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Golden-Thompson Type Inequalities and Their Equality Cases

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In this paper we state some log-majorization results for matrices and their applications to matrix norm inequalities. The equality cases in these inequalities are characterized. Full details of Section 2 are presented in [2], [9].

1. Preliminaries

Let $\vec{a} = (a_1, \dots, a_n)$ and $\vec{b} = (b_1, \dots, b_n)$ be vectors in $\mathbf{R}^n$. The *weak majorization* (or the *submajorization*) $\vec{a} \prec_w \vec{b}$ means that

$$\sum_{i=1}^k a_i^* \le \sum_{i=1}^k b_i^*, \qquad 1 \le k \le n,$$

where $(a_1^*, \dots, a_n^*)$ and $(b_1^*, \dots, b_n^*)$ are the decreasing rearrangements of $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$, respectively. The *majorization* $\vec{a} \prec \vec{b}$ means that $\vec{a} \prec_w \vec{b}$ and the equality holds for $k = n$ in the above, i.e. $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$; in other words, $\vec{a}$ is a convex combination of the vectors obtained by permuting the components of $\vec{b}$ (see [1, Theorem 1.3]). When $\vec{a}, \vec{b} \ge 0$ (i.e. $a_i \ge 0$, $b_i \ge 0$ for $1 \le i \le n$), we define the *weak log-majorization* $\vec{a} \underset{(\log)}{\prec_w} \vec{b}$ if

$$\prod_{i=1}^k a_i^* \le \prod_{i=1}^k b_i^*, \qquad 1 \le k \le n,$$

and the *log-majorization* $\vec{a} \underset{(\log)}{\prec} \vec{b}$ if $\vec{a} \underset{(\log)}{\prec_w} \vec{b}$ and $\prod_{i=1}^n a_i = \prod_{i=1}^n b_i$. When $\vec{a}, \vec{b} > 0$, it is obvious that $\vec{a} \underset{(\log)}{\prec} \vec{b}$ [resp. $\vec{a} \underset{(\log)}{\prec_w} \vec{b}$] is equivalent to $\log \vec{a} \prec \log \vec{b}$ [resp. $\log \vec{a} \prec_w \log \vec{b}$].

In this paper we consider $n \times n$ complex matrices. For a Hermitian matrix H let $\vec{\lambda}(H) = (\lambda_1(H), \dots, \lambda_n(H))$ denote the vector of eigenvalues of H in decreasing order (with multiplicities). When H and K are Hermitian matrices, the *majorization* $H \prec K$ [resp. the *weak majorization* $H \prec_w K$] is defined as $\vec{\lambda}(H) \prec \vec{\lambda}(K)$ [resp. $\vec{\lambda}(H) \prec_w \vec{\lambda}(K)$]. We write $A \ge 0$ if a matrix A is positive semidefinite, and $A > 0$ if $A \ge 0$ is positive definite (or invertible). For $A, B \ge 0$ the *log-majorization* $A \underset{(\log)}{\prec} B$ is defined as $\vec{\lambda}(A) \underset{(\log)}{\prec} \vec{\lambda}(B)$. See [1], [13] for majorization theory for vectors and matrices. In particular, we remark that if $A, B \ge 0$ and $A \underset{(\log)}{\prec} B$, then $A \prec_w B$ and hence $\|A\| \le \|B\|$ for any unitarily invariant norm $\|\cdot\|$.

Let $||\cdot||$ be a unitarily invariant norm on $n \times n$ matrices. We say that $||\cdot||$ is *strictly increasing* if $0 \le A \le B$ and $||A|| = ||B||$ imply $A = B$. Let $\Phi : \mathbf{R}^n \to [0, \infty)$ be the symmetric gauge function (see [5], [14]) corresponding to $||\cdot||$, so that $||A|| = \Phi(\vec{\lambda}(A))$ for $A \ge 0$. Then it is easy to see that $||\cdot||$ is strictly increasing if and only if $0 \le \vec{a} \le \vec{b}$ and $\Phi(\vec{a}) = \Phi(\vec{b})$ imply $\vec{a} = \vec{b}$. For instance, the Schatten p-norms $||X||_p = (\sum_{i=1}^n \lambda_i(|X|)^p)^{1/p}$ are strictly increasing when $1 \le p < \infty$, while the Ky Fan norms $||X||_{(k)} = \sum_{i=1}^k \lambda_i(|X|)$ are not so when $1 \le k < n$. Note that $||A||_{(k)}$ for $A \ge 0$ is nothing but the kth partial trace $\mathrm{Tr}_k A = \sum_{i=1}^k \lambda_i(A)$.

2. Golden-Thompson type inequalities

For every $A, B \ge 0$ the log-majorization $(A^{1/2}BA^{1/2})^r \underset{(\log)}{\prec} A^{r/2}B^r A^{r/2}$ for $r \ge 1$ was proved by Araki [3], which is equivalent to say that

$$(A^{p/2}B^p A^{p/2})^{1/p} \underset{(\log)}{\prec} (A^{q/2}B^q A^{q/2})^{1/q}, \qquad 0 < p \le q. \tag{2.1}$$

This shows the following:

Proposition 2.1. *If $A, B \ge 0$ and $||\cdot||$ is a unitarily invariant norm, then $||(A^{p/2}B^p A^{p/2})^{1/p}||$ is an increasing function of $p > 0$.*

This implies norm inequalities of Golden-Thompson type. In fact, if H and K are Hermitian matrices, then

$$||e^{H+K}|| \le ||(e^{pH/2}e^{pK}e^{pH/2})^{1/p}||, \qquad p > 0,$$

for any unitarily invariant norm, and the above right-hand side decreases to the left-hand as $p \downarrow 0$. The above inequality in case of $p = 1$ was formerly given by Lenard [12] and Thompson [18]. Moreover the specialization to the trace norm is the famous Golden-Thompson trace inequality ([8], [17]).

The next theorem characterizes the equality case in the Golden-Thompson type inequality given by Proposition 2.1.

Theorem 2.2. *Let $A, B \geq 0$ and $||\cdot||$ be a strictly increasing unitarily invariant norm. Then the following conditions are equivalent:*

(i) $||(A^{p/2}B^pA^{p/2})^{1/p}||$ *is not strictly increasing in* $p > 0$*;*

(ii) $||(A^{p/2}B^pA^{p/2})^{1/p}||$ *is constant for* $p > 0$*;*

(iii) $(A^{p/2}B^pA^{p/2})^{1/p} = (A^{q/2}B^qB^{q/2})^{1/q}$ *for some* $0 < p < q$*;*

(iv) $(A^{p/2}B^pA^{p/2})^{1/p}$ *is constant for* $p > 0$*;*

(v) $AB = BA$.

Remark. In case of $A, B > 0$ Friedland and So [7, Theorem 3.1] characterized the situation when $\mathrm{Tr}_k(A^{p/2}B^pA^{p/2})^{1/p}$ is not stricly increasing in $p > 0$. This characterization is a bit complicated because of the non-strict increasingness of Tr_k.

Theorem 2.2 reads as follows when $A, B > 0$ and $||\cdot||$ is the trace norm. This corollary was already stated in [7]. The equivalence between (iii) and (iv) below determines the equality case in the original Golden-Thompson trace inequality. A proof of this equivalence is found also in [15].

Corollary 2.3. *Let H and K be Hermitian. Then the following conditions are equivalent:*

(i) $\mathrm{Tr}(e^{pH/2}e^{pK}e^{pH/2})^{1/p}$ *is not strictly increasing;*

(ii) $\mathrm{Tr}(e^{pH/2}e^{pK}e^{pH/2})^{1/p}$ *is constant;*

(iii) $\mathrm{Tr}\, e^He^K = \mathrm{Tr}\, e^{H+K}$*;*

(iv) $HK = KH$.

For $0 \leq \alpha \leq 1$ and $A, B > 0$ the *α-power mean* $A\#_\alpha B$ is defined by

$$A\#_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2},$$

which can be extended to $A, B \geq 0$ as

$$A\#_\alpha B = \lim_{\varepsilon \downarrow 0}(A + \varepsilon I)\#_\alpha(B + \varepsilon I).$$

This α-power mean is the operator mean (see [11]) corresponding to the operator monotone function t^α. In particular when $\alpha = 1/2$, $A\#_{1/2}B = A\#B$ is called the geometric mean. Moreover $A\#_0B = A$ and $A\#_1B = B$. For every $A, B \geq 0$ and $0 \leq \alpha \leq 1$ we proved in [2] that $(A\#_\alpha B)^r \underset{(\log)}{\succ} A^r\#_\alpha B^r$ holds for $r \geq 1$; or equivalently

$$(A^p\#_\alpha B^p)^{1/p} \underset{(\log)}{\succ} (A^q\#_\alpha B^q)^{1/q}, \qquad 0 < p \leq q.$$

So we have:

Proposition 2.4. *If $A, B \geq 0$, $0 \leq \alpha \leq 1$, and $||\cdot||$ is a unitarily invariant norm, then $||(A^p \#_\alpha B^p)^{1/p}||$ is a decreasing function of $p > 0$.*

Particularly when $A = e^H$ and $B = e^K$ with Hermitian matrices H, K and $||\cdot||$ is the trace norm, we have for $p, q > 0$

$$\mathrm{Tr}(e^{pH} \#_\alpha e^{pK})^{1/p} \leq \mathrm{Tr}\, e^{(1-\alpha)H+\alpha K} \leq \mathrm{Tr}(e^{(1-\alpha)qH/2} e^{\alpha qK} e^{(1-\alpha)qH/2})^{1/q}$$

(see [2, Corollary 2.3] and also [10, Theorem 3.4]). The above second inequality for $q = 1$ becomes the Golden-Thompson trace inequality, and it is fairly reasonable to consider the first inequality as complementary to the Golden-Thompson one. So the norm inequality given by Proposition 2.4 are considered as the complementary counterpart of the Golden-Thompson type one.

Let us here characterize, in parallel to Theorem 2.2, the situation when equality occurs in this inequality in case of $A, B > 0$.

Theorem 2.5. *Let $A, B > 0$, $0 < \alpha < 1$, and $||\cdot||$ be a strictly increasing unitarily invariant norm. Then the following conditions are equivalent:*

(i) $||(A^p \#_\alpha B^p)^{1/p}||$ *is not strictly decreasing in* $p > 0$;

(ii) $||(A^p \#_\alpha B^p)^{1/p}|| = ||\exp\{(1-\alpha)\log A + \alpha \log B\}||$ *for all* $p > 0$;

(iii) $(A^p \#_\alpha B^p)^{1/p} = (A^q \#_\alpha B^q)^{1/q}$ *for some* $0 < p < q$;

(iv) $(A^p \#_\alpha B^p)^{1/p} = \exp\{(1-\alpha)\log A + \alpha \log B\}$ *for all* $p > 0$;

(v) $AB = BA$.

Remark. In contrast with Theorem 2.2 we cannot extend Theorem 2.5 to $A, B \geq 0$; in fact, if P and Q are any orthoprojections and $0 < \alpha < 1$, then we have $(P^p \#_\alpha Q^p)^{1/p} = P \wedge Q$ (independently of $p > 0$) by [11, Theorem 3.7].

The following was shown in [2] (see also [10]) by differentiating $\mathrm{Tr}(e^{pH} \#_\alpha e^{pK})^{1/p}$ by α at $\alpha = 0$.

Proposition 2.6. *For every $A, B \geq 0$, $\frac{1}{p}\mathrm{Tr}\, A \log(A^{p/2} B^p A^{p/2})$ is an increasing function of $p > 0$ and decreases to $\mathrm{Tr}\, A(\log A + \log B)$ as $p \downarrow 0$.*

In the following we characterize the situation when equality occurs in the logarithmic trace inequality given by Proposition 2.6.

Theorem 2.7. *Let $A \geq 0$ and $B > 0$. Then the following conditions are equivalent:*

(i) $\frac{1}{p}\operatorname{Tr} A\log(A^{p/2}B^pA^{p/2})$ *is not strictly increasing in* $p > 0$*;*

(ii) $\frac{1}{p}\operatorname{Tr} A\log(A^{p/2}B^pA^{p/2}) = \operatorname{Tr} A(\log A + \log B)$ *for all* $p > 0$*;*

(iii) $AB = BA$*.*

Remark. When $A, B \geq 0$ (instead of $B > 0$), $\operatorname{Tr} A\log(A^{p/2}B^pA^{p/2})$ can be $-\infty$ for all $p > 0$, while of course Theorem 2.7 holds if the support projection of A is dominated by that of B.

Furthermore, we have for an arbitrary matrix T

$$|e^T| \underset{(\log)}{\prec} e^{\operatorname{Re} T} \leq e^{|\operatorname{Re} T|} \prec_w e^{|T|}, \tag{2.2}$$

where $|X| = (X^*X)^{1/2}$ and $\operatorname{Re} X = (X + X^*)/2$ for a matrix X. The log-majorization in (2.2) was proved by Cohen [6] (see also [2]), generalizing the trace inequality of Bernstein [4]. The latter in (2.2) follows from the well-known weak majorization $|\operatorname{Re} T| \prec_w |T|$ (see [13, p. 240, p. 244]) and the preservation of weak majorization under an increasing convex function (see [1, Corollary 2.2], [13, p. 116]). So we have:

Proposition 2.8. *If T is an arbitrary matrix and $||\cdot||$ is a unitarily invariant norm, then*

$$||e^T|| \leq ||e^{\operatorname{Re} T}|| \leq ||e^{|\operatorname{Re} T|}|| \leq ||e^{|T|}||.$$

The next theorem clarifies when the equality cases occur in the norm inequalities of Proposition 2.8.

Theorem 2.9. *Let T be a matrix and $||\cdot||$ be a strictly increasing unitarily invariant norm. Then:*

(1) $||e^T|| = ||e^{\operatorname{Re} T}||$ *if and only if T is normal.*

(2) $||e^{|\operatorname{Re} T|}|| = ||e^{|T|}||$ *if and only if T is Hermitian.*

(3) $||e^T|| = ||e^{|\operatorname{Re} T|}||$ *if and only if T is normal and* $\operatorname{Re} T \geq 0$*.*

(4) $||e^{\operatorname{Re} T}|| = ||e^{|T|}||$, $||e^T|| = ||e^{|T|}||$, *and $T \geq 0$ are all equivalent.*

Remarks. (1) When $||\cdot||$ is the Frobenius (or Hilbert-Schmidt) norm, Theorem 2.9(1) reads as follows: $\operatorname{Tr} e^{T^*}e^T = \operatorname{Tr} e^{T^*+T}$ if and only if T is normal. This was already proved in [15].

(2) It is well known (see [1, Theorem 6.7], [13, p. 240]) that $\lambda_k(\operatorname{Re} T) \leq \lambda_k(|T|)$ for $1 \leq k \leq n$. The equality case $\lambda_k(\operatorname{Re} T) = \lambda_k(|T|)$ for fixed k was characterized by So

and Thompson [16]. Further it was shown in [16] that $\vec{\lambda}(\mathrm{Re}\,T) = \vec{\lambda}(|T|)$, $\vec{\lambda}(|e^T|) = e^{\vec{\lambda}(|T|)}$, $\mathrm{Tr}|e^T| = \mathrm{Tr}\, e^{|T|}$, and $T \geq 0$ are all equivalent. Theorem 2.9 considerably refines this result.

3. Golden-Thompson type inequalities for three or four matrices

In this section we discuss norm inequalities of Golden-Thompson type for three or four matrices which are commuting except one. Also the equality cases are characterized.

Proposition 3.1. *If $A_1, A_2, B \geq 0$ and $A_1A_2 = A_2A_1$, then*

$$|A_1BA_2| \underset{(\log)}{\succ} (A_1A_2)^{1/2}B(A_1A_2)^{1/2} \sim B^{1/2}A_1A_2B^{1/2}, \tag{3.1}$$

where $\sim$ dentes the unitary equivalence.

Proof. By a technique of compound matrices used in [2], it suffices to show that $|A_1BA_2| \leq I$ implies $(A_1A_2)^{1/2}B(A_1A_2)^{1/2} \leq I$. We may assume $A_2 > 0$. Then since $A_2BA_1^2BA_2 \leq I$, we get $BA_1^2B \leq A_2^{-2}$ and so $(A_1BA_1)^2 \leq (A_1A_2^{-1})^2$, which implies $A_1BA_1 \leq A_1A_2^{-1}$. Hence $(A_1A_2)^{1/2}B(A_1A_2)^{1/2} \leq I$ and the first part is proved. The second part is obvious. □

By the log-majorization (2.1), the above (3.1) further implies that

$$|A_1BA_1| \underset{(\log)}{\succ} (B^{p/2}(A_1A_2)^pB^{p/2})^{1/p}, \qquad 0 < p \leq 1.$$

Corollary 3.2. *Let $A_1, A_2 \geq 0$ with $A_1A_2 = A_2A_1$, and $||\cdot||$ be a unitarily invariant norm. If $B \geq 0$ then*

$$||A_1BA_2|| \geq ||B^{1/2}A_1A_2B^{1/2}||. \tag{3.2}$$

Moreover for any B

$$||A_1B^*BA_2|| \geq ||BA_1A_2B^*||. \tag{3.3}$$

Proof. (3.2) is a consequence of (3.1). When B is replaced by B^*B in (3.1), we have

$$|A_1B^*BA_2| \underset{(\log)}{\succ} (A_1A_2)^{1/2}B^*B(A_1A_2)^{1/2} \sim BA_1A_2B^*,$$

showing (3.3). □

Proposition 3.3. *If $A_1, A_2, A_3, B \geq 0$ and $A_iA_j = A_jA_i$, then*

$$|A_1BA_2BA_3| \underset{(\log)}{\succ} (B^{1/2}(A_1A_2A_3)^{1/2}B^{1/2})^2.$$

Proof. We have

$$\begin{aligned} |A_1BA_2BA_3| &\underset{(\log)}{\succ} (A_1A_3)^{1/2}BA_2B(A_1A_3)^{1/2} \\ &= |A_2^{1/2}B(A_1A_3)^{1/2}|^2 \\ &\underset{(\log)}{\succ} (B^{1/2}(A_1A_2A_3)^{1/2}B^{1/2})^2, \end{aligned}$$

using (3.1) twice.

The following corollary is a generalization of the Golden-Thompson inequality.

Corollary 3.4. *If H_1, H_2, H_3, K are Hermitian and $H_iH_j = H_jH_i$, then*

$$||e^{H_1}e^{K}e^{H_2}|| \geq ||e^{H_1+H_2+K}|| \tag{3.4}$$

and

$$||e^{H_1}e^{K}e^{H_2}e^{K}e^{H_3}|| \geq ||e^{H_1+H_2+H_3+2K}|| \tag{3.5}$$

for any unitarily invariant norm.

Proof. Propositions 3.1 and 3.3 together with (2.1) imply that

$$|e^{H_1}e^{K}e^{H_2}| \underset{(\log)}{\succ} (e^{pK/2}e^{p(H_1+H_2)}e^{pK/2})^{1/p}, \qquad 0 < p \leq 1,$$

$$|e^{H_1}e^{K}e^{H_2}e^{K}e^{H_3}| \underset{(\log)}{\succ} (e^{pK}e^{p(H_1+H_2+H_3)}e^{pK})^{1/p}, \qquad 0 < p \leq 1/2.$$

Taking the limits of the right-hand sides as $p \downarrow 0$, we have

$$|e^{H_1}e^{K}e^{H_2}| \underset{(\log)}{\succ} e^{H_1+H_2+K},$$

$$|e^{H_1}e^{K}e^{H_2}e^{K}e^{H_3}| \underset{(\log)}{\succ} e^{H_1+H_2+H_3+2K},$$

showing (3.4) and (3.5). □

Question. If $H_1, \ldots, H_n, K$ are Hermitian and $H_iH_j = H_jH_i$, then

$$|e^{H_1}e^{K}e^{H_2}\cdots e^{K}e^{H_n}| \underset{(\log)}{\succ} e^{H_1+\cdots+H_n+(n-1)K} \ ?$$

In the sequel let us characterize the equality cases in the norm inequalities (3.2), (3.4), and (3.5). First note [9, Lemma 2.2] that if $A, B \geq 0$ and $||\cdot||$ is a stricly increasing unitarily invariant norm, then $A \underset{(\log)}{\succ} B$ and $||A|| = ||B||$ imply $A \sim B$.

For commuting $A_1, A_2 \geq 0$, let Q be the join of the support projections of A_1, A_2. Then both sides of (3.2) are determined by QBQ; in fact we have

$$||A_1BA_2|| = ||A_1QBQA_2||,$$

$$||B^{1/2}A_1A_2B^{1/2}|| = ||(A_1A_2)^{1/2}B(A_1A_2)^{1/2}|| = ||(A_1A_2)^{1/2}QBQ(A_1A_2)^{1/2}||.$$

So to characterize the equality case of (3.2), we may assume without loss of generality that $Q = I$, i.e. $A_1 + A_2 > 0$.

Theorem 3.5. *Let $A_1, A_2, B \geq 0$ with $A_1A_2 = A_2A_1$ and $A_1 + A_2 > 0$, and P be the support projection of A_1. Assume that $||\cdot||$ is a strictly increasing unitarily invariant norm. Then $||A_1BA_2|| = ||B^{1/2}A_1A_2B^{1/2}||$ if and only if B commutes with P and $PA_1^{-1}A_2$.*

Proof. Suppose that B commute with P and $PA_1^{-1}A_2$. Let $PA_1^{-1}A_2 = \sum_{k=1}^{m} \alpha_k P_k$ be the spectral decomposition of $PA_1^{-1}A_2$, where $P = \sum_{k=1}^{m} P_k$ and α_k are all distinct. Then A_1, A_2, B commute with all P and P_k, $1 \leq k \leq m$. Since $(I - P)A_1 = 0$, we get

$$(I-P)|A_1BA_2|^2 = A_2B(I-P)A_1^2BA_2 = 0,$$

so that

$$(I-P)|A_1BA_2| = 0 = (I-P)B^{1/2}A_1A_2B^{1/2}.$$

For $1 \leq k \leq m$, since $P_kA_2 = \alpha_kP_kA_1$, we get

$$\begin{aligned} P_k|A_1BA_2| &= \alpha_kP_kA_1BA_1 \\ &\sim \alpha_kP_kB^{1/2}A_1^2B^{1/2} \\ &= P_kB^{1/2}A_1A_2B^{1/2}. \end{aligned}$$

Therefore $|A_1BA_2| \sim B^{1/2}A_1A_2B^{1/2}$, which implies $||A_1BA_2|| = ||B^{1/2}A_1A_2B^{1/2}||$.

Conversely suppose $||A_1BA_2|| = ||B^{1/2}A_1A_2B^{1/2}||$. It follows from (3.1) that $|A_1BA_2| \sim B^{1/2}A_1A_2B^{1/2}$ and hence

$$\operatorname{Tr} A_1BA_2^2BA_1 = \operatorname{Tr}(B^{1/2}A_1A_2B^{1/2})^2 = \operatorname{Tr} A_1A_2BA_1A_2B. \tag{3.6}$$

Now we may assume that $A_1 = \mathrm{diag}(s_1, \dots, s_n)$ and $A_2 = \mathrm{diag}(t_1, \dots, t_n)$. Let $B = [b_{ij}]$. Then

$$\mathrm{Tr}\, A_1 B A_2^2 B A_1 = \sum_{i,j=1}^{n} s_i^2 t_j^2 |b_{ij}|^2, \tag{3.7}$$

$$\mathrm{Tr}\, A_1 A_2 B A_1 A_2 B = \sum_{i,j=1}^{n} s_i s_j t_i t_j |b_{ij}|^2. \tag{3.8}$$

By (3.6)–(3.8) we get

$$\sum_{i,j=1}^{n} (s_i t_j - s_j t_i)^2 |b_{ij}|^2 = 0,$$

so that $b_{ij} = 0$ unless $s_i t_j = s_j t_i$. If $s_i = 0$ and $s_j > 0$, then $t_i > 0$ due to $A_1 + A_2 > 0$, so $b_{ij} = 0$. Hence $BP = PB$. If $s_i, s_j > 0$ and $t_i/s_i \neq t_j/s_j$, then $b_{ij} = 0$. This implies that B commutes with $PA_1^{-1}A_2$. □

Theorem 3.6. *Let H_1, H_2, H_3, K be Hermitian with $H_iH_j = H_jH_i$. Assume that $\|\cdot\|$ is a strictly increasing unitarily invariant norm. Then:*

(1) $\|e^{H_1}e^Ke^{H_2}\| = \|e^{H_1+H_2+K}\|$ if and only if K commutes with H_1, H_2.

(2) $\|e^{H_1}e^Ke^{H_2}e^Ke^{H_3}\| = \|e^{H_1+H_2+H_3+K}\|$ if and only if K commutes with H_1, H_2, H_3.

Proof. We show "only if" parts (the converse parts are obvious).

(1) Suppose $\|e^{H_1}e^Ke^{H_2}\| = \|e^{H_1+H_2+K}\|$. Since

$$|e^{H_1}e^Ke^{H_2}| \underset{(\log)}{\succ} e^{K/2}e^{H_1+H_2}e^{K/2} \underset{(\log)}{\succ} e^{H_1+H_2+K},$$

we get

$$|e^{H_1}e^Ke^{H_2}| \sim e^{K/2}e^{H_1+H_2}e^{K/2} \sim e^{H_1+H_2+K}.$$

By Theorem 3.5, the first equivalence implies that e^K commutes with $e^{H_2-H_1}$, i.e. $K(H_2 - H_1) = (H_2 - H_1)K$. The second implies the equality case of the Golden-Thompson inequality, so $K(H_1+H_2) = (H_1+H_2)K$ by Corollary 2.3. Hence K commutes with H_1, H_2.

(2) Suppose $\|e^{H_1}e^Ke^{H_2}e^Ke^{H_3}\| = \|e^{H_1+H_2+H_3+2K}\|$. Since

$$\begin{aligned} |e^{H_1}e^Ke^{H_2}e^Ke^{H_3}| &\underset{(\log)}{\succ} e^{(H_1+H_3)/2}e^Ke^{H_2}e^Ke^{(H_1+H_3)/2} \\ &= |e^{H_2/2}e^Ke^{(H_1+H_3)/2}| \\ &\underset{(\log)}{\succ} e^{H_1+H_2+H_3+2K}, \end{aligned}$$

these terms are all unitarily equivalent. By Theorem 3.5, $e^K e^{H_2} e^K$ commutes with $e^{H_3 - H_1}$. Furthermore by (1), K commutes with H_2 and $H_1 + H_3$. Hence K commutes with H_1, H_2, H_3. □

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