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HAUSDORFF DIMENSION AND FOURIER COEFFICIENTS

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§1. Introduction

The Weierstrass function $W_{a,b}(x) = \sum a^n \cos(b^n \pi x)$ and the Takagi function $T(x) = \sum \frac{1}{2^n}\psi(2^n x)$ (with $\psi(x) = 1 - |2x - 2[x] - 1|$) are well known as nowhere differentiable continuous functions. The graphs of such a function may be fractal sets. A. S. Besicovitch and H. D. Ursell [2] studied the Hausdorff dimension of the graph of the function $f(x) = \sum b_n^{-\delta}\psi(b_n x)$ and determined the Hausdorff dimension in some special cases. The Hausdorff dimension of Weierstrass-type curves have been studied by M.V. Breek and Z.V. Lewis [1] and R.D. Mualdin and S.C. Williams [5]. M. Hata and M. Yamaguchi [3] investigated the generalized Takagi function $f(x) = \sum a_n \psi(2^{n-1}x)$, which is called a Takagi series. The relation between the coefficients $\{a_n\}$ and non-differentiability of this Takagi series has been studied by N. Kono [4]. While the upper bound of Hausdorff dimension of their graphs is known, the exact Hausdorff dimension has not yet been obtained with few exceptions [2]. On the other hand, there have been introduced various dimensions of fractal sets, such as packing dimension, box-counting dimension and so on. The following relation holds among Hausdorff dimension $\dim_H K$, lower box-counting dimension $\underline{\dim}_B K$ and upper box-counting dimension $\overline{\dim}_B K$: $\dim_H K \leq \underline{\dim}_B K \leq \overline{\dim}_B K$.

In this paper we shall consider a generalized Takagi series, i.e. $f(x) = \sum_{n=1}^{\infty} a_n \varphi(2^{n-1}x)$, where φ satisfies some properties (2.1)–(2.4). We shall call this function a super Takagi function. Some of Weierstrass functions are super Takagi functions. We investigate the lower and upper box-counting dimensions of their graphs and show that lower and upper box counting dimensions of the graph of a super Takagi function depend only on the coefficients $\{a_n\}$ but not on φ. In Theorem 1 we give the upper box-counting dimension of its graph. In general, lower and upper box-counting dimensions are not the same. We shall show a condition where lower and upper box-counting dimensions of the graph are the same and hence its box-counting dimension is obtained (Theorem 2). We also show a condition where the upper bound of lower box-counting dimension is strictly smaller than upper box-counting dimension (Theorem 3). For a Takagi series, we obtain the exact lower box-counting dimension of its graph, which is different from its upper box-counting dimension (Theorem 4). This is an extension of a result [2, 3.IV] of A. S. Besicovitch and H. D. Ursell.

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In §4, we consider the operator $T_{\underline{a}}$ depending on the coefficients $\underline{a} = \{a_n\}$, since the box-counting dimension of the graph of a super Takagi function depends only on the coefficients. Since a function f is an infinite series $\sum_{n=1}^{\infty} a_n\varphi(2^{n-1}x)$, its Hausdorff dimension may be greater than 1, while if we take finite sum $f_n = \sum_{k=1}^{n} a_k\varphi(2^{k-1}x)$, its Hausdorff dimension is 1. Though f_n converges to f uniformly, $\dim_H G_{f_n}$ does not converge to $\dim_H G_f$. So we shall find some functional which has much relation with box-counting dimension and which is continuous as $f_n \to f$. The Hölder exponent has much relation with Hausdorff dimension. We examine the property of Hölder exponents (Proposition 3) and show that local Hölder exponent has the desired property (Proposition 4). Theorem 5 shows that local Hölder exponent of f_n converges to that of f as $n \to \infty$. For $\varphi(x) = e^{2\pi ix}$, the above $\{a_n\}$ are Fourier coefficients of f. So Fourier coefficients have much relation with box-counting dimension and also with Hausdorff dimension.

§2. Super Takagi Functions

Let E be the set of continuous functions φ defined on $\mathbf{R}$ satisfying the following conditions (2.1) – (2.4):

(2.1) $\varphi(x) = \varphi(x+1) = \varphi(1-x)$

(2.2) $\varphi(x) + \varphi(1/2 - x) = 1$

(2.3) $\varphi(0) = 0$

(2.4) φ is uniformly Lipshitz continuous on $\mathbf{R}$, i.e., there exists $K > 0$ such that

$$|\varphi(x) - \varphi(y)| \le K|x-y| \qquad (\text{for } x, y \in \mathbf{R}).$$

M. Hata and M. Yamaguti generalized the Takagi function as follows:

$$f(x) = \sum_{n=1}^{\infty} a_n\psi(2^{n-1}x),$$

where $\{a_n\} \in l_1$, $\psi(x) = 1 - |2x - 2[x] - 1|$ and $[x]$ is the Gauss number of x. They called the generalized Takagi function a Takagi series. We shall generalize the Takagi series as follows:

$$f(x) = \sum_{n=1}^{\infty} a_n\varphi(2^{n-1}x),$$

where $\varphi \in E$ and $\{a_n\} \in l_1$.

We shall call this generalized function a super Takagi function. Some of Weierstrass functions are super Takagi functions.

In the study of linear difference equations, M. Hata and M. Yamaguti [3] noticed that a Takagi series f has the following relation

(2.5)
$$f\left(\frac{2i-1}{2^k}\right) - \frac{1}{2}\left\{f\left(\frac{i}{2^{k-1}}\right) + f\left(\frac{i-1}{2^{k-1}}\right)\right\} = a_k$$
$$(1 \le i \le 2^{k-1}, k = 1, 2, \cdots).$$

Super Takagi functions do not necessarily satisfy the relation (2.5), but satisfy the following relation (2.6) as shown in the following lemma.

Lemma 1. *For a super Takagi function* $f(x) = \sum_{n=1}^{\infty} a_n \varphi(2^{n-1}x)$,

$$\sum_{j=1}^{2^{n-1}} \{2f(\frac{2j-1}{2^n}) - f(\frac{j}{2^{n-1}}) - f(\frac{j-1}{2^{n-1}})\} = 2^n a_n. \tag{2.6}$$

Proof. Since $\sum_{j=1}^{2^{n-1}} \{2\varphi(\frac{2j-1}{2^n}) - \varphi(\frac{j}{2^{n-1}}) - \varphi(\frac{j-1}{2^{n-1}})\} = 0 \ \ (n \geq 2)$ and $2\varphi(\frac{1}{2}) - \varphi(0) - \varphi(1) = 2$ holds, we have

$$\begin{aligned}
&\sum_{j=1}^{2^{n-1}} \{2f(\frac{2j-1}{2^n}) - f(\frac{j}{2^{n-1}}) - f(\frac{j-1}{2^{n-1}})\} \\
&= \sum_{k=1}^{n} a_k 2^{k-1} \sum_{j=1}^{2^{n-k}} \{2\varphi(\frac{2j-1}{2^{n-k+1}}) - \varphi(\frac{j}{2^{n-k}}) - \varphi(\frac{j-1}{2^{n-k}})\} \\
&= 2^n a_n.
\end{aligned}$$

□

§3. Box-counting dimension

Let F be a non-empty bounded subset of $[0,1] \times \mathbf{R}$. Consider the collection of squares in the 2^{-n}-coordinate mesh of $[0,1] \times \mathbf{R}$, i.e. squares of the form

$$[\frac{m_1}{2^n}, \frac{m_1+1}{2^n}] \times [\frac{m_2}{2^n}, \frac{m_2+1}{2^n}]$$

where m_1 and m_2 are integers. Let $N_n(F)$ be the numbers of such 2^{-n}-squares that intersect F. Then the lower and upper box-counting dimensions of F respectively are defined as

$$\underline{\dim}_B F = \underline{\lim}_{n} \frac{\log N_n(F)}{\log 2^n} \qquad \text{and} \qquad \overline{\dim}_B F = \overline{\lim}_{n} \frac{\log N_n(F)}{\log 2^n}.$$

If they are equal, their common value is called the box-counting dimension $\dim_B F$ of F. Let G_f be the graph of f on [0,1], i.e.

$$G_f = \{(x,y) |\ 0 \leq x \leq 1,\ y = f(x)\}.$$

Then the following shows an estimate of $N_n(G_f)$.

Lemma 2.

$$2^n(\max_{1\le k\le n}(2^k|a_k|)\vee 1)\le N_n(G_f)\le 2^n(2+K\sum_{k=1}^{n}2^k|a_k|+\|\varphi\|_\infty 2^{n+1}\sum_{k=n+1}^{\infty}|a_k|),$$

where $\|\varphi\|_\infty=\sup\{|\varphi(x)|:\ x\in\mathbf{R}\}$.

Proof. The number of 2^{-n}-squares that intersect $G_f\cap[\frac{j}{2^n},\frac{j+1}{2^n}]\times\mathbf{R}$ is more than $2^n|f(\frac{j+1}{2^n})-f(\frac{j}{2^n})|\vee 1$. So

$$N_n(G_f)\ge\sum_{j=1}^{2^n}(2^n|f(\frac{j}{2^n})-f(\frac{j-1}{2^n})|\vee 1)$$

$$\ge 2^n\max_{1\le k\le n}\{|\sum_{j=1}^{2^{k-1}}(2f(\frac{2j-1}{2^k})-f(\frac{2j}{2^k})-f(\frac{2j-2}{2^k}))|\vee 1\}.$$

By using lemma 1, we have

$$N_n(G_f)\ge 2^n(\max_{1\le k\le n}2^k|a_k|\vee 1).$$

On the other hand, by virtue of the relation

$$|f(x+h)-f(x)|\le K\sum_{k=1}^{n}2^{k-1}|a_k||h|+2\|\varphi\|_\infty\sum_{k=n+1}^{\infty}|a_k|$$

we have

$$N_n(G_f)\le 2^n(2+K\sum_{k=1}^{n}2^{k-1}|a_k|+\|\varphi\|_\infty 2^{n+1}\sum_{k=n+1}^{\infty}|a_k|).\qquad\square$$

By using Lemma 2, a formula for $\overline{\dim}_B G_f$ is obtained, which shows that $\overline{\dim}_B G_f$ depends only on $\{a_n\}$ but not on φ.

Theorem 1. *Let* $f(x)=\sum_{n=1}^{\infty}a_n\varphi(2^{n-1}x)$ *be a super Takagi function. Then*

$$\overline{\dim}_B G_f=1\vee(2+\overline{\lim_n}\log_2|a_n|^{1/n}).$$

Proof. By using lemma 2, we get for any $\varepsilon>0$

$$N_n(G_f)\le 2^n\{M_1+M_2(2\,\overline{\lim_n}|a_n|^{1/n}+\varepsilon)^{n+1}\}$$

for sufficiently large n, where M_1 and M_2 are positive numbers which are independent of n. So we get the conclusion. $\square$

As for the lower box-counting dimension, it is not so easy to get its value. We shall show the region of its value.

Proposition 1. *For a super Takagi function* $f(x) = \sum_{n=1}^{\infty} a_n \varphi(2^{n-1}x)$, *we have*

$$1 + (0 \vee \underline{\lim}_n \max_{1\le k\le n} \log_2(2^k|a_k|)^{1/n}) \le \underline{\dim}_B G_f$$
$$\le 1 + \{0 \vee \underline{\lim}_n \log_2(\sum_{k=1}^{n} 2^k|a_k| + 2^n \sum_{k=n+1}^{\infty} |a_k|)^{1/n}.$$

Proof. By using Lemma 2 and the relation

$$\underline{\lim}_n \frac{\log(2 + K\sum_{k=1}^{n} 2^k|a_k| + \|\varphi\|_\infty 2^{n+1}\sum_{k=n+1}^{\infty}|a_k|)}{\log 2^n}$$
$$= 0 \vee \underline{\lim}_n \log_2(\sum_{k=1}^{n} 2^k|a_k| + 2^n \sum_{k=n+1}^{\infty} |a_k|)^{1/n},$$

we get the conclusion. □

In general, the upper and lower box-counting dimensions are not the same. The following shows a case where the box-counting dimension exists.

Theorem 2. *Let* $f(x) = \sum_{n=1}^{\infty} a_n \varphi(2^{n-1}x)$ *be a super Takagi function. If there exists a subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying*

$$\overline{\lim}_n |a_n|^{1/n} = \lim_k |a_{n_k}|^{1/n_k} = \gamma \quad \text{and} \quad \lim_k \frac{n_{k+1}}{n_k} = 1$$

then

$$\dim_B G_f = \begin{cases} 2 + \log_2 \gamma & (1/2 < \gamma < 1) \\ 1 & (0 < \gamma \le 1/2). \end{cases}$$

Proof. By using Proposition 1 and the relation

$$\underline{\lim}_n \max_{1\le j\le n} \log_2(2^j|a_j|)^{1/n} \ge \underline{\lim}_k \inf_{0\le l< n_{k+1}-n_k} \log_2(2^{n_k}|a_{n_k}|)^{\frac{1}{n_k+l}}$$
$$\ge 1 + \log_2 \gamma$$

for $\gamma > \frac{1}{2}$, we get the conclusion. □

Remark. *If* $a_n = t^n$ *with* $1/2 < |t| < 1$, *that is,* $f(x) = \sum_{n=1}^{\infty} t^n \varphi(2^{n-1}x)$, *then* $\dim_B G_f = 2 + \log_2 |t|$.

Accordingly, the box-counting dimension $\dim_B G_f$ *of the Takagi function* $f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n}\psi(2^{n-1}x)$ *is 1.*

The following gives a case where upper and lower box-counting dimensions are different.

Theorem 3. *For a super Takagi function $f(x) = \sum_{n=1}^{\infty} a_n \varphi(2^{n-1}x)$, let $\{a_{n_k}\}$ and $\{a_{m_k}\}$ be disjoint subsequences of $\{a_n\}$ such that $\{a_{n_k}\} \cup \{a_{m_k}\} = \{a_n\}$. Put $\gamma = \overline{\lim}_n |a_n|^{1/n}$, $d = \overline{\lim}_k |a_{m_k}|^{1/m_k}$ and $\mu = \overline{\lim}_k n_{k+1}/n_k$.*

If $\lim_k |a_{n_k}|^{1/n_k} = \gamma$, $d < 1/2 < \gamma < 1$ and $\mu > 1$, then

$$\underline{\dim}_B G_f \leq 2 - \frac{\mu\delta}{1 + (\mu - 1)\delta} \quad (< 2 - \delta = \overline{\dim}_B G_f), \tag{3.1}$$

where $\delta = -\log_2 \gamma$.

Proof. Let $\{n_{k_j}\}$ be a subsequence of $\{n_k\}$ such that $\lim_{j\to\infty} n_{k_j+1}/n_{k_j} = \mu$. For any ε $(0 < \varepsilon < \min\{\mu - 1, 1 - \gamma, 1/2 - d\})$ and for sufficiently large j, put $l_j = [(\mu - \varepsilon - 1)\delta n_{k_j}]$. Then by using $d < 1/2 < \gamma$, we have

$$n_{k_j} + l_j \leq n_{k_j}(1 + (\mu - \varepsilon - 1)\delta) < n_{k_j+1} \tag{3.2}$$

and

$$\begin{aligned}&\sum_{k=1}^{n_{k_j}+l_j} 2^k |a_k| + 2^{n_{k_j}+l_j} \sum_{k=n_{k_j}+l_j+1}^{\infty} |a_k| \\ &\leq \; M_1 + M_2(2\gamma + 2\varepsilon)^{n_{k_j}} + M_3 2^{n_{k_j}+l_j}(\gamma + \varepsilon)^{n_{k_j+1}},\end{aligned}$$

where M_1, M_2 and M_3 are independent of j. So by using Proposition 1 and (3.2), we have

$$\begin{aligned}\underline{\dim}_B G_f &\leq 1 + \underline{\lim}_j \log_2\Big(\sum_{k=1}^{n_{k_j}+l_j} 2^k |a_k| + 2^{n_{k_j}+l_j} \sum_{k=n_{k_j}+l_j+1}^{\infty} |a_k|\Big)^{\frac{1}{n_{k_j}+l_j}} \\ &\leq 1 + \frac{1 - \delta + \varepsilon\mu}{1 + (\mu - \varepsilon - 1)\delta}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitraly, we have

$$\underline{\dim}_B G_f \leq 1 + \frac{1 - \delta}{1 + (\mu - 1)\delta} = 2 - \frac{\mu\delta}{1 + (\mu - 1)\delta}. \qquad \square$$

The inequality in (3.1) can be replaced by equality in a special case of Takagi series as follows. In order to show this equality, we will show a lemma concerning the lower estimate of $N_n(G_f)$, which is more precise than in Lemma 2 in a special case of Takagi series.

Lemma 3. *Let f be a Takagi series, i.e.*

$$f(x) = \sum_{n=1}^{\infty} a_n \psi(2^{n-1}x) \qquad \text{where} \quad \psi(x) = 1 - |2x - 2[x] - 1|.$$

Let $\{a_{n_k}\}$ and $\{a_{m_k}\}$ be disjoint subsequences of $\{a_n\}$ such that $\{a_{n_k}\} \cup \{a_{m_k}\} = \{a_n\}$. If

$$\overline{\lim_n} |a_n|^{1/n} = \lim_k |a_{n_k}|^{1/n_k} = \gamma \qquad \text{and} \qquad \overline{\lim_n} |a_{m_k}|^{1/m_k} = d < \gamma$$

are satisfied, then for any $\tilde{\gamma}$ ($\frac{d+\gamma}{2} < \tilde{\gamma} < \gamma$) and $\tilde{d}$ ($\frac{d+\gamma}{2} > \tilde{d} > d$),

$$N_{n_k+l}(G_f) \geq 2^{2n_k+l-1}\tilde{\gamma}^{n_k}\{1 + 2^l \tilde{\gamma}^{n_{k+1}-n_k}(1 - (n_{k+1} - n_k)\frac{\tilde{d}^{n_k+1}}{\tilde{\gamma}^{n_{k+1}}})\}$$

for sufficiently large k and $1 \leq l \leq n_{k+1} - n_k$.

Proof. Put

$$M_{j_1 \cdot j_2} = \sup\{|f(x) - f(y)| : \frac{j_1 - 1}{2^{n_k}} + \frac{j_2 - 1}{2^{n_k+l}} \leq x, y \leq \frac{j_1 - 1}{2^{n_k}} + \frac{j_2}{2^{n_k+l}}\}$$

for $1 \leq j_1 \leq 2^{n_k}$ and $1 \leq j_2 \leq 2^l$. Then

$$\begin{aligned} M_{j_1, j_2} \geq & |f_{n_k}(\frac{j_1}{2^{n_k}}) - f_{n_k}(\frac{j_1 - 1}{2^{n_k}})| \ \frac{2^{-(n_k+l)} - 2^{-n_{k+1}}}{2^{-n_k}} + \tilde{\gamma}^{n_{k+1}} \\ & - 2(n_{k+1} - n_k)\tilde{d}^{n_k+1} \end{aligned}$$

holds for sufficiently large k. By using Lemma 1, we have

$N_{n_k+l}(G_f) \geq \sum_{j_1=1}^{2^{n_k}} \sum_{j_2=1}^{2^l} M_{j_1 \cdot j_2}/2^{-(n_k+l)}$

$\geq \{(1 - 2^{-n_{k+1}+n_k+l})2^{n_k}|a_{n_k}| + 2^{n_k+l}(\tilde{\gamma}^{n_{k+1}} - 2(n_{k+1} - n_k)\tilde{d}^{n_k+1})\}2^{n_k+l}$

By relations $n_{k+1} - n_k - l \geq 1$, $\tilde{d}/\tilde{\gamma} < 1$ and $|a_{n_k}| \geq \tilde{\gamma}^{n_k}$, we get the conclusion. □

Theorem 4. *Let f be a Takagi series and let $\{a_{n_k}\}$ and $\{a_{m_k}\}$ be disjoint subsequences of $\{a_n\}$ such that $\{a_{n_k}\} \cup \{a_{m_k}\} = \{a_n\}$. Put*

$$\gamma = \overline{\lim_n} |a_n|^{1/n} \qquad \text{and} \qquad d = \overline{\lim_k} |a_{m_k}|^{1/m_k}.$$

If $\lim_k \frac{n_{k+1}}{n_k} = \mu > 1$, $\gamma > \frac{1}{2}$ *and* $d < \min\{\frac{1}{2}, \gamma^\mu\}$, *then*

$$\underline{\dim}_B G_f = 2 - \frac{\mu\delta}{1 + (\mu - 1)\delta},$$

where $\delta = -\log_2 \gamma$.

Proof. By using Lemma 3 and the relation $\lim_k (n_{k+1} - n_k) d^{n_k} / \gamma^{n_{k+1}} = 0$ under the assumption, we have

$$\underline{\dim}_B G_f \geq \underline{\lim}_k \inf_{0 \leq l < n_{k+1} - n_k} \frac{2n_k + l + \log_2 \gamma^{n_k}(1 + 2^l \gamma^{n_{k+1} - n_k})}{n_k + l}.$$

Let $\tilde{l}$ and $\bar{d}$ satisfy the relation $n_k + l = n_k(1 + \tilde{l})$ and $d = 2^{-\bar{d}}$. Then $0 \leq \tilde{l} < \mu + \varepsilon - 1, \bar{d} > \delta\mu$ and

$$\begin{aligned} \underline{\dim}_B G_f &\geq \underline{\lim}_k \inf_l \frac{n_k(2 + \tilde{l} - \delta) + \log_2(1 + 2^{(\tilde{l} - \delta(\mu - 1))n_k})}{n_k(1 + \tilde{l})} \\ &= \begin{cases} 1 + \frac{1-\delta}{1+\tilde{l}} & \tilde{l} \leq \delta(\mu - 1) \\ 2 - \frac{\mu\delta}{1+\tilde{l}} & \tilde{l} > \delta(\mu - 1), \end{cases} \end{aligned}$$

which implies $\underline{\dim}_B G_f \geq 2 - \frac{\mu\delta}{1+(\mu-1)\delta}$. Since the opposite inequality is obtained in Theorem 3, we have $\underline{\dim}_B G_f = 2 - \frac{\mu\delta}{1+(\mu-1)\delta}$. □

Remark. *In [2] A.S. Besicovitch and H.D. Ursell gave the Hausdorff dimension* $\dim_H G_f$ *of the function* $f(x) = \sum_{n=1}^\infty b_n^{-\delta} \psi(b_n x)$ *as follows:*

(*) *If* $b_{n+1} = b_n^\mu$ *and* $b_1 > 1$, *then* $\dim_H G_f = 2 - \frac{\mu\delta}{1+(\mu-1)\delta}$.

If $b_n = 2^{t_n}$ *with* $t_n \in \mathbf{N}$ $(\forall n \in \mathbf{N})$, *then the Besicovitch's result (*) is obtained as a special case of Theorem 4, by putting* $a_k = 2^{-\delta t_n}$ *for* $k = t_n$ *with some* $n \in \mathbf{N}$ *and* $a_k = 0$ *for* $k \in \mathbf{N} \setminus \cup_{n=1}^\infty \{t_n\}$.

§4. Hölder exponent

From the result in §3, we see that the box-counting dimension $\dim_H G_f$ for a super Takagi function $f(x) = \sum_{n=1}^\infty a_n \varphi(2^{n-1}x)$ depends only on $\{a_n\}$ but not on $\varphi \in E$. So we shall consider the relation between the box counting dimension and the property of an operator. Let an operator S on $C_b(\mathbf{R})$ be defined by

$$Sg(x) = g(2x) \qquad \text{for} \qquad g \in C_b(\mathbf{R}) \text{ and } x \in \mathbf{R}.$$

For $\underline{a} = \{a_n\} \in l_1$, define $T_{\underline{a},n}$, and $T_{\underline{a}}$ by

$$T_{\underline{a},n} = \sum_{k=1}^{n} a_k S^{k-1} \qquad \text{and} \qquad T_{\underline{a}} = \sum_{k=1}^{\infty} a_k S^{k-1}.$$

Then the function $f(x) = \sum_{n=1}^{\infty} a_n \varphi(2^{n-1}x)$ in §3 is $T_{\underline{a}}\varphi$. The Hausdorff dimension and box-counting dimension of the graph $G_{T_{\underline{a},n}\varphi}$ of $T_{\underline{a},n}\varphi$ is 1 for any $\underline{a} \in l_1$ and $\varphi \in E$, while those of $G_{T_{\underline{a}}\varphi}$ become greater than 1 for some $\underline{a} \in l_1$. Though $T_{\underline{a},n}\varphi$ converges to $T_{\underline{a}}\varphi$ as $n \to \infty$ in $C_b(\mathbf{R})$, these dimensions of $G_{T_{\underline{a},n}\varphi}$ do not converge to those of $G_{T_{\underline{a}}\varphi}$. So we shall find some functional which has much relation with box-counting dimension and which is continuous as $T_{\underline{a},n} \to T_{\underline{a}}$ in some sense.

Let $H(f)$ be the Hölder exponent of a function f on $[0,1]$, that is,

$$H(f) = \sup\{\alpha; \exists c > 0 \text{ s.t. } |f(x) - f(y)| \le c|x-y|^{\alpha} \text{ for any } x, y \in [0,1]\}.$$

Then the following relation between the Hausdorff dimension $\dim_H G_f$ and $H(f)$ is obtained by A.S. Besicovitch and H.D. Ursell [2]:

$$\dim_H G_f \le 2 - H(f).$$

If f is not a constant function, then $H(f) \le 1$. For $\varphi \in E$, $H(T_{\underline{a},n}\varphi) = 1$ for any $n \in \mathrm{N}$ and $H(T_{\underline{a}}\varphi) < 1$ for some $\underline{a} \in l_1$. So $H(T_{\underline{a},n}\varphi)$ does not converge to $H(T_{\underline{a}}\varphi)$ as the same as Hausdorff dimension. As for the Hölder exponent of f, the following is obtained.

Proposition 3. *For $\varphi \in E$ and $\{a_n\} \in l_1$, we have the following*

$(1) H(T_{\underline{a},n}\varphi) = 1$

$(2) (-\log_2 \sup_n |a_n|^{1/n}) \wedge 1 \le H(T_{\underline{a}}\varphi) \le (-\log_2 \overline{\lim_n} |a_n|^{1/n}) \wedge 1$

$(3) \lim_n H(T_{\underline{a}}\varphi - T_{\underline{a},n}\varphi) = (-\log_2 \overline{\lim_n} |a_n|^{1/n}) \wedge 1 \ \ (= 2 - \overline{\dim}_B G_{T_{\underline{a}}\varphi})$

(4)*if* $a_n = \gamma^n$ *with* $1/2 \le \gamma < 1$, *then* $H(T_{\underline{a}}\varphi) = -\log_2 \gamma$.

Proof. (1) is easily obtained.

(2) Put $\gamma_0 = \sup |a_k|^{1/k}$ and $\delta_0 = -\log_2 \gamma_0$. For any $h > 0$ find $n \in \mathrm{N}$ such that $2^{-(n+1)} \le h < 2^{-n}$. Then by the relation

$$|T_{\underline{a}}\varphi(x+h) - T_{\underline{a}}\varphi(x)| \le K \sum_{k=1}^{n} \gamma_0^k 2^{k-1} h + 2\|\varphi\|_\infty \sum_{k=n+1}^{\infty} \gamma_0^k$$

$$\le \begin{cases} (\frac{K}{2\gamma_0 - 1} + \frac{2\|\varphi\|_\infty}{1-\gamma_0}) h^{\delta_0} & \text{if } 1/2 < \gamma_0 < 1 \\ (\frac{K\gamma_0}{1-2\gamma_0} + \frac{2\|\varphi\|_\infty}{1-\gamma_0}) h & \text{if } 0 < \gamma_0 < 1/2, \end{cases}$$

we have $H(T_{\underline{a}}\varphi) \geq (\delta_0) \wedge 1$.

On the other hand,

$$|T_{\underline{a}}\varphi(\frac{j}{2^n}) - T_{\underline{a}}\varphi(\frac{j-1}{2^n})| \geq |a_n| = (\frac{1}{2^n})^{-\log_2 |a_n|^{1/n}}$$

for some j $(1 \leq j \leq 2^n)$. So we have $H(T_{\underline{a}}\varphi) \leq \delta \wedge 1$ by putting $\gamma = \overline{\lim}_n |a_n|^{1/n}$ and $\delta = -\log_2 \gamma$ and by using the fact $H(T_{\underline{a}}\varphi) \leq 1$.

(3) and (4) are obtained from (2). □

Since we considered global Hölder exponent H, $H(T_{\underline{a},n}\varphi)$ is independent of n as the case of Hausdorff dimension. So we shall consider local Hölder exponent and the following functional H_n, H_∞ on $C_b(\mathbf{R})$. For $f \in C_b(\mathbf{R})$, define

$$H_n(f) = \sup\{\alpha : |f(x+h) - f(x)| \leq |h|^\alpha \text{ for any } x, |h| < 1/2^n\}$$
$$H_\infty(f) = \lim_n H_n(f) \quad (= \sup H_n(f))$$
$$H_n(T_{\underline{a},n}) = \sup\{H_n(T_{\underline{a},n}\varphi); \varphi \in E\} \qquad \text{and}$$
$$H_\infty(T_{\underline{a}}) = \sup\{H_\infty(T_{\underline{a}}\varphi); \varphi \in E\}.$$

Then $H_n(T_{\underline{a},n})$ depends on n as follows.

Proposition 4.

(1) For $\varphi \in E$,

$$(-\log_2 |a_n|^{1/n}) \wedge 1 \geq H_n(T_{\underline{a},n}\varphi) \geq (1 - \log_2(\frac{K}{2}\sum_{k=1}^{n} 2^k |a_k|)^{1/n}) \wedge 1,$$

where K is the Lipschitz constant of φ.

(2) $$(-\log_2 |a_n|^{1/n}) \wedge 1 \geq H_n(T_{\underline{a},n}) \geq (1 - \log_2(\sum_{k=1}^{n} 2^k |a_k|)^{1/n}) \wedge 1.$$

Proof. (1) For h with $(|h| < 1/2^n)$,

$$|T_{\underline{a},n}\varphi(x+h) - T_{\underline{a},n}\varphi(x)| \leq \frac{K}{2}\sum_{k=1}^{n} 2^k |a_k||h| = 2^{nc_n}|h|,$$

where $c_n = \log_2(\frac{K}{2}\sum_{k=1}^{n} 2^k|a_k|)^{1/n}$. So $H_n(T_{\underline{a},n}\varphi) \geq (1-c_n) \wedge 1$. On the other hand for some $x = j/2^n$ $(0 \leq j \leq 2^n - 1)$, $|T_{\underline{a},n}\varphi(x + \frac{1}{2^n}) - T_{\underline{a}.n}\varphi(x)| \geq (\frac{1}{2^n})^{-\log_2 |a_n|^{1/n}}$ implies that $(-\log_2 |a_n|^{1/n}) \geq H_n(T_{\underline{a},n}\varphi)$.

(2) follows from (1) and the fact that $K \geq 2$ for $\varphi \in E$. □

By using the estimate of $H_n(T_{\underline{a},n})$ in Proposition 4, we show that inferior limit of $H_n(T_{\underline{a},n})$ is $2 - \overline{\dim}_B G_{T_{\underline{a}}\varphi}$.

Theorem 5.

$$\text{(1)}\ H_\infty(T_{\underline{a}}) = (-\log_2 \overline{\lim_n} |a_n|^{1/n}) \wedge 1 \ \ (= 2 - \overline{\dim}_B G_{T_{\underline{a}}\varphi})$$
$$\text{(2)}\ \underline{\lim_n} H_n(T_{\underline{a},n}) = H_\infty(T_{\underline{a}}).$$

Proof. (1) Put $\gamma = \overline{\lim}_n |a_n|^{1/n}$. Suppose $1 > \gamma > 1/2$.
For any $\tilde{\gamma} > \gamma$,

$$|T_{\underline{a}}\varphi(x+h) - T_{\underline{a}}\varphi(x)| \leq K\frac{(2\tilde{\gamma})^{n+1}}{2\tilde{\gamma}-1}|h| + \frac{\tilde{\gamma}^{n+1}}{1-\tilde{\gamma}}2\|\varphi\|_\infty$$

for sufficiently large $n \in \mathbf{N}$ and h with $2^{-(n+1)} \leq |h| < 2^{-n}$. By putting $\tilde{\delta} = -\log_2 \tilde{\gamma}$, we have $|T_{\underline{a}}\varphi(x+h) - T_{\underline{a}}\varphi(x)| \leq M|h|^{\tilde{\delta}}$, where M depends only on $\tilde{\gamma}$, K and $\|\varphi\|_\infty$, not on n. If $M \leq 1$, $|T_{\underline{a}}\varphi(x+h) - T_{\underline{a}}\varphi(x)| \leq |h|^{\tilde{\delta}}$. If $M > 1$, put $\beta_n = \frac{1}{n}\log_2 M$. Then $\beta > 0$, $|T_{\underline{a}}\varphi(x+h) - T_{\underline{a}}\varphi(x)| \leq |h|^{\tilde{\delta}-\beta_n}$, and $\tilde{\delta} - \beta_n > 0$ for sufficiently large n. For any y with $|y| < 2^{-n}$, there exists m such that $m \geq n$, $2^{-(m+1)} \leq |y| < 2^{-m}$. So $|T_{\underline{a}}\varphi(x+y) - T_{\underline{a}}\varphi(x)| \leq |y|^{\tilde{\delta}-\beta_m} \leq |y|^{\tilde{\delta}-\beta_n}$, which implies $H_n(T_{\underline{a}}\varphi) \geq \tilde{\delta} - \beta_n$. Since $\lim_n \beta_n = 0$, we have $H_\infty(T_{\underline{a}}\varphi) \geq \tilde{\delta}$. So $H_\infty(T_{\underline{a}}) \geq (-\log_2 \overline{\lim}_n |a_n|)$. In case of $0 < \gamma \leq 1/2$, $H_\infty(T_{\underline{a}}) \geq 1$ is obtained in the same way. The opposite inequality $H_\infty(T_{\underline{a}}) \leq (-\log_2 \overline{\lim}_n |a_n|^{1/n}) \wedge 1$ is proved in a similar way to the proof of Proposition 4.

(2) follows from (1) and Proposition 4. □

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