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Dimension Theory of the C*-algebras of Lie Groups

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Dedicated to Professor Masamichi Takesaki on his sixtieth birthday

§1 Introduction Dimension theory of topological spaces takes a decisive role in geometry as well as topology. However, no appropriate definition of dimension is established in C*-algebra theory as noncommutative topological spaces. Rieffel[5] defined a notion of stable rank which may be considered as a noncommutative version of the complex dimension of topological spaces in connection with the cancellation law of C*-modules. Soon after this, Brown and Pedersen [1] considered a noncommutative version of the real dimension of topological spaces which they called real rank. Although they studied fundamental properties of stable (real) rank, nevertheless nobody can find a simple C*-algebra with stable (real) rank greater than one. Moreover there are several peculiar phenomena of stable (real) rank in simple C*-algebras. On the other hand, we know lots of non simple C*-algebras whose stable rank are greater than one, for example abelian or group C*-algebras. Here are two problems raised by Rieffel [5], one of which says that "Describe the stable rank of the reduced C*-algebras of Lie groups in terms of the group structure of G" and the other one is "Compute the stable rank of the reduced C*-algebras of the free groups with n-generators ($n \geq 2$)".

In this paper, we shall report partial results for the above problems. More precisely, we have a good estimation of the stable rank of the reduced C*-algebras of semisimple Lie groups by using their real rank. We then obtain that the stable rank of the free group C*-algebras is one, which solves one of the Rieffel's problems. However, many difficulties appear to compute stable rank in the case of solvable Lie groups because of their representation theory. We only can compute it for simply connected nilpotent Lie groups though their primitive case is solved by Sheu [7].

§2 Semisimple Cases Let G be a connected semisimple Lie group and let $G = KAN$ be a Iwasawa decomposition of G. Put $rr(G) = \dim_{\mathbb{R}} A$ and we call it the real rank of G. We use A.Wassermann's description of $C_r^*(G)$ as follows:

Lemma 2.1 ([10]) Suppose G is linear, then we have that

$$C_r^*(G) \simeq \oplus_{(P,\omega)} C_o(\hat{A}'/W) \rtimes R_\omega$$

where (P,ω) is a pair of a cuspidal parabolic subgroup P of G and a dicrete series ω of M for the Langland's decomposition $P = MA'N'$ of P, $\hat{A}'$ is the dual group of A', and the stabilizer group W_ω of W at ω can be decomposed as the semidirect product $W \rtimes R_\omega$ of a normal subgroup W by a finite abelian group R_ω.

We also need the following proposition in order to show our main result for semisimple Lie groups:

Lemma 2.2 ([8]) Let (A,G,α) be a C*-dynamical system where G is a finite abelian group. Then we have that

$$\mathrm{sr}(A^\alpha) \wedge 2 \leq \mathrm{sr}(A \rtimes G) \leq \mathrm{sr}(A^\alpha)$$

where A^α is the C*-algebra of all the fixed points of A under α.

Applying Lemma 2.1 and Lemma 2.2, we obtain the following theorem:

Theorem 2.3 ([8]) Suppose G is a connected linear semisimple Lie group and let rr(G) be the real rank of G, then we estimate that

$$([\mathrm{rr}(G)/2] + 1) \wedge 2 \leq \mathrm{sr}(C_r^*(G)) \leq [\mathrm{rr}(G)/2] + 1.$$

As a corollary, we easily deduce the following fact:

Corollary 2.4 ([8]) Let G be one of $SO_o(n,m)$, $SU(n,m)$ and $SP(n,m)$ $(n \geq 1, m = 1,2,3)$. Then we have that

$$\mathrm{sr}(C_r^*(G)) = m \text{ if } m = 1,2 \text{ , and } = 2 \text{ if } m = 3.$$

Moreover let F_4 be the exceptional Lie group of type F_4. Then we have that

$$sr(Cr^*(F_4)) = 1.$$

In connection with the above corollary, it would be of independent interest to consider discrete subgroups of semisimple Lie groups of real rank one because of hyperbolic geometry.

Let us now take a unimodular locally compact group G and a unimodular closed subgroup H of G. Then there exists a G-invariant Borel measure μ of G/H such that

$$\int_G f(g)\, d_G(g) = \int_{G/H} \int_H f(gh)\, d_H(h) d\mu(gH)$$

for all $f \in C_c(G)$. We manipulate the relation between $sr(Cr^*(G))$ and $sr(Cr^*(H))$ in the following fashon:

Proposition 2.5 ([8]) Let G, H, and μ be as above and suppose $\mu(G/H) > 0$, then we have that

$$sr(Cr^*(H)) \le sr(Cr^*(G)).$$

Remark According to Schulz [6], there exist many pairs (G,H) of groups G and H with the property that G/H is finite and $sr(Cr^*(G)) \le sr(Cr^*(H))$. Together with the above proposition, the equality holds for Schulz's examples.

Combining Corollary 2.4 and Proposition 2.5, we easily deduce the following fact:

Corollary 2.6 ([8]) Let Γ be a lattice of a connected linear semisimple Lie group of real rank one, then it follows that

$$sr(Cr^*(\Gamma)) = 1.$$

Let us consider a hyperbolic manifold M and denote by $\pi_1(M)$ the fundamental group of M. Since M is hyperbolic, there exists a connected linear semisimple Lie group G of real rank one such that $\pi_1(M) \subset G$. By Corollary 2.6, we have the following:

Corollary 2.7 ([8]) Let M be a hyperbolic manifold and $\pi_1(M)$ the fundamental group of M, then it implies that

$$sr(Cr^*(\pi_1(M))) = 1 .$$

As a good application of the above corollary, let F_n be the free group with n-generators ($n \geq 2$). Since there exists a hyperbolic manifold M such that $\pi_1(M) = F_n$. Therefore we obtain the following corollary which solves one of the problems posed by Rieffel [5]:

Corollary 2.8 ([8]) Let F_n the free group with n-generators ($n \geq 2$). Then we have that

$$sr(Cr^*(F_n)) = 1 .$$

The following statement is easily seen by Rieffel's results [5]:

Corollary 2.9 ([8]) The cancellation law holds for $Cr^*(F_n)$ and all invertible elements of $Cr^*(F_n)$ are dense in $Cr^*(F_n)$.

§3 Solvable Cases In comparison with semisimple Lie groups, nothing is known for stable rank in solvable Lie groups except a special case of nilpotent Lie groups, which is due to Sheu [7]. It says that if $G = \mathbb{R}^d \rtimes \mathbb{R}$ ($d \geq 0$) is a nilpotent Lie group and let $(\mathfrak{g}^*, G, Ad^*)$ be the dynamical system of the coadjoint action Ad^* of G on the real dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G, then

$$sr(Cr^*(G)) = \dim_{\mathbb{C}} (\mathfrak{g}^*)^G = [\dim_{\mathbb{R}} (\mathfrak{g}^*)^G / 2] + 1$$

where $(\mathfrak{g}^*)^G$ is the subspace of all the fixed points of $\mathfrak{g}^*$ under Ad^*. Thus for example let H^3 be the 3-dimensional Heisenberg group over $\mathbb{R}$, then $sr(Cr^*(H^3)) = 2$. However taking G the real ax+b group, we then easily see that $sr(Cr^*(G)) = 2$ whereas $\dim_{\mathbb{C}} (\mathfrak{g}^*)^G = 1$. Since the ax+b group is an exponential non nilpotent Lie group, the next theorem seems to be the best estimation among connected simply connected Lie groups:

Theorem 3.1 ([9]) Let G be a connected simply connected nilpotent Lie group and let ($\mathfrak{g}^*$, G, Ad*) be the dynamical system of the coadjoint action Ad* of G on the real dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$ of G, then

$$\mathrm{sr}(\mathrm{Cr}^*(G)) = \dim_{\mathbf{C}} (\mathfrak{g}^*)^G$$

where $(\mathfrak{g}^*)^G$ is the subspace of all the fixed points of $\mathfrak{g}^*$ under Ad*.

The proof of the above theorem are based on the Kirillov's polarization method [3] for constructing irreducible representations and a deep analysis of the universal enveloping algebra of $\mathfrak{g}$ due to Dixmier [2]. More precisely, given a m = 0,1,2,..., let Ω_m be the set of all $[\phi] \in \mathfrak{g}^*/G$ with $\dim_{\mathbf{R}} [\phi] = m$ where $\mathfrak{g}^*/G$ is the coadjoint orbit space of $\mathfrak{g}^*$ by Ad* of G. Then $\mathfrak{g}^*/G$ is the disjoint union of Ω_m ($m \geq 0$) and Ω_0 is a closed subset of $\mathfrak{g}^*/G$ homeomorphic to $(\mathfrak{g}^*)^G$. Let I be the closed *-ideal of Cr*(G) which corresponds to $(\mathfrak{g}^*/G)\backslash\Omega_0$. Then we have the following lemma by Kirillov's polarization method [3]:

Lemma 3.2 ([9]) $\mathrm{Cr}^*(G)/I \simeq C_0(\Omega_0)$ and $\dim \rho = \aleph_0$ for all $\rho \in \mathrm{Irr}\, I$.

Since G is nilpotent, theuniversal enveloping algebra U($\mathfrak{g}$) of $\mathfrak{g}$ is Noetherian, henceforth so is the primitive ideal space PrimU($\mathfrak{g}$) of U($\mathfrak{g}$). By Dixmier [2], we have the following lemma:

Lemma 3.3 ([2]) There exists a finite composition series $\{I_k\}$ of I such that I_k/I_{k-1} is a C*-algebra of continuous trace class for all $k \geq 1$.

We also use Nistor's result [4] concerning stable rank as follows:

Lemma 3.4 ([4]) Let A be a C*-algebra and I a closed *-ideal of A. Suppose I is a C*-algebra of continuous trace class and $\dim \rho = \aleph_0$ for all $\rho \in \mathrm{Irr}\, I$, then we have that

$$\mathrm{sr}(A) \leq \mathrm{sr}(A/I) \vee 2 .$$

We finally use the following lemma in order to show Theorem 3.1:

Lemma 3.5 ([9]) Suppose G is a connected simply connected nilpotent Lie group such that

$$\dim_{\mathbb{R}} (\mathfrak{g}^*)^G = 1,$$

then we have that

$$G \simeq \mathbb{R}.$$

Considering again the real ax+b group G, we can compute that

$$\max_{[\phi]\in \mathfrak{g}^*/G} \dim_{\mathbb{C}} [\phi] = 2.$$

This is also true for $G = H^3$. We then pose the followin conjecture:

Conjecture Let G be an exponential Lie group and let Z be the center of G. Let $\mathfrak{g}$, $\mathfrak{z}$ be the Lie algebras of G, Z respectively. Then we would have that

$$sr(Cr^*(G)) = \max_{[\phi]\in (\mathfrak{g}/\mathfrak{z})^*/(G/Z)} \dim_{\mathbb{C}} [\phi] \vee \dim_{\mathbb{C}} \mathfrak{z}$$

where $(\mathfrak{g}/\mathfrak{z})^*/(G/Z)$ means the orbit space of $(\mathfrak{g}/\mathfrak{z})^*$ by Ad* of G/Z.

In the case of solvable Lie groups of non type I, there is no way to proceed our plan for computing stable rank of group C*-algebras although we know that

$$sr(Cr^*(M^n)) = 1$$

for all the n-dimensional Mautner groups M^n ($n \geq 5$).

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