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Distributions associated with C^*-dynamical systems

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For a given $C^*(\mathrm{or} W^*)$ -dynamical system (A, α, R^n), We generalize elements of the C^*-algebra A to construct a class of objects (called C^*-distributions). Any member of this class can be differentiable in the C^*-distribution sense and has its Fourier transform in these objects. Hence all elements of A , even ones which are not differentiable by the original derivations δ coming from the action α, can be differentiable and have Fourier transforms in this wider class.

The main result in this paper is the structure theorem for these new objects. We will also describe applications to differential equations on C^*-algebras. We will finally present a Payley-Wiener-Schwartz type theorem for C^*-algebras. For simplicity, we treat the case $n = 1$ and assume that A is a C^*-algebra. We have similar discussions for W^*-algebras with R^n-actions.

Full details of this paper will appear in [Ma2].

Let $\mathcal{D} = \mathcal{D}(R)$ be the topological vector space of all C^∞ -functions on R with compact support. Let $\mathcal{S} = \mathcal{S}(R)$ be the topological vector space of all rapiddly decreasing functions on R.

1. Definition (C^*-Distribution spaces).

$$\mathcal{D}'_\alpha(A) = \{v : \mathcal{D} \to A \text{ continuous linear} \,; \alpha_t(v(\phi)) = v(\tau_t\phi), \quad \phi \in \mathcal{D}, t \in R\}$$

$$\mathcal{S}'_\alpha(A) = \{u : \mathcal{S} \to A \text{ continuous linear} \,; \alpha_t(u(\phi)) = u(\tau_t\phi), \quad \phi \in \mathcal{S}, t \in R\}$$

$$\hat{\mathcal{S}}'_\alpha(A) = \{u : \mathcal{S} \to A \text{ continuous linear} \,; \alpha_t(u(\phi)) = u(e_t\phi), \quad \phi \in \mathcal{S}, t \in R\}$$

where τ_t is translation by $t \in R$ and e_t is the function on R defined by $e_t(s) = e^{its}$.

Each member of the above spaces is called a C^*-distribution.

Example 1. For an element $x \in A$ and a non-negative integer k, we set

$$\xi^k_x(\phi) = \int_R \alpha_t(x) D^k\phi(t)dt, \qquad \eta^k_x(\phi) = \int_R \alpha_t(x) D^k\hat{\phi}(t)dt, \qquad \phi \in \mathcal{S},$$

then $\xi^k_x \in \mathcal{S}'_\alpha(A)$ and $\eta^k_x \in \hat{\mathcal{S}}'_\alpha(A)$, where $D^k\phi$ is the k-th derivative of ϕ.

Example 2. If A is unital, then the integral I and the ordinary delta function δ

$$I(\phi) = \int_R \phi(t)dt, \qquad \delta(\phi) = \phi(0) \qquad \phi \in \mathcal{S}$$

define elements of $\mathcal{S}'_\alpha(A)$ and $\hat{\mathcal{S}}'_\alpha(A)$ respectively.

The C^*-algebra A is embedded into $\mathcal{S}'_\alpha(A) (\subset \mathcal{D}'_\alpha(A))$ and $\hat{\mathcal{S}}'_\alpha(A)$ through the maps ξ and η defined by

$$\xi_x(\phi) = \int_R \alpha_t(x)\phi(t)dt, \qquad \eta_x(\phi) = \int_R \alpha_t(x)\hat{\phi}(t)dt, \qquad x \in A.$$

2. Fourier transforms and several operations.

Definitions.

Fourier transform $\wedge : \mathcal{S}'_\alpha(A) \to \hat{\mathcal{S}}'_\alpha(A)$ is defined by $\hat{u}(\phi) = u(\hat{\phi})$ where $\hat{\phi}$ is the Fourier transform of ϕ.

In particular, for $x \in A$, we define "the Fourier transform " $\hat{x}$ of x as the menber of the C^*-distributions by η_x, namely,

$$\hat{x} = \eta_x.$$

Furthermore, we can naturally define "the convolution product " $\hat{x} * \hat{y}$ between $\hat{x}$ and $\hat{y}$, for $x, y \in A$ by

$$\hat{x} * \hat{y} = (\hat{xy}), \qquad x, y \in A.$$

Fourier inverse transform $\vee : \hat{\mathcal{S}}'_\alpha(A) \to \mathcal{S}'_\alpha(A)$ is defined by $\check{u}(\phi) = u(\check{\phi})$ where $\check{\phi}$ is the Fourier inverse transform of ϕ.

Differentiation D on $\mathcal{D}'_\alpha(A)$ is defined by $(Du)(\phi) = u(D\phi)$, $u \in \mathcal{D}'_\alpha(A)$. Hence $D\xi_x = \xi_{\delta(x)}$, $x \in A$.

Multiplication on $\hat{\mathcal{S}}'_\alpha(A)$ *by a slowly increasing function* f *on* R is defined by $(fu)(\phi) = u(f\phi), u \in \hat{\mathcal{S}}'_\alpha(A)$.

Convolution on $\mathcal{S}'_\alpha(A)$ *by a rapidly decreasing distribution* E *on* R is defined by $(E * u)(\phi) = u(E * u), u \in \mathcal{S}'_\alpha(A)$.

Property.

(i) $(\hat{fu}) = \hat{f} * \hat{u}$, where f is a slowly increasing function on R and $u \in \hat{\mathcal{S}}'_\alpha(A)$.

(ii) $(E \hat{*} u) = \hat{E}\hat{u}$, where E is a rapidly decreasing distribution on R and $u \in \mathcal{S}'_\alpha(A)$.

(iii) $(\hat{Pv}) = P(-D)\hat{v}$, $\quad (P(\hat{D})u) = P\hat{u}$, where P is a polynomial on R and $v \in \hat{\mathcal{S}}'_\alpha(A)$, $u \in \mathcal{S}'_\alpha(A)$.

(iv) $\check{\delta} = I$, $\quad \hat{I} = \delta$.

(v) $\delta * u = u$, $\quad u \in \mathcal{S}'_\alpha(A)$ where δ is the ordinary delta function on R.

3. Orders of C^*-distributions.

L^1-order $L^1(u)$ and *range-order* $r(u)$ of $u \in \mathcal{D}'_\alpha(A)$ are defined by

$$L^1(u) = Min\{M \in N \cup \{0\}; \forall K \subset R \text{ compact}, \exists c_K \geq 0 \text{ such that}$$

$$\|u(\phi)\| \leq c_K \int_R |D^M\phi(t)|dt, \quad \forall \phi \in \mathcal{D}, \mathrm{supp}(\phi) \subset K\},$$

$$r(u) = Max\{k \in N \cup \{0\}; u(\mathcal{D}) \subset \delta^k(A^\infty)\}.$$

Lemma 3.1. For $u \in \mathcal{D}'_\alpha(A)$ with $L^1(u) = k$, there exist positive constants $c_0, c_1, ..., c_k$ such that

$$\|u(\phi)\| \leq \sum_{l=0}^{k} \int_R c_l |(D^l\phi)(t)|dt, \quad \forall \phi \in \mathcal{D}.$$

Corollary 3.2. $\mathcal{S}'_\alpha(A) = \mathcal{D}'_\alpha(A)$.

4. Several lemmas and main theorem.

The followings are lemmas to prove the main theorem (Theorem 4.7).

Lemma 4.1. $\forall u \in \mathcal{D}'_\alpha(A), \quad L^1(u) < \infty$.

Lemma 4.2. $\forall u \in \mathcal{D}'_\alpha(A)$ with $r(u) \geq 1, \quad \exists v \in \mathcal{D}'_\alpha(A)$ satisfying

$$Dv = u, \qquad L^1(v) = L^1(u) - 1, \quad r(v) = r(u) - 1.$$

Lemma 4.3. $\forall u \in \mathcal{D}'_\alpha(A)$ with $L^1(u) = 0, \quad \exists$(unique) $x \in A''; u = \xi_x$.

Corollary 4.4. $\forall u \in \mathcal{D}'_\alpha(A)$ with $L^1(u) \leq r(u), \quad \exists x \in A'', \exists M \in N \cup 0; u = D^M\xi_x$.

Lemma 4.5. $\forall u \in \mathcal{D}'_\alpha(A), \quad \exists x \in A, \exists v \in \mathcal{D}'_\alpha(A)$ satisfying

$$u = \xi_x + v, \qquad L^1(v) = L^1(u), \quad r(v) \geq 1.$$

By using Lemma 4.2 and Lemma 4.5, we have

Lemma 4.6. $\forall u \in \mathcal{D}'_\alpha(A)$ with $L^1(u) \geq 1, \quad \exists x \in A, \exists v \in \mathcal{D}'_\alpha(A)$ satisfying

$$u = \xi_x + Dv, \qquad L^1(v) = L^1(u) - 1.$$

By repeating Lemma 4.6, L^1-order of a C^*-distribution finally reduces to 0. Since we know the structure of C^*-distributions of L^1-order 0 as in Lemma 4.3, we reach the following structure theorem.

Theorem 4.7. Any element of C^*-distributions $\mathcal{D}'_\alpha(A)$ can be represented as a finite linear combination of finite order derivatives of elements of A''(a weak closure of A on a Hilbert space). Namely, $\forall u \in \mathcal{D}'_\alpha(A), \exists x_0, x_1, ..., x_m \in A''$;

$$u(\phi) = \sum_{k=0}^{m} \int_R \alpha_t(x_k)(D^k\phi)(t)dt, \quad \forall\phi \in \mathcal{D}.$$

5. Applications to differential equations.

Proposition 5.1. Let $P(t)$ be a polynomial of variable t. Any given $a \in A$, there exists $u_a \in \mathcal{S}'_\alpha(A)$ such that

$$P(D)u_a = a$$

if the ordinary differential equation $P(D)E = \delta$ has a fundamental solution E of rapidly decreasing distribution on R. In paticular, if the E can be taken as an integrable function on R, the above C^*-distribution u_a can be taken in elements of the C^*-algebra A.

6. Spectrum and support.

Spectrum $\mathrm{Sp}_\alpha(u)$ of $u \in \mathcal{D}'_\alpha(A)$ is defined by

$$\mathrm{Sp}_\alpha(u) = \{p \in R | u(\phi) = 0 \Rightarrow \hat{\phi}(p) = 0, \forall\phi \in \mathcal{D}\}.$$

Support $\mathrm{supp}_\alpha(v)$ of $v \in \hat{\mathcal{S}}'_\alpha(A)$ is defined by

$$\mathrm{supp}_\alpha(v) = \{p \in R | v(\phi) = 0 \Rightarrow \phi(p) = 0, \forall\phi \in \mathcal{S}\}.$$

We notice that

$$\mathrm{Sp}_\alpha(\xi_x) = \mathrm{Sp}_\alpha(x): \text{ the } \alpha-\text{spectrum of } x \in A \text{ in the sense of W.Arveson}$$

and

$$\mathrm{Sp}_\alpha(u) = \mathrm{supp}_\alpha(\hat{u}), \quad u \in \mathcal{S}'_\alpha(A).$$

In particular,

$$\mathrm{Sp}_\alpha(x) = \mathrm{supp}_\alpha(\hat{x}), \quad x \in A.$$

7. Payley-Wiener-Schwartz type theorem.

The classical Payley-Winer-Schwartz theorem states that a smooth function (distribution) has compact support if and only if its Fourier transform can be extended on complex numbers as an entire function of exponential type. We shall give an analogue of this for C^*-algebras.

Lemma 7.1. For $u \in \mathcal{S}'_\alpha(A)$, if $Sp_\alpha(u)$ is compact, there exists a unique element $x \in A$ such that $u = \xi_x$.

As $Sp_\alpha(x) = supp_\alpha(\hat{x}), x \in A$, the following theorem is regarded as a C^*-algebra version of the classical Payley-Wiener-Schwartz theorem.

Theorem 7.2. For $x \in A$, $Sp_\alpha(x)$ is compact if and only if the A-valued function $x(t) = \alpha_t(x), t \in R$ can be extended on the complex number C as an A-valued entire function such that there exist a positive constant γ and a non-negative integer N such that

$$\|x(z)\| \leq \gamma(1+|z|)^N e^{r|Imz|}, \qquad z \in C,$$

where r is the radius of a ball with center the origin which contains $Sp_\alpha(x)$.

References.

[Ar] W.B.Arveson, *On groups of automorphisms of operator algebras*, J. Functional Anal. **15** (1974), 217–247.

[Ma1] K.Matsumoto, *Periodic distributions on C^*-algebras, preprint.*

[Ma2] K.Matsumoto, *Distributions on operator algebras associated with R^n-actions (tentative title), in preparation.*

[Sch1] L.Schwartz, "Theorie des distributions," Hermann, Paris, 1956.

[Sch2] L.Schwartz, "Methodes mthematiques pour les sciences physiques," Hermann, Paris, 1961.

[Ru] W.Rudin, "Functional analysis," Mcgraw-Hill, 1973.