^V *which restricts to the same subdivision of the boundary of* 2^V, *then* $h(\Gamma') \geq h(\Gamma)$.

Proof: By the principle of Inclusion-Exclusion, the definition of local h-vectors implies that $h(\Gamma', x) = \sum_{W \subseteq V} l_V(\Gamma'_W, x)$. But since Γ' and Γ agree on the boundary of 2^V, we have $\Gamma'_W = \Gamma_W$ for all proper faces $W \subset V$. Thus $h(\Gamma', x) - h(\Gamma, x) = l_V(\Gamma', x) - l_V(\Gamma, x)$. But Γ' is quasigeometric, so $l_V(\Gamma', x) \geq 0$, and we are given that $l_V(\Gamma, x) = 0$. Thus $h(\Gamma', x) - h(\Gamma, x) \geq 0$, as desired. □

6 Regular Subdivisions and Intersection Homology

Now let us consider the connection between local h-vectors and intersection homology theory. Let X denote the complex toric variety associated with the fan Σ_X of cones on the faces of the simplex 2^V. If Γ is a subdivision of 2^V, let Y denote the complex toric variety associated with the fan Σ_Y of cones on the faces of Γ. Then by [Danilov], there exists a proper morphism of toric varieties $Y \to X$, induced by the subdivision map $\Sigma_Y \to \Sigma_X$. From this fact, Stanley deduced the following ([Stan92, Theorem 5.2]):

Theorem 6.1 *If* Γ *is a regular subdivision of* 2^V, *then* $l_V(\Gamma)$ *is unimodal.*

The idea of the proof is as follows:
Because of the existence of a proper morphism from Y to X, the intersection homology of Y decomposes into direct summands, each of which corresponds to the "fiber" of Y over some strata of X. If we use the stratification $X = \cup_{W \subseteq V} X^W$ (where X^W denotes the inverse image under the moment map, of the face dual to W), then we get the Poincare series $F(IH(Y;Q),x) = \sum_{W \subseteq V} \phi^W(x)$, where the ϕ^W's correspond to the direct summands mentioned above. Since $F(IH(Y;Q);x) = h(\Gamma, x^2)$, we can assume the ϕ^W's are polynomials in x^2 as well. *i.e.*, $h(\Gamma, x^2) = \sum_{W \subseteq V} \phi^W(x^2)$. Thus by Inclusion-Exclusion we have $l_V(\Gamma, x) = \phi^V(x)$.

Now since Γ is regular, $Y \to X$ is projective, so by the Hard Lefschetz property of the decomposition theorem, each $\phi^W(x)$ is unimodal. Thus $l_V(\Gamma)$ is unimodal. □

A consequence of this theorem is that if Δ is the boundary of a convex simplicial polytope, and Δ' is a regular subdivision of Δ, then $g(\Delta') \geq g(\Delta)$, where $g(\Delta) := (h_0, h_1 - h_0, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$ and $h_i = h_i(\Delta)$ for all i. ([Stan92, Corollary 5.3])

Up to this point we have defined local h-vectors only for simplicial subdivisions of simplices. It is possible to generalize the definition for arbitrary polyhedral subdivisions of arbitrary polytopes, so that the connection to intersection homology holds for *rational* subdivisions, but in other cases not much is known other than symmetry. See [Stan92, Part II] for the definition of generalized local h-vectors (in terms of the incidence algebra of the face poset of the polytope) and related results.

In conclusion we give the following theorems of Chan, which characterize local h- vectors in two main cases:

Theorem 6.2 *Let $l = (l_0, l_1, \ldots, l_d) \in \mathbf{Z}^{d+1}$. Then $l = l_V(\Gamma)$ for some subdivision Γ of 2^V (where $\#V = d$) if and only if $l_0 = 0$, $l_1 \geq 0$, and $l_i = l_{d-i}$ for all i.*

The "only if" direction follows from results mentioned earlier (due to Stanley), and the "if" direction depends on three basic constructions which appear in [Chan92]. If l is unimodal in addition to satisfying the other conditions, the constructions yield a regular subdivision, so we have the following result as well.

Theorem 6.3 *Let $l = (l_0, l_1, \ldots, l_d) \in \mathbf{Z}^{d+1}$. Then $l = l_V(\Gamma)$ for some regular subdivision Γ of 2^V (where $\#V = d$) if and only if $l_0 = 0$, $l_i = l_{d-i}$ for all i, and $l_0 \leq l_1 \leq \ldots \leq l_{\lfloor \frac{d}{2} \rfloor} \geq \ldots \geq l_d$.*

Conjecture (Stanley): If Γ is quasigeometric, then $l_V(\Gamma)$ is unimodal.

If the conjecture is true, then Theorem 6.3 also characterizes local h-vectors of quasigeometric subdivisions, since regular implies quasigeometric.

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