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On uniformly convex functions and uniformly smooth functions

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1 Introduction

In 1983, Zălinescu [10] studied the uniformly convex functions giving some characterizations and examples of such functions. He showed that if a proper, lower semicontinuous and convex function defined on a reflexive Banach space is uniformly convex on the whole Banach space then its conjugate function is uniformly Fréchet differentiable on the interior of the domain of the conjugate function and that the converse is true under some condition. Let $\psi : [0,\infty) \to [0,\infty]$ be a function. He also characterized the uniform convexity of the function $x \mapsto \int_0^{\|x\|} \psi(t)\,dt$ defined on bounded balls in a Banach space. On the other hand, it is well known that in a Hilbert space H,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{1.1}$$

for all $x, y \in H$ and $0 \leq \lambda \leq 1$. Lim [5], B. Prus and R. Smarzewski [6], R. Smarzewski [7] and Xu [8, 9] have studied inequalities that are analogous to (1.1) in a Banach space. These inequalities are related to uniform convexity and uniform Fréchet differentiability of the functional $x \mapsto \|x\|^p$.

In this paper, we study uniformly convex functions and uniformly smooth functions in the framework of the nonstandard analysis [3]. Let E be a real normed linear space, let $f : E \to (-\infty,\infty]$ be a function and let Y be a subset of E such that there exists $\varepsilon > 0$ with $Y + \{x \in E : \|x\| \leq \varepsilon\} \subset \operatorname{dom} f$. We mean that f is uniformly smooth on Y if

$$\lim_{t\to 0} \frac{f(y+tu) - f(y)}{t}$$

exists uniformly for $y \in Y$ and $u \in E$ with $\|u\| = 1$. If f is convex and for each $y \in Y$, $\sup_{\|u\|=1} \overline{\lim_{t\to 0}} \left|\frac{f(y+tu) - f(y)}{t}\right| < \infty$, the uniform smoothness coincides with the uniform Fréchet differentiability. We show that in a Banach space, a proper, lower semicontinuous and convex function f is uniformly convex on the whole Banach space if and only if its conjugate function is uniformly Fréchet differentiable on $R(\partial f)$. Let $\varphi : [0,\infty) \to (-\infty,\infty]$ be a function. We characterize the uniform convexity and the uniformly smoothness of the function $x \mapsto \varphi(\|x\|)$ on bounded balls in a normed linear space. We also show sufficient conditions which ensure the uniform convexity and the uniform smoothness of the function $x \mapsto \varphi(\|x\|)$ on a whole normed linear space.

2 Nonstandard analysis

We adopt the notational conventions and the framework for the nonstandard analysis described in [3]. For convenience, we state some definitions. We denote the set of all real numbers and the set of all positive real numbers by $\mathbb{R}$ and $\mathbb{R}_+$ respectively. Let a and b be elements in $^*\mathbb{R}$. We define symbols $\simeq$, $\gtrsim$, $\lesssim$, $\gnsim$ and $\lnsim$ as follows:

$$\begin{aligned}
&a \simeq b \text{ if for any } \varepsilon \in \mathbb{R}_+, |a-b| < \varepsilon;\\
&a \gtrsim b \text{ if } a > b \text{ or } a \simeq b;\\
&a \lesssim b \text{ if } a < b \text{ or } a \simeq b;\\
&a \gnsim b \text{ if } a > b \text{ and } a \not\simeq b;\\
&a \lnsim b \text{ if } a < b \text{ and } a \not\simeq b.
\end{aligned}$$

We recall that a is finite if there exists a standard positive real number c with $|a| \leq c$ and a is infinite if a is not finite. Let E be a normed linear space and let x and y be elements in $^*E$. We write $x \simeq y$ if $\|x-y\| \simeq 0$ and we denote by $\mu(x)$ the set $\{z \in {}^*E : z \simeq x\}$.

3 Preliminaries

Throughout this paper, all vector spaces are real, o denotes the origin of a vector space and if E is a normed linear space then $E^\#$ denotes its dual. Let E be a normed linear space. We write $\langle x^\#, x\rangle$ in place of $x^\#(x)$ for $x \in E$ and $x^\# \in E^\#$. Let C and D be subsets of E. $C+D$ denotes the set $\{x+y \in E : x \in C, y \in D\}$. C is said to be convex if $\lambda x + (1-\lambda)y \in C$ for all $x, y \in C$ and $0 \leq \lambda \leq 1$. For a positive real number a, $S_E(a)$ and $B_E(a)$ denote $\{x \in E : \|x\| = a\}$ and $\{x \in E : \|x\| \leq a\}$ respectively. E is said to be uniformly convex if for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that for any $x, y \in S_E(1)$,

$$\|y-x\| \geq \varepsilon \text{ implies } \|\frac{x+y}{2}\| \leq 1-\delta.$$

E is said to be uniformly smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists uniformly for $x, y \in S_E(1)$. The modulus of convexity and smoothness of E are defined respectively by

$$\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : x, y \in E, \|x\| = \|y\| = 1, \|x-y\| \geq \varepsilon\}, \quad 0 \leq \varepsilon \leq 2,$$

and

$$\rho(\tau) = \sup\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\}, \quad \tau > 0.$$

It is easy to see that E is uniformly convex if and only if $\delta(\varepsilon) > 0$ for any $\varepsilon \in (0,2]$ and E is uniformly smooth if and only if $\lim_{\tau \downarrow 0} \frac{\rho(\tau)}{\tau} = 0$. In the nonstandard representation, E is uniformly convex if and only if for any $x, y \in {}^*E$ such that $\|x\|$ and $\|y\|$ are finite and $\|x\| \simeq \|y\|$,

$$x \not\simeq y \text{ implies } \left\|\frac{x+y}{2}\right\| \lesssim\!\!\!\!\!/ \;\; \frac{\|x\|+\|y\|}{2}$$

and E is uniformly smooth if and only if

$$\frac{\|x+u\|-\|x\|}{\|u\|} \simeq \frac{\|x-u\|-\|x\|}{-\|u\|}$$

for any $x \in {}^*E$ such that $\|x\|$ is finite and $\|x\| \not\simeq 0$, and for any $u \in {}^*E \setminus \{o\}$ with $u \simeq o$. Let $f : E \to (-\infty, \infty]$ be a function. $\operatorname{dom} f$ denotes the set $\{x \in E : f(x) < \infty\}$. f is said to be proper if $\operatorname{dom} f \neq \emptyset$. Let X be a convex subset of E. f is said to be convex on X if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for any $x, y \in \operatorname{dom} f \cap X$ and for any $\lambda \in [0,1]$. f is said to be strictly convex on X if the above inequality is strict. Let $g : E \to (-\infty, \infty]$ be a proper and convex function. $g^\# : E^\# \to (-\infty, \infty]$ denotes the conjugate function of g which is defined by

$$g^\#(x^\#) = \sup\{\langle x^\#, x\rangle - g(x) : x \in E\}, \quad x^\# \in E^\#$$

and $g^{\#\#} : E \to (-\infty, \infty]$ denotes the second conjugate function of g which is defined by

$$g^{\#\#}(x) = \sup\{\langle x^\#, x\rangle - g^\#(x^\#) : x^\# \in E^\#\}, \quad x \in E.$$

It is well known (cf. [1]) that $g = g^{\#\#}$ if and only if g is lower semicontinuous. The subdifferential of g at $x \in E$ is the set

$$(\partial g)(x) = \{x^\# \in E^\# : g(y) \geq g(x) + \langle x^\#, y-x\rangle \text{ for all } y \in E\}.$$

By ∂g, we mean the set $\{(x, x^\#) \in E \times E^\# : x^\# \in (\partial g)(x)\}$ and by $R(\partial g)$, we mean the set $\bigcup\{(\partial g)(x) : x \in E\}$. It is well known (cf. [1]) that $(x, x^\#) \in \partial g$ if and only if $\langle x^\#, x\rangle = g(x) + g^\#(x^\#)$. Let φ be a real valued convex function defined on an open interval I of $\mathbb{R}$. It is also well known (cf. [4]) that φ is continuous on I, and if φ is differentiable on I then its derivative φ' is continuous on I.

4 Uniformly convex functions and uniformly smooth functions

We start this section by some definitions. Let $g : E \to (-\infty, \infty]$ be a function and let X be a convex subset of E. g is said to be uniformly convex on X if for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that

$$\|y - x\| \geq \varepsilon \text{ implies } g\Big(\frac{x+y}{2}\Big) \leq \frac{g(x)+g(y)}{2} - \delta$$

for any $x, y \in \operatorname{dom} g \cap X$. Let $h : E \to (-\infty, \infty]$ be a function and let Y be a subset of E such that there exists $\varepsilon \in \mathbb{R}_+$ with $Y + B_E(\varepsilon) \subset \operatorname{dom} h$. We define that h is uniformly smooth on Y if

$$\lim_{t\to 0} \frac{h(y+tu)-h(y)}{t}$$

exists uniformly for $y \in Y$ and $u \in S_E(1)$. We recall that h is uniformly Fréchet differentiable on Y if for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that for any $y \in Y$, there exists $y^\# \in E^\#$ such that

$$0 < |t| \le \delta \text{ and } u \in S_E(1) \text{ implies } \left|\frac{h(y+tu)-h(y)}{t} - \langle y^\#, u\rangle\right| \le \varepsilon.$$

If h is convex and for each $y \in Y$, $\sup_{\|u\|=1} \overline{\lim_{t\to 0}} \left|\frac{h(y+tu)-h(y)}{t}\right| < \infty$, the uniform smoothness of h on Y coincides with what h is uniformly Fréchet differentiable on Y. In the nonstandard representation, h is uniformly smooth on Y if and only if

$$t \simeq 0 \text{ and } s \simeq 0 \text{ implies } \frac{h(y+tu)-h(y)}{t} \simeq \frac{h(y+su)-h(y)}{s}$$

for all $y \in {}^*Y$, $u \in {}^*S_E(1)$ and $t, s \in {}^*\mathbb{R} \setminus \{0\}$. If h is convex then h is uniformly smooth on Y if and only if

$$u \simeq o \text{ implies } \frac{h(y+u)-h(y)}{\|u\|} \simeq \frac{h(y-u)-h(y)}{-\|u\|}$$

for all $y \in {}^*Y$ and $u \in {}^*E \setminus \{o\}$. Concerning uniform convexity and uniform smoothness, we have the following propositions. The first one is Remark 2.6 in [10].

PROPOSITION 1 (Zălinescu). Let E be a normed linear space and let X be a convex subset of E. Let $f : E \to (-\infty, \infty]$ be a proper and convex function. Then the following are equivalent;

(i) f is uniformly convex on X, i.e., for any $x, y \in {}^*(\operatorname{dom} f \cap X)$,

$$y \not\simeq x \text{ implies } f\Big(\frac{x+y}{2}\Big) \lessapprox \frac{f(x)+f(y)}{2},$$

(ii) for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that for any $x, y \in \operatorname{dom} f \cap X$ and $0 \le \lambda \le 1$,

$$\|y-x\| \ge \varepsilon \text{ implies } f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda y) f(y) - \lambda(1-\lambda)\delta,$$

i.e., for any $x, y \in {}^*(\operatorname{dom} f \cap X)$ and for any $\lambda \in {}^*(0,1)$,

$$y \not\simeq x \text{ implies } \frac{f(\lambda x + (1-\lambda)y)}{\lambda(1-\lambda)} \lessapprox \frac{\lambda f(x) + (1-\lambda) f(y)}{\lambda(1-\lambda)}.$$

PROPOSITION 2. Let E be a normed linear space and let $f : E \to (-\infty, \infty]$ be a function. Let Y be a subset of E such that there exists $\varepsilon \in \mathbb{R}_+$ with $Y + B_E(\varepsilon) \subset \operatorname{dom} f$, and let f be uniformly smooth on Y. Then for any $y, z \in {}^*Y$ with $y \neq z$ and for any $\lambda \in {}^*(0,1)$,

$$y \simeq z \ \text{ implies } \ \frac{f(\lambda y + (1-\lambda)z)}{\lambda(1-\lambda)\|y-z\|} \simeq \frac{\lambda f(y) + (1-\lambda)f(z)}{\lambda(1-\lambda)\|y-z\|},$$

i.e., for any $\varepsilon \in \mathbb{R}_+$, there exists $\delta \in \mathbb{R}_+$ such that for any $y, z \in Y$ and $0 \leq \lambda \leq 1$,

$$\|z-y\| \leq \delta \ \text{ implies } \ \left|\lambda f(y) + (1-\lambda)f(z) - f(\lambda y + (1-\lambda)z)\right| \leq \lambda(1-\lambda)\varepsilon\|y-z\|.$$

PROOF. Since f is uniformly smooth on Y, we have

$$\frac{f(x+tu) - f(x)}{t} \simeq \frac{f(x+su) - f(x)}{s} \tag{4.1}$$

for all $x \in {}^*Y$, $u \in {}^*S_E(1)$ and $t, s \in {}^*\mathbb{R} \setminus \{0\}$ with $t \simeq 0$ and $s \simeq 0$. Let y and z be any elements of ${}^*Y$ such that $y \neq z$ and $y \simeq z$, and let λ be any element of ${}^*(0,1)$. We may assume $\lambda \in {}^*(0, \frac{1}{2}]$. From (4.1), we get

$$\begin{aligned}
\frac{f(y) - f(z)}{\|y-z\|} &= \frac{f\left(z + \|y-z\| \cdot \dfrac{y-z}{\|y-z\|}\right) - f(z)}{\|y-z\|} \\
&\simeq \frac{f\left(z + \lambda\|y-z\| \cdot \dfrac{y-z}{\|y-z\|}\right) - f(z)}{\lambda\|y-z\|} \\
&= \frac{f(\lambda y + (1-\lambda)z) - f(z)}{\lambda\|y-z\|},
\end{aligned}$$

and hence we have

$$\frac{f(\lambda y + (1-\lambda)z)}{\lambda\|y-z\|} \simeq \frac{\lambda f(y) + (1-\lambda)f(z)}{\lambda\|y-z\|}.$$

Since $\lambda \not\simeq 1$, we obtain

$$\frac{f(\lambda y + (1-\lambda)z)}{\lambda(1-\lambda)\|y-z\|} \simeq \frac{\lambda f(y) + (1-\lambda)f(z)}{\lambda(1-\lambda)\|y-z\|}.$$

By the transfer principle, we obtain the standard representation. □

Let φ be a real valued convex function defined on an open interval I of $\mathbb{R}$. It is well known that if φ is strictly convex on I then for any bounded and closed interval $J(\subset I)$, φ is uniformly convex on J, and if φ is differentiable on I then for any bounded and closed interval $J(\subset I)$, φ is uniformly smooth on J. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function which is uniformly smooth on $\mathbb{R}$. It is easy to see that if $t, s \in {}^*\mathbb{R} \setminus \{0\}$, $t \neq s$ and $t \simeq s$ then $\psi'(t) \simeq \frac{\psi(s)-\psi(t)}{s-t} \simeq \psi'(s)$.

Next, we show relation between a proper, lower semicontinuous convex function defined on a Banach space and its conjugate function. The following was partly obtained by Zălinescu [10]. In the following, the proofs of (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (ii) are essentially same as the proofs of Zălinescu.

THEOREM 1. Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Then the following conditions are equivalent;

(i) f is uniformly convex on E,

(ii) for any $(x, x^\#) \in {}^*(\partial f)$ and for any $y \in {}^*E$,

$$y \not\simeq x \ \text{ implies } \ f(y) \gtrsim f(x) + \langle x^\#, y - x \rangle,$$

(iii) for any $(x, x^\#) \in {}^*(\partial f)$ and for any $y \in {}^*E$,

$$y \not\simeq x \ \text{ implies } \ \frac{f(y) - f(x)}{\|y - x\|} \gtrsim \frac{\langle x^\#, y - x \rangle}{\|y - x\|},$$

(iv) for any $(x, x^\#) \in {}^*(\partial f)$, for any $u^\# \in {}^*E^\# \setminus \{o\}$ with $u^\# \simeq o$ and for any $y \in {}^*E$,

$$\langle x^\# + u^\#, y - x \rangle + f(x) \geq f(y) \ \text{ implies } \ y \simeq x,$$

(v) ${}^*(R(\partial f)) + \mu(o) \subset {}^*(\operatorname{dom} f)$, and for any $(x, x^\#) \in {}^*(\partial f)$ and for any $u^\# \in {}^*E^\# \setminus \{o\}$ with $u^\# \simeq o$,

$$\frac{f^\#(x^\# + u^\#) - f^\#(x^\#)}{\|u^\#\|} \simeq \Big\langle \frac{u^\#}{\|u^\#\|}, x \Big\rangle,$$

i.e., there exists $\varepsilon \in \mathbb{R}_+$ such that $R(\partial f) + B_E(\varepsilon) \subset \operatorname{dom} f$, and $f^\#$ is uniformly Fréchet differentiable on $R(\partial f)$.

PROOF. (i) $\Rightarrow$ (ii). Let $(x, x^\#)$ be any element of ${}^*(\partial f)$ and let y be any element of ${}^*E$ with $y \not\simeq x$. We may assume $y \in {}^*(\operatorname{dom} f)$. Since $y \not\simeq x$, we have $f(\frac{x+y}{2}) \lesssim \frac{f(x)+f(y)}{2}$. Hence, by $(x, x^\#) \in {}^*(\partial f)$, we get

$$\begin{aligned} f(y) &\gtrsim 2f\Big(\frac{x+y}{2}\Big) - f(x) \\ &= f(x) + 2\Big(f\Big(\frac{x+y}{2}\Big) - f(x)\Big) \\ &\geq f(x) + 2\Big\langle x^\#, \frac{x+y}{2} - x \Big\rangle \\ &= f(x) + \langle x^\#, y - x \rangle. \end{aligned}$$

Therefore (ii) is valid.

(ii) $\Rightarrow$ (iii). Let $(x, x^\#)$ be any element of ${}^*(\partial f)$ and let y be any element of ${}^*E$ with $y \not\simeq x$. We may assume that $y \in {}^*(\operatorname{dom} f)$. If $\|y - x\|$ is finite, it is clear that (iii) is valid. Let $\|y - x\|$ be infinite. Put $u = x + \frac{y-x}{\|y-x\|}$. Then we have $\|u - x\| = 1$ and, by the convexity of f,

$$\frac{f(y) - f(x)}{\|y - x\|} \geq f(u) - f(x).$$

Hence, by (ii), we get

$$\begin{aligned}\frac{f(y)-f(x)}{\|y-x\|} &\geq f(u)-f(x)\\ &\gtrsim_{\not\approx} \langle x^{\#}, u-x\rangle\\ &= \frac{\langle x^{\#}, y-x\rangle}{\|y-x\|}.\end{aligned}$$

Therefore (iii) holds.

(iii) $\Rightarrow$ (iv). Let $(x, x^{\#})$ be any element of $^*(\partial f)$, let $u^{\#}$ be any element of $^*E^{\#} \setminus \{o\}$ with $u^{\#} \simeq o$ and let y be any element of $^*E$ such that $\langle x^{\#}+u^{\#}, y-x\rangle + f(x) \geq f(y)$. Suppose $y \not\simeq x$. Then we get

$$\begin{aligned}\frac{\langle x^{\#}+u^{\#}, y-x\rangle}{\|y-x\|} &\geq \frac{f(y)-f(x)}{\|y-x\|}\\ &\gtrsim_{\not\approx} \frac{\langle x^{\#}, y-x\rangle}{\|y-x\|}.\end{aligned}$$

So we have $\|u^{\#}\| \gtrsim_{\not\approx} 0$. This contradicts $u^{\#} \simeq o$. Therefore $y \simeq x$.

(iv) $\Rightarrow$ (v). Let $(x, x^{\#})$ be any element of $^*(\partial f)$ and let $u^{\#}$ be any element of $^*E^{\#} \setminus \{o\}$ with $u^{\#} \simeq o$. First, we prove that $x^{\#}+u^{\#} \in {}^*(\operatorname{dom} f)$. By the definition of $f^{\#}$, we have

$$\begin{aligned}f^{\#}(x^{\#}+u^{\#}) = \sup\{&\langle x^{\#}+u^{\#}, y\rangle - f(y) : y \in {}^*E,\\ &\langle x^{\#}+u^{\#}, y\rangle - f(y) \geq \langle x^{\#}+u^{\#}, x\rangle - f(x)\}.\end{aligned}$$

So, let $y \in {}^*E$ be any element such that $\langle x^{\#}+u^{\#}, y\rangle - f(y) \geq \langle x^{\#}+u^{\#}, x\rangle - f(x)$. Then, by (iv), we get $y \simeq x$. Hence we obtain

$$\begin{aligned}\langle x^{\#}+u^{\#}, y\rangle - f(y) &= \big(\langle x^{\#}, y\rangle - f(y)\big) + \langle u^{\#}, y\rangle\\ &\leq f^{\#}(x^{\#}) + \big(\langle u^{\#}, y-x\rangle + \langle u^{\#}, x\rangle\big)\\ &\leq f^{\#}(x^{\#}) + \|u^{\#}\|\|y-x\| + \langle u^{\#}, x\rangle\\ &< f^{\#}(x^{\#}) + 1 + \langle u^{\#}, x\rangle.\end{aligned}$$

So we have $f^{\#}(x^{\#}+u^{\#}) < f^{\#}(x^{\#}) + 1 + \langle u^{\#}, x\rangle < \infty$, i.e., $x^{\#}+u^{\#} \in {}^*(\operatorname{dom} f)$. By the definition of $f^{\#}$, we can choose $z \in {}^*E$ which satisfies

$$\frac{f^{\#}(x^{\#}+u^{\#}) - (\langle x^{\#}+u^{\#}, z\rangle - f(z))}{\|u^{\#}\|} \simeq 0$$

and

$$\langle x^{\#}+u^{\#}, z\rangle - f(z) \geq \langle x^{\#}+u^{\#}, x\rangle - f(x).$$

We have $z \simeq x$ by (iv). Since $(x, x^\#) \in {}^*(\partial f)$, we get

$$\begin{aligned}
&\langle x^\# + u^\#, z - x\rangle + f(x) - f(z) \\
&\le \langle x^\# + u^\#, z - x\rangle + f(x) - (\langle x^\#, z - x\rangle + f(x)) \\
&= \langle u^\#, z - x\rangle \\
&\le \|u^\#\|\|z - x\|.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&\frac{f^\#(x^\# + u^\#) - f^\#(x^\#)}{\|u^\#\|} \\
&\simeq \frac{\langle x^\# + u^\#, z\rangle - f(z) - f^\#(x^\#)}{\|u^\#\|} \\
&= \frac{\langle x^\# + u^\#, z\rangle - f(z) - (\langle x^\#, x\rangle - f(x))}{\|u^\#\|} \\
&= \frac{\big(\langle x^\# + u^\#, z - x\rangle + f(x) - f(z)\big) + \langle u^\#, x\rangle}{\|u^\#\|} \\
&\simeq \|z - x\| + \frac{\langle u^\#, x\rangle}{\|u^\#\|} \\
&\simeq \Big\langle \frac{u^\#}{\|u^\#\|}, x\Big\rangle.
\end{aligned}$$

Therefore $f^\#$ is Fréchet differentiable on $R(\partial f)$.

(v) $\Rightarrow$ (ii). Let $(x, x^\#)$ be any element of ${}^*(\partial f)$ and let y be any element of ${}^*E$ with $y \not\simeq x$. Since $y \not\simeq x$, there exists a standard positive real number ε such that $\|y - x\| \ge 2\varepsilon$. By the transfer principle, there exists standard positive real number δ such that for any $u^\# \in {}^*E^\#$,

$$0 < \|u^\#\| \le \delta \ \text{ implies } \ \frac{f^\#(x^\# + u^\#) - f^\#(x^\#) - \langle u^\#, x\rangle}{\|u^\#\|} \le \varepsilon.$$

Hence we get

$$\begin{aligned}
f(y) &= f^{\#\#}(y) \\
&= \sup\{\langle y^\#, y\rangle - f^\#(y^\#) : y^\# \in {}^*E^\#\} \\
&\ge \sup\{\langle x^\# + u^\#, y\rangle - f^\#(x^\# + u^\#) : u^\# \in {}^*E^\#, 0 < \|u^\#\| \le \delta\} \\
&\ge \sup\{\langle x^\# + u^\#, y\rangle - \big(f^\#(x^\#) + \langle u^\#, x\rangle + \varepsilon\|u^\#\|\big) : u^\# \in {}^*E^\#, 0 < \|u^\#\| \le \delta\} \\
&= \sup\{\langle u^\#, y - x\rangle - \varepsilon\|u^\#\| : u^\# \in {}^*E^\#, 0 < \|u^\#\| \le \delta\} + \langle x^\#, y\rangle - f^\#(x^\#) \\
&= \sup\{\langle u^\#, y - x\rangle - \varepsilon\|u^\#\| : u^\# \in {}^*E^\#, 0 < \|u^\#\| \le \delta\} + \langle x^\#, y\rangle - \big(\langle x^\#, x\rangle - f(x)\big) \\
&\ge \delta\|y - x\| - \varepsilon\delta + f(x) + \langle x^\#, y - x\rangle \\
&\gtrsim f(x) + \langle x^\#, y - x\rangle.
\end{aligned}$$

Therefore (ii) is valid.

(ii) $\Rightarrow$ (i). Let y and z be any element of $^*(\mathrm{dom}\, f)$ such that $y \not\simeq z$. By Theorem 2 in [2], there exists $(x, x^\#) \in {}^*(\partial f)$ such that

$$f\Big(\frac{y+z}{2}\Big) \simeq f(x) + \Big\langle x^\#, \frac{y+z}{2} - x\Big\rangle.$$

By (ii), we have $\frac{y+z}{2} \simeq x$ and hence $y \not\simeq x$ and $z \not\simeq x$. So we have $f(y) \gtrsim f(x) + \langle x^\#, y - x\rangle$ and $f(z) \gtrsim f(x) + \langle x^\#, z - x\rangle$. Hence we get

$$\begin{aligned} f\Big(\frac{y+z}{2}\Big) &\simeq f(x) + \Big\langle x^\#, \frac{y+z}{2} - x\Big\rangle \\ &= \frac{f(x) + \langle x^\#, y - x\rangle + f(x) + \langle x^\#, z - x\rangle}{2} \\ &\lesssim \frac{f(y) + f(z)}{2}. \end{aligned}$$

Therefore f is uniformly convex on E. □

REMARK. If a Banach space is reflexive, Zălinescu [10] showed that if $f : E \to (-\infty, \infty]$ is a proper and lower semicontinuous function which is uniformly convex on E, then f is uniformly Fréchet differentiable on $\mathrm{Int}(\mathrm{dom}\, f^\#)$. In the case, $R(\partial f)$ is open and hence $R(\partial f) = \mathrm{Int}(\mathrm{dom}\, f^\#)$.

5 Characterization of uniform convexity and uniform smoothness on bounded balls

In this section, we characterize the uniform convexity and uniform smoothness of $\varphi(\|\cdot\|)$ on bounded balls in a normed linear space. The following is essentially same as Theorem 4.1.(ii) in [10]. Compare these statements.

THEOREM 2 (Zălinescu). Let E be a normed linear space and let $\varphi : [0, \infty) \to [0, \infty]$ be an increasing function. Let $M = \sup(\mathrm{dom}\, \varphi) > 0$. Then $\varphi(\|\cdot\|)$ is uniformly convex on $B_E(a)$ for any $a \in (0, M)$ if and only if φ is strictly convex on $\mathrm{dom}\, \varphi$ and E is uniformly convex.

PROOF. Let φ be strictly convex on $\mathrm{dom}\, \varphi$ and let E be uniformly convex. Let a be any real number which satisfies $0 < a < M$. We remark that φ is uniformly convex on $[0, a]$. Let $x, y \in {}^*B_E(a)$ such that $x \not\simeq y$. If $\|x\| \not\simeq \|y\|$, the uniform convexity of φ yields

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \le \varphi\Big(\frac{\|x\| + \|y\|}{2}\Big) \lesssim \frac{\varphi(\|x\|) + \varphi(\|y\|)}{2}.$$

Next suppose that $\|x\| \simeq \|y\|$. Since $x \not\simeq y$, the uniform convexity of E yields

$$\Big\|\frac{x+y}{2}\Big\| \lesssim \frac{\|x\| + \|y\|}{2}.$$

So we have

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \lesssim\!\!\!\!\!\not\;\;\; \varphi\Big(\frac{\|x\|+\|y\|}{2}\Big) \leq \frac{\varphi(\|x\|)+\varphi(\|y\|)}{2}.$$

We show the necessity. Let $\varphi(\|\cdot\|)$ be uniformly convex on $B_E(a)$ for any $a \in (0, M)$. Fix an element $x_0 \in E$ such that $\|x_0\| = 1$. Let r and s be standard real numbers such that $r, s \in [0, M]$ and $r \neq s$. Since r and s are different, we have $r \not\simeq s$. Assume $r, s < M$. In virtue of the uniformly convexity of $\varphi(\|\cdot\|)$ on $B_E(\max\{|t|, |s|\})$, we get

$$\begin{aligned}\varphi\Big(\frac{r+s}{2}\Big) &= \varphi\Big(\Big\|\frac{rx_0+sx_0}{2}\Big\|\Big)\\ &\lesssim\!\!\!\!\!\not\;\;\; \frac{\varphi(\|rx_0\|)+\varphi(\|sx_0\|)}{2}\\ &= \frac{\varphi(r)+\varphi(s)}{2}.\end{aligned}$$

Hence we obtain $\varphi\big(\frac{r+s}{2}\big) < \frac{\varphi(r)+\varphi(s)}{2}$. If $M \neq \infty$, $M \in \operatorname{dom}\varphi$ and r or s is equal to M, we can also show $\varphi(\frac{r+s}{2}) < \frac{\varphi(r)+\varphi(s)}{2}$ from the convexity of φ. Next we show that E is uniformly convex. Suppose not. Let b be a standard real number such that $0 < b < M$. Then there are $x, y \in {}^*S_E(b)$ such that $x \not\simeq y$ and $\|\frac{x+y}{2}\| \simeq b$. The uniform convexity of $\varphi(\|\cdot\|)$ on $B_E(b)$ yields

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \lesssim\!\!\!\!\!\not\;\;\; \frac{\varphi(\|x\|)+\varphi(\|y\|)}{2} = \varphi(b).$$

But, by the continuity of φ on $\operatorname{Int}(\operatorname{dom}\varphi)$, we have

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \simeq \varphi(b),$$

which is a contradiction. Therefore E is uniformly convex. □

Using our theorem and Proposition 1, we have the following. Compare this with Theorem 2 in [9].

THEOREM 3. Let E be a normed linear space and let a be a positive real number. Let $\varphi : [0, \infty) \to [0, \infty)$ be a function such that $\varphi(0) = 0$ and strictly convex on $[0, \infty)$. Then E is uniformly convex if and only if there exists an increasing function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$, $g(t) > 0$ for all $t > 0$ and

$$\varphi(\|\lambda x + (1-\lambda)y\|) \leq \lambda\varphi(\|x\|) + (1-\lambda)\varphi(\|y\|) - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_E(a)$ and $0 \leq \lambda \leq 1$.

The following is the dual version of Theorem 2, which characterizes the uniform Fréchet differentiability of $\varphi(\|\cdot\|)$ on bounded balls.

THEOREM 4. Let E be a normed linear space and let $\varphi : [0,\infty) \to [0,\infty]$ be a strictly increasing and convex function such that $\varphi(0) = 0$ and $\varphi'(0) = 0$. Let $M = \sup(\operatorname{dom}\varphi) > 0$. Then $\varphi(\|\cdot\|)$ is uniformly Fréchet differentiable on $B_E(a)$ for any $a \in (0, M)$ if and only if φ is differentiable on $(0, M)$ and E is uniformly smooth.

PROOF. Suppose that φ is differentiable on $(0, M)$ and E is uniformly smooth. Let a be any real number such that $0 < a < M$ and let $x, u \in {}^*E$ such that $\|x\| \le a, u \simeq o$ and $u \ne o$. We remark that φ is uniformly smooth on $[0, a]$. If $x \simeq o$, we get

$$\begin{aligned}\frac{\varphi(\|x+u\|) - \varphi(\|x\|)}{\|u\|} &\simeq \frac{\varphi(\|x+u\|) - \varphi(\|x\|)}{\|x+u\| - \|x\|}\frac{\|x+u\| - \|x\|}{\|u\|} \\ &\simeq \varphi'(\|x\|)\frac{\|x+u\| - \|x\|}{\|u\|} \\ &\simeq \varphi'(0)\frac{\|x+u\| - \|x\|}{\|u\|} \\ &= 0\end{aligned}$$

and similarly,

$$\frac{\varphi(\|x-u\|) - \varphi(\|x\|)}{-\|u\|} \simeq 0.$$

If $x \not\simeq o$, we get

$$\begin{aligned}\frac{\varphi(\|x+u\|) - \varphi(\|x\|)}{\|u\|} &= \frac{\varphi(\|x+u\|) - \varphi(\|x\|)}{\|x+u\| - \|x\|}\frac{\|x+u\| - \|x\|}{\|u\|} \\ &\simeq \frac{\varphi(\|x-u\|) - \varphi(\|x\|)}{\|x-u\| - \|x\|}\frac{\|x-u\| - \|x\|}{-\|u\|} \\ &= \frac{\varphi(\|x-u\|) - \varphi(\|x\|)}{-\|u\|}.\end{aligned}$$

Hence we have

$$\frac{\varphi(\|x+u\|) - \varphi(\|x\|)}{\|u\|} \simeq \frac{\varphi(\|x-u\|) - \varphi(\|x\|)}{-\|u\|}.$$

On the other hand, it is easy to see that $x \in B_E(a)$, $u \in {}^*E \setminus \{0\}$ and $u \ne o$ implies

$$\left|\frac{\varphi(\|x+u\|) - \varphi(\|x\|)}{\|u\|}\right| \lesssim \varphi'(\|x\|).$$

Therefore $\varphi(\|\cdot\|)$ is uniformly Fréchet differentiable on $B_E(a)$. Next we show the necessity. Let $\varphi(\|\cdot\|)$ be uniformly Fréchet differentiable on $B_E(a)$ for any $a \in (0, M)$. Fix an element $x_0 \in E$ such that $\|x_0\| = 1$. Let r be a standard real number with $0 < r < M$ and let $s \in {}^*\mathbb{R}$ such that $s \simeq 0$ and $s \ne 0$. Then we have

$$\frac{\varphi(r+s) - \varphi(r)}{s} = \frac{\varphi(\|rx_0 + sx_0\|) - \varphi(\|rx_0\|)}{\|sx_0\|}$$

$$\begin{aligned}
&\simeq \frac{\varphi(\|rx_0 - sx_0\|) - \varphi(\|rx_0\|)}{-\|sx_0\|} \\
&= \frac{\varphi(r-s) - \varphi(r)}{-s},
\end{aligned}$$

which shows that φ is differentiable on $(0, M)$. Next we prove that E is uniformly smooth. Let b be a standard real number such that $0 < b < M$. Let $x \in {}^*S_E(b)$, and let $u \in {}^*E$ such that $u \simeq o$ and $u \neq o$. Then we get

$$\begin{aligned}
\frac{\|x+u\| - \|x\|}{\|u\|} &= \frac{\|x+u\| - \|x\|}{\varphi(\|x+u\|) - \varphi(\|x\|)} \frac{\varphi(\|x+u\|) - \varphi(\|x\|)}{\|u\|} \\
&\simeq \frac{\|x-u\| - \|x\|}{\varphi(\|x-u\|) - \varphi(\|x\|)} \frac{\varphi(\|x-u\|) - \varphi(\|x\|)}{-\|u\|} \\
&\simeq \frac{\|x-u\| - \|x\|}{-\|u\|}.
\end{aligned}$$

Therefore E is uniformly smooth. □

By the same argument, we have the following.

THEOREM 5. Let E be a normed linear space and let a be a positive real number. Let $\varphi : [0, \infty) \to \mathbb{R}$ be a function such that $\varphi(0) = 0$, $\varphi'(0) = 0$ and $\varphi \not\equiv 0$ on $[0, a]$. Then $\varphi(\|\cdot\|)$ is uniformly smooth on $B_E(a)$ if and only if φ is uniformly smooth on $[0, a]$ and E is uniformly smooth.

Using Theorem 4 and Proposition 2, we have the following. Compare this with Theorem 2′ in [9].

THEOREM 6. Let E be a normed linear space and let a be a positive real number. Let $\varphi : [0, \infty) \to [0, \infty)$ be a strictly increasing and convex function such that $\varphi(0) = 0$ and $\varphi'(0) = 0$. Then E is uniformly smooth if and only if there exists an increasing function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$, $\lim_{t \downarrow 0} \frac{g(t)}{t} = 0$ and

$$\varphi(\|\lambda x + (1-\lambda)y\|) \geq \lambda\varphi(\|x\|) + (1-\lambda)\varphi(\|y\|) - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_E(a)$ and $0 \leq \lambda \leq 1$.

6 On uniform convexity and uniform smoothness on whole space

In this section, we show sufficient conditions which guarantee the uniform convexity and the uniform smoothness of the function $\varphi(\|\cdot\|)$ on a whole normed linear space. We begin with uniformly convex case. We need the following lemma.

LEMMA 1. Let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing and convex function. If $R, M, \delta, \varepsilon$ be real numbers such that $0 \le R \le M$ and $0 < \delta \le \varepsilon$, then

$$\varphi\Big(R+\frac{\delta}{2}\Big) - \varphi(R) \le \frac{\varphi(M+\varepsilon)-\varphi(M)}{2}.$$

PROOF. Let $R, M, \delta, \varepsilon$ be real numbers such that $0 \le R \le M$ and $0 < \delta \le \varepsilon$. Since φ is increasing, we have

$$\varphi\Big(R+\frac{\delta}{2}\Big) - \varphi(R) \le \varphi\Big(R+\frac{\varepsilon}{2}\Big) - \varphi(R). \tag{6.1}$$

By the convexity of φ and

$$\begin{cases} R+\dfrac{\varepsilon}{2} = \dfrac{\frac{\varepsilon}{2}}{M+\frac{\varepsilon}{2}-R}\Big(M+\dfrac{\varepsilon}{2}\Big) + \Big(1-\dfrac{\frac{\varepsilon}{2}}{M+\frac{\varepsilon}{2}-R}\Big)R \\ M = \dfrac{M-R}{M+\frac{\varepsilon}{2}-R}\Big(M+\dfrac{\varepsilon}{2}\Big) + \Big(1-\dfrac{M-R}{M+\frac{\varepsilon}{2}-R}\Big)R, \end{cases}$$

we get

$$\begin{cases} \varphi(R+\dfrac{\varepsilon}{2}) \le \dfrac{\frac{\varepsilon}{2}}{M+\frac{\varepsilon}{2}-R}\varphi\Big(M+\dfrac{\varepsilon}{2}\Big) + \Big(1-\dfrac{\frac{\varepsilon}{2}}{M+\frac{\varepsilon}{2}-R}\Big)\varphi(R) \\ \varphi(M) \le \dfrac{M-R}{M+\frac{\varepsilon}{2}-R}\varphi\Big(M+\dfrac{\varepsilon}{2}\Big) + \Big(1-\dfrac{M-R}{M+\frac{\varepsilon}{2}-R}\Big)\varphi(R). \end{cases}$$

Adding these inequalities, we obtain

$$\varphi\Big(R+\frac{\varepsilon}{2}\Big) - \varphi(R) \le \varphi\Big(M+\frac{\varepsilon}{2}\Big) - \varphi(M). \tag{6.2}$$

Next, by the convexity of φ, we get

$$\begin{aligned} \varphi\Big(M+\frac{\varepsilon}{2}\Big) &= \varphi\Big(\frac{1}{2}(M+\varepsilon)+\frac{1}{2}M\Big) \\ &\le \frac{1}{2}\varphi(M+\varepsilon)+\frac{1}{2}\varphi(M), \end{aligned}$$

and hence

$$\varphi\Big(M+\frac{\varepsilon}{2}\Big) - \varphi(M) \le \frac{\varphi(M+\varepsilon)-\varphi(M)}{2}. \tag{6.3}$$

Therefore (6.1), (6.2) and (6.3) yield

$$\varphi\Big(R+\frac{\delta}{2}\Big) - \varphi(R) \le \frac{\varphi(M+\varepsilon)-\varphi(M)}{2}. \quad \square$$

THEOREM 7. Let E be a normed linear space and let $\varphi : [0,\infty) \to [0,\infty)$ be a function such that it is uniformly convex on $[0,\infty)$ and $\varphi(0) = 0$. If for some positive real number c, the modulus of convexity δ satisfies $\delta(\varepsilon) \ge c\varphi(\varepsilon)$ for any $\varepsilon \in [0,2]$ and

$$\varliminf_{\substack{s\to\infty \\ t\downarrow 0}} \big(\varphi(t)\varphi(s) - \varphi(ts)\big) \ge 0,$$

then $\varphi(\|\cdot\|)$ is uniformly convex on E.

PROOF. Let c be a positive real number such that $\delta(\varepsilon) \geq c\varphi(\varepsilon)$ for any $\varepsilon \in [0,2]$ and let

$$\varliminf_{\substack{s\to\infty \\ t\downarrow 0}} \Big(\varphi(t)\varphi(s) - \varphi(ts)\Big) \geq 0 \tag{6.4}$$

be satisfied. Let $x, y \in {}^*E$ such that $x \not\simeq y$. If $\|x\| \not\simeq \|y\|$, the uniform convexity of φ yields

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \leq \varphi\Big(\frac{\|x\|+\|y\|}{2}\Big) \lessapprox \frac{\varphi(\|x\|)+\varphi(\|y\|)}{2}.$$

Next suppose $\|x\| \simeq \|y\|$. If $\|x\|$ and $\|y\|$ are finite, by the uniform convexity of E, we have $\|\frac{x+y}{2}\| \lessapprox \frac{\|x\|+\|y\|}{2}$. Hence we obtain

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \lessapprox \varphi\Big(\frac{\|x\|+\|y\|}{2}\Big) \leq \frac{\varphi(\|x\|)+\varphi(\|y\|)}{2}.$$

Let $\|x\|$ and $\|y\|$ be infinite. Without loss of generality we may assume $\|x\| \leq \|y\|$. Put

$$M = \|x\|, M+\varepsilon = \|y\|, R = \Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\| \text{ and } R + \frac{\delta}{2} = \Big\|\frac{x+y}{2}\Big\|.$$

Since $\delta \leq \varepsilon$ and $R \leq M$, we have

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) - \varphi\Big(\Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\|\Big) \leq \frac{\varphi(\|y\|) - \varphi(\|x\|)}{2} \tag{6.5}$$

by Lemma 1. If we prove

$$\varphi\Big(\Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\|\Big) \lessapprox \varphi(\|x\|) \tag{6.6}$$

then this inequality and (6.5) yield

$$\varphi\Big(\Big\|\frac{x+y}{2}\Big\|\Big) \lessapprox \frac{\varphi(\|x\|)+\varphi(\|y\|)}{2},$$

which completes the proof. Suppose (6.6) is false. Then the convexity of φ, $\varphi(0) = 0$ and $\varphi(t) \geq 0$ for all $t \geq 0$ yield

$$\varphi\Big(\Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\|\Big) \leq \frac{\Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\|}{\|x\|}\varphi(\|x\|) \leq \varphi(\|x\|)$$

and hence

$$\frac{\Big\|\frac{x + \frac{\|x\|}{\|y\|}y}{2}\Big\|}{\|x\|}\varphi(\|x\|) \simeq \varphi(\|x\|).$$

On the other hand, $\delta(\varepsilon) \geq c\varphi(\varepsilon)$ for any $\varepsilon \in [0,2]$ yields

$$1 - \Big\|\frac{\frac{x}{\|x\|} + \frac{y}{\|y\|}}{2}\Big\| \geq c\varphi\Big(\Big\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\Big\|\Big).$$

So we have

$$\varphi(\|x\|) \geq \frac{\left\|\frac{x+\frac{\|x\|}{\|y\|}y}{2}\right\|}{\|x\|}\varphi(\|x\|) + c\varphi\Big(\Big\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\Big\|\Big)\varphi(\|x\|)$$
$$\simeq \varphi(\|x\|) + c\varphi\Big(\Big\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\Big\|\Big)\varphi(\|x\|).$$

If $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \gtrsim 0$, it is clear that $\varphi\Big(\Big\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\Big\|\Big)\varphi(\|x\|) \gtrsim 0$. If $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| \simeq 0$, then by (6.4), we get

$$\varphi\Big(\Big\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\Big\|\Big)\varphi(\|x\|) \gtrsim \varphi\Big(\Big\|x - \frac{\|x\|}{\|y\|}y\Big\|\Big) \gtrsim 0.$$

Hence we have $\varphi(\|x\|) \gtrsim \varphi(\|x\|)$, which is a contradiction. Therefore (6.6) is valid. □

The following is due to Xu [9]. In his paper, he wrote $p > 1$, but if $1 < p < 2$, there exists no normed linear space such that $\|\cdot\|^p$ is uniformly convex on the whole space.

THEOREM 8 (Xu). Let $p \geq 2$ be a fixed real number. Let E be a normed linear space. Then the following are equivalent;

(i) there exists a constant $c > 0$ such that $\delta(\varepsilon) \geq c \cdot \varepsilon^p$ for all $0 \leq \varepsilon \leq 2$,

(ii) the functional $\|\cdot\|^p$ is uniformly convex on E,

(iii) there exists a constant $d > 0$ such that

$$\|\lambda x + (1-\lambda)y\|^p + \lambda(1-\lambda)d\|x-y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p$$

for all $x, y \in E$ and $0 \leq \lambda \leq 1$.

PROOF. (i) $\Rightarrow$ (ii). Put $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = t^p$ for $t \geq 0$. It is easy to see that φ is uniformly convex on $[0,\infty)$, $\varphi(0) = 0$, and $\delta(\varepsilon) \geq c\varphi(\varepsilon)$ for all $0 \leq \varepsilon \leq 2$. The definition of φ implies $\varphi(t)\varphi(s) - \varphi(ts) = 0$ for all $t, s \geq 0$. Hence, by our theorem, $\|\cdot\|^p$ is uniformly convex on E.

(ii) $\Rightarrow$ (iii). Let x and y be any elements of $^*E$ such that $x \neq y$ and let λ be any element of $^*(0,1)$. Since $\|\cdot\|^p$ is uniformly convex, by Proposition 1,

$$\frac{\left\|\lambda\frac{x}{\|x-y\|} + (1-\lambda)\frac{y}{\|x-y\|}\right\|^p}{\lambda(1-\lambda)} \lesssim \frac{\lambda\left\|\frac{x}{\|x-y\|}\right\|^p + (1-\lambda)\left\|\frac{y}{\|x-y\|}\right\|^p}{\lambda(1-\lambda)}.$$

By the transfer principle, there exists a standard positive real number d such that

$$\frac{\left\|\lambda\frac{x}{\|x-y\|} + (1-\lambda)\frac{y}{\|x-y\|}\right\|^p}{\lambda(1-\lambda)} + d \leq \frac{\lambda\left\|\frac{x}{\|x-y\|}\right\|^p + (1-\lambda)\left\|\frac{y}{\|x-y\|}\right\|^p}{\lambda(1-\lambda)}$$

for all $x, y \in E$ and $\lambda \in (0,1)$, i.e.,

$$\|\lambda x + (1-\lambda)y\|^p + \lambda(1-\lambda)d\|x-y\|^p \le \lambda\|x\|^p + (1-\lambda)\|y\|^p$$

for all $x, y \in E$ and $0 \le \lambda \le 1$.

(iii) $\Rightarrow$ (i). Let x and y be any elements of ${}^*S_E(1)$ with $x \neq y$. By (iii), we have

$$\left\|\frac{x+y}{2}\right\|^p + \frac{1}{4}d\|x-y\|^p \le 1,$$

and hence

$$\frac{1}{\|x-y\|^p} - \frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|^p \ge \frac{1}{4}d. \tag{6.7}$$

We claim that there exists a standard positive real number c such that

$$\frac{1}{\|u-v\|^p} - \frac{1}{\|u-v\|^p}\left\|\frac{u+v}{2}\right\| \ge c$$

for all $u, v \in S_E(1)$. Suppose not, i.e., there exist $x, y \in {}^*S_E(1)$ such that

$$\frac{1}{\|x-y\|^p} \simeq \frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|.$$

For any standard natural number n, we have

$$\begin{aligned}\left|\frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|^n - \frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|^{n+1}\right| &= \left\|\frac{x+y}{2}\right\|^n\left|\frac{1}{\|x-y\|^p} - \frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|\right| \\ &\simeq 0.\end{aligned}$$

Hence we obtain

$$\frac{1}{\|x-y\|^p} \simeq \frac{1}{\|x-y\|^p}\left\|\frac{x+y}{2}\right\|^n$$

for any standard natural number n, which contradicts (6.7). Therefore (i) is valid. □

The dual version of Theorem 7 is the following.

THEOREM 9. Let E be a normed linear space and let $\varphi : [0,\infty) \to [0,\infty)$ be a function such that it is uniformly smooth on $[0,\infty)$, $\varphi(0) = 0$ and $\varphi'(0) = 0$. If for some positive real number c, the modulus of smoothness ρ satisfies $\rho(\tau) \le c\varphi(\tau)$ for all $\tau > 0$ and

$$\lim_{\substack{a \downarrow 0 \\ A \to \infty}} \left|\varphi'(A)\frac{\varphi(\frac{a}{A})}{\frac{a}{A}}\right| = 0,$$

then $\varphi(\|\cdot\|)$ is uniformly smooth on E. Moreover, if φ is convex then $\varphi(\|\cdot\|)$ is uniformly Fréchet differentiable on E.

PROOF. Let c be a positive real number such that $\rho(\tau) \leq c\varphi(\tau)$ for all $\tau > 0$, and let

$$\lim_{\substack{a \downarrow 0 \\ A \to \infty}} \left| \varphi'(A) \frac{\varphi(\frac{a}{A})}{\frac{a}{A}} \right| = 0 \tag{6.8}$$

be satisfied. Let $x \in {}^*E$, $u \in {}^*S_E(1)$ and $t, s \in {}^*\mathbb{R} \setminus \{0\}$ with $t \simeq 0$ and $s \simeq 0$. First we assume $x = o$. Then we get

$$\begin{aligned} \frac{\varphi(\|x+tu\|) - \varphi(\|x\|)}{t} &= \frac{\varphi(t) - \varphi(0)}{t} \\ &\simeq \varphi'(0) \\ &= 0. \end{aligned}$$

Hence we obtain
$$\frac{\varphi(\|x+tu\|) - \varphi(\|x\|)}{t} \simeq \frac{\varphi(\|x+su\|) - \varphi(\|x\|)}{s}.$$
Next we assume $x \neq o$. Then we get

$$\begin{aligned} &\frac{\varphi(\|x+tu\|) - \varphi(\|x\|)}{t} - \frac{\varphi(\|x+su\|) - \varphi(\|x\|)}{s} \\ &= \frac{\varphi(\|x+tu\|) - \varphi(\|x\|)}{\|x+tu\| - \|x\|} \frac{\|x+tu\| - \|x\|}{t} - \frac{\varphi(\|x+su\|) - \varphi(\|x\|)}{\|x+su\| - \|x\|} \frac{\|x+su\| - \|x\|}{s} \\ &\simeq \varphi'(\|x\|)\Big(\frac{\|x+tu\| - \|x\|}{t} - \frac{\|x+su\| - \|x\|}{s}\Big). \end{aligned}$$

If $\|x\|$ is finite, $\varphi'(\|x\|)$ is finite and hence we can derive

$$\frac{\varphi(\|x+tu\|) - \varphi(\|x\|)}{t} \simeq \frac{\varphi(\|x+su\|) - \varphi(\|x\|)}{s}$$

from the uniform smoothness of E. So we may assume that $\|x\|$ is infinite. Let $\alpha = \max\{|t|, |s|\}$. Since $\rho(\tau) \leq c\varphi(\tau)$ for all $\tau > 0$, we have

$$\frac{\|\frac{x+\alpha u}{\|x\|}\| + \|\frac{x-\alpha u}{\|x\|}\|}{2} - 1 \leq c\varphi\Big(\frac{\alpha}{\|x\|}\Big),$$

i.e.,

$$\frac{\|x+\alpha u\| + \|x-\alpha u\| - 2\|x\|}{\alpha} \leq 2c \frac{\varphi(\frac{\alpha}{\|x\|})}{\frac{\alpha}{\|x\|}}.$$

So (6.8) yields

$$\begin{aligned} \left|\varphi'(\|x\|)\Big(\frac{\|x+tu\| - \|x\|}{t} - \frac{\|x+su\| - \|x\|}{s}\Big)\right| &\leq \left|\varphi'(\|x\|)\frac{\|x+\alpha u\| + \|x-\alpha u\| - 2\|x\|}{\alpha}\right| \\ &\leq 2c\left|\varphi'(\|x\|)\frac{\varphi(\frac{\alpha}{\|x\|})}{\frac{\alpha}{\|x\|}}\right| \\ &\simeq 0. \end{aligned}$$

Therefore we obtain

$$\frac{\varphi(\|x+tu\|)-\varphi(\|x\|)}{t} \simeq \frac{\varphi(\|x+su\|)-\varphi(\|x\|)}{s},$$

which implies that $\varphi(\|\cdot\|)$ is uniformly smooth on E. □

The following is also due to Xu [9]. In his paper, he wrote $q > 1$, but if $q > 2$, there exists no normed linear space such that $\|\cdot\|^q$ is uniformly Fréchet differentiable on the whole space.

THEOREM 10 (Xu). Let q be a fixed real number with $1 < q \leq 2$. Let E be a normed linear space. Then the following are equivalent;

(i) there exists a constant $c > 0$ such that $\rho(\tau) \leq c \cdot \tau^q$ for all $\tau > 0$,

(ii) the functional $\|\cdot\|^q$ is uniformly Fréchet differentiable on E,

(iii) there exists a constant $d > 0$ such that

$$\|\lambda x + (1-\lambda)y\|^q + \lambda(1-\lambda)d\|x-y\|^q \geq \lambda\|x\|^q + (1-\lambda)\|y\|^q$$

for all $x, y \in E$ and $0 \leq \lambda \leq 1$.

PROOF. (i) $\Rightarrow$ (ii). Put $\varphi : [0,\infty) \to [0,\infty)$ by $\varphi(t) = t^q$ for $t \geq 0$. It is easy to see that φ is uniformly smooth on $[0,\infty)$, $\varphi(0) = 0$ and $\varphi'(0) = 0$. The inequality $\rho(\tau) \leq c \cdot \tau^q$ for all $\tau > 0$ implies that $\rho(\tau) \leq c\varphi(\tau)$ for all $\tau > 0$. By the definition of φ, we have

$$\lim_{\substack{a \downarrow 0 \\ A \to \infty}} \left| \varphi'(A) \frac{\varphi(\frac{a}{A})}{\frac{a}{A}} \right| = \lim_{a \downarrow 0} q a^{q-1} = 0.$$

So, by our theorem, $\|\cdot\|^q$ is uniformly Fréchet differentiable on E.

(ii) $\Rightarrow$ (iii). Let M be any infinite element of ${}^*\mathbb{R}_+$. Let x and y be any elements of ${}^*E$ with $x \neq y$. Then $\frac{x}{M\|x-y\|} \simeq \frac{y}{M\|x-y\|}$. Let $\lambda \in {}^*(0,1)$. Since $\|\cdot\|^q$ is uniformly Fréchet differentiable on E, by Proposition 2, we have

$$\frac{\|\frac{\lambda x}{M\|x-y\|} + \frac{(1-\lambda)y}{M\|x-y\|}\|^q}{\lambda(1-\lambda)} \simeq \frac{\lambda\|\frac{x}{M\|x-y\|}\|^q + (1-\lambda)\|\frac{y}{M\|x-y\|}\|^q}{\lambda(1-\lambda)}.$$

Hence we obtain

$$\frac{\lambda\|x\|^q + (1-\lambda)\|y\|^q - \|\lambda x + (1-\lambda)y\|^q}{\lambda(1-\lambda)} \frac{1}{M^q\|x-y\|^q} \simeq 0.$$

So we have

$$\frac{\lambda\|x\|^q + (1-\lambda)\|y\|^q - \|\lambda x + (1-\lambda)y\|^q}{\lambda(1-\lambda)} \frac{1}{M^q\|x-y\|^q} \leq 1,$$

i.e.,

$$\lambda\|x\|^q + (1-\lambda)\|y\|^q - \|\lambda x + (1-\lambda)y\|^q \leq M^q\lambda(1-\lambda)\|x-y\|^q.$$

Therefore, by the transfer principle, (iii) is valid.

(iii) $\Rightarrow$ (i). Let $x \in S_E(1)$ and $u \in E \setminus \{o\}$. Since $\|x+u\| \geq 1$ or $\|x-u\| \geq 1$, we have $\|x+u\| + \|x-u\| \leq \|x+u\|^q + \|x-u\|^q$. From (iii), we can derive

$$\begin{aligned}\frac{\|x+u\|+\|x-u\|}{2} &\leq \frac{\|x+u\|^q+\|x-u\|^q}{2} \\ &\leq \left\|\frac{(x+u)+(x-u)}{2}\right\|^q + \frac{1}{2}\cdot\frac{1}{2}\cdot d\|(x+u)-(x-u)\|^q \\ &= \|x\|^q + 2^{q-2}d\|u\|^q.\end{aligned}$$

Hence we obtain

$$\frac{\|x+u\|+\|x-u\|}{2} - 1 \leq 2^{q-2}d\|u\|^q,$$

which implies that $\rho(\tau) \leq 2^{q-2}d\cdot\tau^q$ for all $\tau > 0$. $\square$

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