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The intersection cohomology and toric varieties

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Introduction

Toric varieties over the field **C** of complex numbers give rise to normal complex analytic spaces with not too complicated singularities. The intersection cohomology due to Goresky-MacPherson [13], [14], [15] is then applicable to these complex analytic spaces and produces interesting invariants for the toric varieties and the corresponding fans. However, the computation of these invariants has been possilbe only when we resort to highly nontrivial theorems such as the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber (cf. [1]) and the purity theorem of Deligne-Gabber (cf. [8]).

There have been attempts to carry out the computation directly in terms of the fans without recourse to these nontrivial theorems by McMullen [29], Stanley [40], the author [31], [33], [34] and others.

Ishida [20], [21], [19] finally succeeded in describing the intersection complex and the intersection cohomology groups (with respect to general perversities) of a toric variety entirely in terms of the corresponding fan. Moreover, he could show a version of the decomposition theorem as well as vanishing theorems in this new formulation.

Unfortunately, however, it does not seem possible at the moment to remove the *rationality assumption* on the fan. It is highly desirable to obtain analogous results, for instance, for simplicial cone decompositions with markings (cf. [31]).

We here try to describe the intersection complexes and intersection cohomology (with respect to the middle perversity) of toric varieties and interesting consequences.

In Section 1, we have a brief review of the relationship among toric varieties, fans and integral convex polytopes. In Section 2, we describe the Poincaré polynomial for the intersection cohomology of toric varieties due, in various forms, to Bernstein-Khovanskii-MacPherson (cf. Stanley [39]), Denef-Loeser [9], Fieseler [10], Joshua [22], Kirwan [25] and Stanley [40]. Section 3 is devoted to a brief account of the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber (cf. [1]). In Section 4 we describe Ishida's vanishing theorems, called the first and second diagonal theorems, and apply them in Section 5 to the Chow rings for simplicial fans. We then see the the relevance of the strong Lefschetz theorem for projective toric varieties.

1 Rational convex polytopes and fans

Throughout, $N \cong \mathbf{Z}^r$ is a free $\mathbf{Z}$-module of rank r, and $M := \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ is its dual with the canonical bilinear pairing $\langle\ ,\ \rangle : M \times N \to \mathbf{Z}$. We let $N_{\mathbf{R}} := N \otimes_{\mathbf{Z}} \mathbf{R}$, $M_{\mathbf{R}} := M \otimes_{\mathbf{Z}} \mathbf{R}$ and $M_{\mathbf{Q}} := M \otimes_{\mathbf{Z}} \mathbf{Q}$.

By the theory of toric varieties (see, for instance, Danilov [6], Fulton [12] and Oda [30]), the following three sets are in bijective correspondence among themselves:

- The set of r-dimensional integral (resp. rational) convex polytopes $\Box$ in $M_{\mathbf{R}}$, i.e., convex polytopes all of whose vertices belong to M (resp. $M_{\mathbf{Q}}$).
- The set of pairs (Δ, ψ) consisting of a finite complete fan Δ for N and a map $\psi : N_{\mathbf{R}} \to \mathbf{R}$ which is $\mathbf{Z}$-valued (resp. $\mathbf{Q}$-valued) on the lattice N, and which is piecewise linear and strictly upper convex with respect to Δ.
- The set of pairs (X, D) of a projective toric variety X over the field $\mathbf{C}$ of complex numbers and an ample Cartier divisor (resp. ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor) D on X.

Indeed, to each integral (resp. rational) convex polytope $\Box \subset M_{\mathbf{R}}$, we associate its support function $\psi_\Box : N_{\mathbf{R}} \to \mathbf{R}$ defined by

$$\psi_\Box(n) := \inf_{m \in \Box} \langle m, n \rangle \qquad \text{for } n \in N_{\mathbf{R}},$$

which is clearly $\mathbf{Z}$-valued (resp. $\mathbf{Q}$-valued) on N. There turns out to exist the coarsest fan Δ for N such that $\psi_\Box$ is piecewise linear and strictly upper convex with respect to Δ. We then obtain the corresponding projective toric variety $X := T_N\,\mathrm{emb}(\Delta)$ over $\mathbf{C}$ and an ample Cartier divisor (resp. ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor) D on X determined by the support function $\psi_\Box$. There exists an order reversing bijection from the set of nonempty faces of $\Box$ to Δ, which assigns to each face F of $\Box$ the cone

$$\sigma := \{n \in N_{\mathbf{R}} \mid \psi_\Box(n) = \langle m, n \rangle, \quad \forall m \in \operatorname{rel\,int} F\} \in \Delta.$$

Under the correspondence among the three sets, the following properties are known to be equivalent:

- $\Box$ is simple, i.e., each vertex is incident to exactly r edges.
- Δ is simplicial, i.e., each $\sigma \in \Delta$ is a simplicial cone.
- X is an orbifold, i.e., X has at worst quotient singularities.

2 Intersection cohomology

In general, let X be an r-dimensional complete normal irreducible algebraic variety, and denote by X^{an} the complex analytic space associated to X. As we explain in Section 3, we can consider the intersection cohomology groups

$$IH^j(X^{\mathrm{an}}, \mathbf{Q}) \qquad j \in \mathbf{Z}$$

with respect to the *middle perversity*, which are known to satisfy the following properties:

- $IH^j(X^{\mathrm{an}}, \mathbf{Q}) = 0$ unless $0 \leq j \leq 2r$.
- They are topological invariants but not necessarily homotopy invariants.
- The direct sum $IH^{\cdot}(X^{\mathrm{an}}, \mathbf{Q}) := \bigoplus_j IH^j(X^{\mathrm{an}}, \mathbf{Q})$ does not have any natural ring structure in general.
- The Poincaré duality holds.
- The weak and strong Lefschetz theorems hold.
- The Lefschetz fixed point theorem holds.
- The Hodge decomposition exists.
- $IH^j(X^{\mathrm{an}}, \mathbf{Q})$ coincides with the ordinary cohomology group $H^j(X^{\mathrm{an}}, \mathbf{Q})$ when X is an orbifold.

Let us now restrict ourselves to the case where X is the r-dimensional complete toric variety associated to a finite complete fan Δ for N. Note that X need not be projective. The crudest information we can get out of the intersection cohomology groups is the *Poincaré polynomial*

$$P_\Delta(t) := \sum_{j=0}^{2r} \left(\dim_{\mathbf{Q}} IH^j(X^{\mathrm{an}}, \mathbf{Q})\right) t^j.$$

We have $P_\Delta(t) = t^{2r} P_\Delta(1/t)$ by the Poincaré duality.

The proof of the following theorem uses a highly nontrivial theorem called the *decomposition theorem* which we explain in Section 3. We here follow the formulation due to Fieseler [10].

Theorem 2.1 (Bernstein-Khovanskii-MacPherson (cf. Stanley [39]), Denef-Loeser [9], Fieseler [10], Joshua [22], Kirwan [25] and Stanley [40]) *Let X be the r-dimensional complete toric variety associated to a finite complete fan Δ. Then we have*

$$P_\Delta(t) = \sum_{\sigma\in\Delta} (t^2-1)^{r-\dim\sigma} \operatorname{trunc}_{(\dim\sigma-1)} \left((1-t^2)P_{\overline{\partial\sigma}}(t)\right),$$

where $\overline{\partial\sigma}$ is the complete fan defined from σ as follows: For a primitive element $n_0 \in N \cap \operatorname{rel\,int}(\sigma)$, consider the free $\mathbf{Z}$-module $\bar{N} := (N \cap \mathbf{R}\sigma)/\mathbf{Z}n_0$ of rank equal to $\dim\sigma - 1$ and let $\overline{\partial\sigma}$ be the fan for $\bar{N}$ defined by

$$\overline{\partial\sigma} := \{(\tau + \mathbf{R}n_0)/\mathbf{R}n_0 \mid \tau \prec \sigma,\ \tau \neq \sigma\}.$$

$P_{\overline{\partial\sigma}}(t)$ is independent of the choice of n_0.

In particular, we have

$$IH^j(X^{\mathrm{an}}, \mathbf{Q}) = 0 \qquad \text{for } j \text{ odd.}$$

Lemma 2.2 *If σ is simplicial (for instance, $\dim\sigma \le 2$), then*

$$P_{\overline{\partial\sigma}}(t) = \frac{1-t^{2\dim\sigma}}{1-t^2} \qquad \text{hence} \quad \operatorname{trunc}_{(\dim\sigma-1)}\left((1-t^2)P_{\overline{\partial\sigma}}(t)\right) = 1.$$

Corollary 2.3 *If Δ is simplicial, then*

$$P_\Delta(t) = \sum_{\sigma\in\Delta} (t^2-1)^{r-\dim\sigma} = \sum_{p=0}^{r} {}^{\#}\Delta(p)(t^2-1)^{r-p},$$

where ${}^{\#}\Delta(p)$ is the cardinality of the set $\Delta(p)$ of p-dimensional cones in Δ.

When Δ is associated to an r-dimensional rational convex polytope $\square \subset M_{\mathbf{R}}$, the coefficients $h_j := \dim_{\mathbf{Q}} IH^{2j}(X^{\mathrm{an}}, \mathbf{Q})$ of $P_\Delta(t)$ form the so-called h-vector $(h_0, h_1, \ldots, h_r)$ of $\square$ and satisfy the Dehn-Sommerville equalities $h_j = h_{r-j}$ for all j by the Poincaré duality. If $\square$ is simple so that Δ is simplicial, then the above corollary describes the relationship between the h-vector and the so-called f-vector $(f_0, f_1, \ldots, f_r)$, where f_j is the number of j-dimensional faces of $\square$, and hence $f_j = {}^{\#}\Delta(r-j)$.

Remark. The r-dimensional toric variety X has a natural action of the algebraic group $T_N \cong (\mathbf{C}^\times)^r$. The equivariant intersection cohomology groups $IH^{\cdot}_{T_N}(X^{\mathrm{an}}, \mathbf{Q})$ considered by Bernstein-Lunts [2], Brylinski [5], Joshua [23] and Kirwan [25] turn out to be simpler for toric varieties than the non-equivariant intersection cohomology groups, and satisfy

$$IH^{\cdot}_{T_N}(X^{\mathrm{an}}, \mathbf{Q}) = \mathrm{Sym}(M_{\mathbf{Q}}) \otimes_{\mathbf{Q}} IH^{\cdot}(X^{\mathrm{an}}, \mathbf{Q}),$$

where $\mathrm{Sym}(M_{\mathbf{Q}})$ is the symmetric algebra for the $\mathbf{Q}$-vector space $M_{\mathbf{Q}}$ with the degrees doubled.

3 The decomposition theorem

We briefly recall the notions of the intersection cohomology and the intersection complexes due to Goresky-MacPherson and Beilinson-Bernstein-Deligne-Gabber. We restrict ourselves to the case of middle perversity. For details, we refer the reader to Beilinson-Bernstein-Deligne [1], Borel et al. [3], Brylinski [4], Deligne [8], Goresky-MacPherson [13], [14], [15], Kirwan [24] and MacPherson [26], [27].

Let X be an r-dimensional normal irreducible algebraic variety of finite type over $\mathbf{C}$. We do not assume X to be complete. For simplicity, we denote by $\mathcal{X} := X^{\mathrm{an}}$ the associated normal complex analytic space.

The *intersection complex* $\mathcal{IC}^{\cdot}_{\mathcal{X}}$ of $\mathbf{Q}_{\mathcal{X}}$-modules with respect to the *middle perversity* is an object in the derived category $D^b_c(\mathbf{Q}_{\mathcal{X}})$ of bounded complexes of $\mathbf{Q}_{\mathcal{X}}$-modules with constructible cohomology sheaves and is defined by

$$\mathcal{IC}^{\cdot}_{\mathcal{X}} := j_{!*}\mathbf{Q}_{\mathcal{X}^\circ}[r] := \mathrm{Image}\left(j_!\mathbf{Q}_{\mathcal{X}^\circ}[r] \longrightarrow j_*\mathbf{Q}_{\mathcal{X}^\circ}[r]\right),$$

where $j : \mathcal{X}^\circ \to \mathcal{X}$ is the open immersion of the smooth part of $\mathcal{X}$ and $j_{!*}$ is called the *intermediate extension*. We here follow the degree convention of Beilinson-Bernstein-Deligne [1], so that the cohomology sheaves satisfy

$$\mathcal{H}^p(\mathcal{IC}^{\cdot}_{\mathcal{X}}) = 0 \qquad \text{unless} \quad -r \leq p \leq r.$$

We also follow their convention of denoting $\mathbf{R}j_*$ and $\mathbf{R}j_!$ between derived categories simply by j_* and $j_!$. $\mathcal{IC}^{\cdot}_{\mathcal{X}}$ is uniquely characterized as an object of $D^b_c(\mathbf{Q}_{\mathcal{X}})$ (i.e., as a complex of $\mathbf{Q}_{\mathcal{X}}$-modules up to quasi-isomorphism) by the following properties:

- $\mathcal{IC}^{\cdot}_{\mathcal{X}}|_{\mathcal{X}^{\circ}} = \mathbf{Q}_{\mathcal{X}^{\circ}}[r]$, where the right hand side denotes the complex with $\mathbf{Q}_{\mathcal{X}^{\circ}}$ at degree $-r$ and 0 elsewhere.
- (The support condition) We have
$$\dim_{\mathbf{C}} \operatorname{supp} \mathcal{H}^p(\mathcal{IC}^{\cdot}_{\mathcal{X}}) < -p \qquad \text{for} \quad -r < p.$$
In particular, $\mathcal{H}^p(\mathcal{IC}^{\cdot}_{\mathcal{X}}) = 0$ for $p \geq 0$.
- (The Verdier self-duality) With respect to the Verdier dualizing functor $\mathbf{D}_{\mathcal{X}}$ for $\mathcal{X}$ (cf. Verdier [42]), we have
$$\mathbf{D}_{\mathcal{X}}(\mathcal{IC}^{\cdot}_{\mathcal{X}}) = \mathcal{IC}^{\cdot}_{\mathcal{X}}.$$

The *intersection cohomology groups* with $\mathbf{Q}$-coefficients and with respect to the *middle perversity* and those with *compact support* are then defined by

$$\begin{aligned} IH^{\cdot}(\mathcal{X}, \mathbf{Q}) &:= \mathbf{H}^{\cdot}(\mathcal{X}, \mathcal{IC}^{\cdot}_{\mathcal{X}}[-r]) \\ IH^{\cdot}_c(\mathcal{X}, \mathbf{Q}) &:= \mathbf{H}^{\cdot}_c(\mathcal{X}, \mathcal{IC}^{\cdot}_{\mathcal{X}}[-r]), \end{aligned}$$

where $\mathbf{H}^{\cdot}$ and $\mathbf{H}^{\cdot}_c$ denote the hypercohomology groups and the hypercohomology groups with compact support, while $[-r]$ denotes the degree shift to the right by r.

When X is an orbifold, then $\mathcal{IC}^{\cdot}_{\mathcal{X}}$ is quasi-isomorphic to $\mathbf{Q}_{\mathcal{X}}[r]$ so that $IH^{\cdot}(\mathcal{X}, \mathbf{Q})$ (resp. $IH^{\cdot}_c(\mathcal{X}, \mathbf{Q})$) coincides with the ordinary cohomology group $H^{\cdot}(\mathcal{X}, \mathbf{Q})$ (resp. the ordinary cohomology group with compact support $H^{\cdot}_c(\mathcal{X}, \mathbf{Q})$).

To state the decomposition theorem, we need to recall the notion of perverse $\mathbf{Q}_{\mathcal{X}}$-modules. A *perverse* $\mathbf{Q}_{\mathcal{X}}$*-module* is an object $K^{\cdot}$ in the derived category $D^b_c(\mathbf{Q}_{\mathcal{X}})$ of bounded complexes of $\mathbf{Q}_{\mathcal{X}}$-modules with constructible cohomology sheaves such that the following conditions are satisfied:

- (The support condition) We have
$$\dim_{\mathbf{C}} \operatorname{supp} \mathcal{H}^p(K^{\cdot}) \leq -p \qquad \text{for all } p \in \mathbf{Z}.$$
In particular, $\mathcal{H}^p(K^{\cdot}) = 0$ for $p > 0$.
- (The depth condition) For any irreducible closed subvariety $Z \subset X$, there exists a Zariski dense open subset $V \subset Z$ such that the local cohomology sheaves satisfy
$$\mathcal{H}_{V^{\mathrm{an}}}(K^{\cdot}) = 0 \qquad \text{for} \quad p < -\dim Z.$$

The category of perverse $\mathbf{Q}_{\mathcal{X}}$-modules is known to be an abelian category which is both Artinian and Noetherian. The *simple objects* are of the intermediate extension form

$$(j_{!*}L)\,[\dim_{\mathbf{C}} V] := \mathrm{Image}\,(j_!L \to j_*L)\,[\dim_{\mathbf{C}} V],$$

where $j : V \to X$ is the immersion of a smooth locally closed subvariety $V \subset X$ and L is a locally constant $\mathbf{Q}_{V^{\mathrm{an}}}$-module (i.e., irreducible local system on V^{an} of $\mathbf{Q}$-vector spaces), while $[\dim_{\mathbf{C}} V]$ denotes the dimension shift to the left by $\dim_{\mathbf{C}} V$. Thus the intersection complex $\mathcal{IC}^{\cdot}_{\mathcal{X}}$ is a simple perverse $\mathbf{Q}_{\mathcal{X}}$-module, since the depth condition follows from the support condition and the Verdier self-duality.

The following decomposition theorem is obtained as a consequence of an analogous theorem for étale perverse sheaves of $\mathbf{Q}_l$-modules on algebraic varieties defined over a finite field, which in turn follows from Deligne's proof of the Weil conjecture in [7]:

Theorem 3.1 (The decomposition theorem of Beilinson-Bernstein-Deligne-Gabber, cf. [1]) *Let $f : X' \to X$ be a proper morphism of irreducible normal algebraic varieties over $\mathbf{C}$ and let $f^{\mathrm{an}} : \mathcal{X}' \to \mathcal{X}$ be the corresponding proper morphism of complex analytic spaces. Then the direct image functor $(f^{\mathrm{an}})_* : D^b_c(\mathbf{Q}_{\mathcal{X}'}) \to D^b_c(\mathbf{Q}_{\mathcal{X}})$ sends a simple perverse $\mathbf{Q}_{\mathcal{X}'}$-module to a* semisimple *(i.e., a direct sum of simple) perverse $\mathbf{Q}_{\mathcal{X}}$-modules.*

Corollary 3.2 *Let $f : X' \to X$ be a proper morphism from an orbifold X' to a normal algebraic variety over $\mathbf{C}$. (For instance, $f : X' \to X$ is a resolution of singularities, or an "orbifoldization", of X.) Then the intersection complex $\mathcal{IC}^{\cdot}_{\mathcal{X}}$ is a direct summand of $(f^{\mathrm{an}})_*\mathcal{IC}^{\cdot}_{\mathcal{X}'}$. In particular, we have injections*

$$\begin{aligned} IH^{\cdot}(\mathcal{X},\mathbf{Q}) &\subset IH^{\cdot}(\mathcal{X}',\mathbf{Q}) = H^{\cdot}(\mathcal{X}',\mathbf{Q}) \\ IH^{\cdot}_c(\mathcal{X},\mathbf{Q}) &\subset IH^{\cdot}_c(\mathcal{X}',\mathbf{Q}) = H^{\cdot}_c(\mathcal{X}',\mathbf{Q}). \end{aligned}$$

Remark. Suppose X is a closed subvariety of a smooth algebraic variety Z over $\mathbf{C}$. Then by the Riemann-Hilbert correspondence in algebraic analysis, the category of perverse $\mathbf{C}_{\mathcal{X}}$-modules is equivalent to the category of algebraic regular holonomic $\mathcal{D}_Z$-modules with support contained in X. The rather mysterious conditions in the definition of perverse $\mathbf{Q}_{\mathcal{X}}$-modules arise naturally in this context. For details, we refer the reader to the references listed at the beginning of this section as well as the literature cited therein.

As for the equivariant intersection cohomology, equivariant intersection complexes and equivariant $\mathcal{D}$-modules, we refer the reader to Bernstein-Lunts [2], Brylinski [5], Hotta [16], Joshua [23] and Kirwan [25].

4 Recent results of Ishida's

In view of the combinatorial importance of the h-vector, it is desirable to describe $P_\Delta(t)$ and $IH^j(X^{\mathrm{an}}, \mathbf{Q})$ directly in terms of Δ or $\Box$ and prove Theorem 2.1 without recourse to the highly nontrivial decomposition theorem. We might then be able to remove the unnatural rationality assumption on $\Box$ and get results valid for *irrational* convex polytopes as well. There have been attempts in this direction by McMullen [29], Stanley [40], the author [31], [33], [34] and others.

Ishida [20], [21], [19] finally succeeded in describing the intersection complex $\mathcal{IC}^{\cdot}_{X^{\mathrm{an}}}$ and the intersection cohomology groups $IH^{\cdot}(X^{\mathrm{an}}, \mathbf{Q})$ (and more generally those with respect to general perversities) of a toric variety X entirely in terms of the corresponding fan Δ. Moreover, he could show a version of the decomposition theorem as well as vanishing theorems in this new formulation. Unfortunately, however, it does not seem possible at the moment to remove the rationality assumption. It is highly desirable to obtain analogous results, for instance, for simplicial cone decompositions with markings (cf. [31]).

Referring the reader to Ishida [20], [21], [19] for details, we here mention only his vanishing theorems, called the first and second diagonal theorems.

(1) (Ishida's first diagonal theorem) Let X be the complete toric variety corresponding to a finite complete fan Δ for $N \cong \mathbf{Z}^r$. Then

$$IH^j(X^{\mathrm{an}}, \mathbf{Q}) = 0 \qquad \text{for } j \text{ odd.}$$

(2) (Ishida's second diagonal theorem) Let π be an r-dimensional strongly convex rational polyhedral cone in $N_{\mathbf{R}} \cong \mathbf{R}^r$. Denote the corresponding r-dimensional affine toric variety and its unique T_N-fixed point by $U_\pi := \mathrm{Spec}(\mathbf{C}[M \cap \pi^\vee])$ and $P := \mathrm{orb}(\pi)$, respectively. Then we have:

(i) $IH^j(U_\pi^{\mathrm{an}}, \mathbf{Q}) = 0$ for j odd.

(ii) $IH^{2p}(U_\pi^{\mathrm{an}}, \mathbf{Q}) = 0$ for $r \leq 2p$.

(iii) $IH^{2p}(U_\pi^{\mathrm{an}} \setminus \{P\}, \mathbf{Q}) = 0$ for $r \leq 2p$.

(iv) $IH^{2p-1}(U_\pi^{\mathrm{an}} \setminus \{P\}, \mathbf{Q}) = 0$ for $2p - 1 \leq r - 1$.

5 The Chow ring of a finite simplicial fan

Let us recall the Chow rings over $\mathbf{Q}$ for simplicial fans (cf. Danilov [6], Park [36], [37] and [30], [31], [33]) and describe consequences for them of Ishida's vanishing theorems mentioned in Section 4.

Let Δ be a finite *simplicial* fan for $N \cong \mathbf{Z}^r$ which need not be complete. We denote $\Delta(1) := \{\rho \in \Delta \mid \dim \rho = 1\}$. For each $\rho \in \Delta(1)$, let $n(\rho)$ be the unique primitive element of N contained in ρ.

Introduce a variable x_ρ for each $\rho \in \Delta(1)$ and denote by

$$S := \mathbf{Q}\,[x_\rho \mid \rho \in \Delta(1)]$$

the polynomial ring over $\mathbf{Q}$ in the variables $\{x_\rho \mid \rho \in \Delta(1)\}$. Let I be the ideal of S generated by the set

$$\{x_{\rho_1}x_{\rho_2}\cdots x_{\rho_s} \mid \rho_1,\ldots,\rho_s \in \Delta(1) \text{ distinct and } \rho_1+\cdots+\rho_s \notin \Delta\}$$

of square-free monomials. On the other hand, let J be the ideal of S generated by the set

$$\left\{ \sum_{\rho\in\Delta(1)} \langle m, n(\rho)\rangle x_\rho \,\middle|\, m \in M \right\}$$

of linear forms.

Definition. The *Chow ring* for a finite simplicial fan Δ is defined to be

$$A = A(\Delta) := S/(I+J).$$

We denote by $v(\rho)$ the image in A of the variable x_ρ. More generally, for each $\sigma \in \Delta$, which is uniquely expressed in the form $\sigma = \rho_1 + \cdots + \rho_p$ with distinct $\rho_1, \ldots, \rho_p \in \Delta(1)$ and $p := \dim \sigma$, we denote $v(\sigma) := v(\rho_1)v(\rho_2)\cdots v(\rho_p)$, which is the image in A of $x_{\rho_1}x_{\rho_2}\cdots x_{\rho_p}$. Note that the multiplication in our definition differs by a multiplicative factor from that in Danilov [6] and Fulton [12].

Proposition 5.1 *The Chow ring $A = A(\Delta)$ for a simplicial fan Δ for $N \cong \mathbf{Z}^r$ is an Artinian graded $\mathbf{Q}$-algebra of the form*

$$A = \bigoplus_{p=0}^{r} A^p \qquad \text{with} \quad A^p = A^p(\Delta) = \sum_{\sigma\in\Delta(p)} \mathbf{Q}v(\sigma)$$

and is generated by A^1 over $A^0 = \mathbf{Q}$. Moreover, we have the following relations:

$$\sum_{\rho\in\Delta(1)} \langle m, n(\rho)\rangle v(\rho) = 0 \qquad \text{for all} \quad m \in M,$$

and, for $\sigma, \sigma' \in \Delta$,

$$v(\sigma)v(\sigma') = \begin{cases} 0 & \text{if } \sigma + \sigma' \notin \Delta \\ v(\sigma+\sigma') & \text{if } \sigma \cap \sigma' = \{0\}, \quad \sigma+\sigma' \in \Delta. \end{cases}$$

The Chow ring $A(\Delta)$ for a finite simplicial and *complete* fan Δ for $N \cong \mathbf{Z}^r$ is known to be a Gorenstein $\mathbf{Q}$-algebra satisfying the duality $\dim_{\mathbf{Q}} A^p(\Delta) = \dim_{\mathbf{Q}} A^{r-p}(\Delta)$ for all $0 \leq p \leq r$. Moreover, we have

$$A^p(\Delta) = H^{2p}(X^{\mathrm{an}}, \mathbf{Q}) = IH^{2p}(X^{\mathrm{an}}, \mathbf{Q}) \qquad \text{for all } p,$$

where X is the complete toric orbifold corresponding to Δ.

For a finite simplicial and complete fan Σ for $\bar{N} \cong \mathbf{Z}^{r-1}$, let us now consider equivariant $\mathbf{P}_1(\mathbf{C})$-bundles over $\bar{X} := T_{\bar{N}}\,\mathrm{emb}(\Sigma)$ and associated $\mathbf{C}$-bundles and $\mathbf{C}^*$-bundles.

For that purpose, let $\eta : \bar{N}_{\mathbf{R}} \to \mathbf{R}$ be an $\mathbf{R}$-valued function which is $\mathbf{Z}$-valued on $\bar{N}$ and piecewise linear with respect to the fan Σ. Denote $N := \bar{N} \oplus \mathbf{Z}n_0 \cong \mathbf{Z}^r$ and consider the graph $g : \bar{N}_{\mathbf{R}} \to N_{\mathbf{R}}$ of η defined by $g(\bar{n}) := \bar{n} + \eta(\bar{n})n_0$. We then let

$$\begin{aligned} \Phi^{\flat} &:= \{g(\bar{\sigma}) \mid \bar{\sigma} \in \Sigma\} \\ \Phi &:= \Phi^{\flat} \coprod \left\{\tau + \mathbf{R}_{\geq 0}n_0 \mid \tau \in \Phi^{\flat}\right\} \\ \tilde{\Phi} &:= \Phi \coprod \left\{\tau + \mathbf{R}_{\geq 0}(-n_0) \mid \tau \in \Phi^{\flat}\right\}. \end{aligned}$$

The projection $N \to \bar{N}$ killing n_0 induces maps of fans $(N, \tilde{\Phi}) \to (\bar{N}, \Sigma)$, $(N, \Phi) \to (\bar{N}, \Sigma)$ and $(N, \Phi^{\flat}) \to (\bar{N}, \Sigma)$ which respectively give an equivariant $\mathbf{P}_1(\mathbf{C})$-bundle $T_N\,\mathrm{emb}(\tilde{\Phi}) \to \bar{X}$, the associated $\mathbf{C}$-bundle $T_N\,\mathrm{emb}(\Phi) \to \bar{X}$ and the associated $\mathbf{C}^*$-bundle $T_N\,\mathrm{emb}(\Phi^{\flat}) \to \bar{X}$.

Proposition 5.2 *$A(\tilde{\Phi})$ is canonically isomorphic to the algebra $A(\Sigma)[\xi]$ over $A(\Sigma)$ generated by an element ξ subject to the relation*

$$\xi(\xi + \bar{\eta}) = 0 \qquad \textit{with} \quad \bar{\eta} := \sum_{\bar{\rho} \in \Sigma(1)} \eta(\bar{n}(\bar{\rho}))\bar{v}(\bar{\rho}) \in A^1(\Sigma),$$

where $\bar{n}(\bar{\rho})$ and $\bar{v}(\bar{\rho})$ for $\bar{\rho} \in \Sigma(1)$ are similar to $n(\rho)$ and $v(\rho)$ previously defined for $\rho \in \Delta(1)$.

Moreover, we have canonical isomorphisms

$$A(\Phi) = A(\Sigma) \qquad \textit{and} \qquad A(\Phi^{\flat}) = A(\Sigma)/A(\Sigma)\bar{\eta}.$$

As in Section 4, let π be an r-dimensional strongly convex rational polyhedral cone in $N_{\mathbf{R}} \cong \mathbf{R}^r$ such that π itself may not be simplicial but all the *proper faces* of π are simplicial. Thus the set $\partial\pi$ of proper faces of π is a simplicial fan for N. The toric variety corresponding to $\partial\pi$ is $U_\pi \setminus \{P\}$, where P is the unique T_N-fixed point in the r-dimensional affine toric variety $U_\pi := \mathrm{Spec}(\mathbf{C}[M \cap \pi^{\vee}])$.

As we saw in Section 4, we now have a proof for the following vanishing theorem entirely in terms of fans, thanks to Ishida [20], [21], [19]:

$$A^p(\partial\pi) = IH^{2p}(U_\pi^{\mathrm{an}} \setminus \{P\}, \mathbf{Q}) = H^{2p}(U_\pi^{\mathrm{an}} \setminus \{P\}, \mathbf{Q}) = 0 \qquad \text{for } r/2 \leq p,$$

as well as

$$IH^{2p-1}(U_\pi^{\mathrm{an}} \setminus \{P\}, \mathbf{Q}) = H^{2p-1}(U_\pi^{\mathrm{an}} \setminus \{P\}, \mathbf{Q}) = 0 \qquad \text{for } 2p-1 \leq r-1.$$

As we now see, this is exactly the consequence of the strong Lefschetz theorem which Stanley [38] used to prove the so-called "g-theorem" conjectured earlier by McMullen [28].

Choose a primitive element $n_0 \in N$ which is contained in the interior of π. Then there certainly exists a decomposition $N = \bar{N} \oplus \mathbf{Z}n_0$ with $\bar{N} \cong \mathbf{Z}^{r-1}$. Since π is assumed to be a strongly convex cone, there exist a complete fan Σ ($\Sigma = \overline{\partial\pi}$ in the notation of Theorem 2.1) for $\bar{N}$ and an $\mathbf{R}$-valued function $\eta : \bar{N}_{\mathbf{R}} \to \mathbf{R}$ which is $\mathbf{Z}$-valued on $\bar{N}$ and is piecewise linear and strictly convex with respect to Σ such that

$$\partial\pi = \{g(\bar{\sigma}) \mid \bar{\sigma} \in \Sigma\} = \Phi^\flat,$$

where $g : \bar{N}_{\mathbf{R}} \to N_{\mathbf{R}}$ is the graph of η defined by $g(\bar{n}) := \bar{n} + \eta(\bar{n})n_0$. Hence

$$A^p(\partial\pi) = A^p(\Sigma)/\bar{\eta}A^{p-1}(\Sigma) \qquad \text{with} \quad \bar{\eta} := \sum_{\bar{\rho} \in \Sigma(1)} \eta(\bar{n}(\bar{\rho}))\bar{v}(\bar{\rho}) \in A^1(\Sigma).$$

Its vanishing in degrees $p \geq r/2$, which is equivalent to the vanishing of $H^{2p-1}(U_\pi^{\mathrm{an}} \setminus \{P\}, \mathbf{Q})$ for $2p-1 \leq r-1$ by the Poincaré duality, is the relevant consequence of the strong Lefschetz theorem for the projective toric variety associated to Σ with respect to the "ample" element $\bar{\eta} \in A^1(\Sigma)$ by [31, Cor. 4.5]. Indeed, we have the following exact sequence for all p, where $\bar{X}$ is the $(r-1)$-dimensional projective toric variety corresponding to the fan Σ:

$$0 \to H^{2p-1}(U_\pi^{\mathrm{an}} \setminus \{P\}, \mathbf{Q}) \to H^{2p-2}(\bar{X}^{\mathrm{an}}, \mathbf{Q}) \xrightarrow{\bar{\eta}} H^{2p}(\bar{X}^{\mathrm{an}}, \mathbf{Q}) \to H^{2p}(U_\pi^{\mathrm{an}} \setminus \{P\}, \mathbf{Q}) \to 0$$

with $A^{p-1}(\Sigma) = H^{2p-2}(\bar{X}^{\mathrm{an}}, \mathbf{Q}) = A^{p-1}(\Phi)$, $A^p(\Sigma) = H^{2p}(\bar{X}^{\mathrm{an}}, \mathbf{Q}) = A^p(\Phi)$ and $A^p(\partial\pi) = H^{2p}(U_\pi^{\mathrm{an}}, \mathbf{Q}) = A^p(\Phi^\flat)$. (See [34, Theorem 3.3 and Corollary 4.5] for details.) Note that the strong Lefschetz theorem asserts the isomorphy of

$$\bar{\eta}^{(r-1)-2j} : A^j(\Sigma) \xrightarrow{\sim} A^{(r-1)-j}(\Sigma) \qquad \text{for all } 0 \leq j \leq (r-1)/2.$$

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