数理解析研究所講究録
第 861 巻 1994 年 77-94

A generalized convexity and a homotopy approach to a quasiconvex minimization

Phan Thien Thach and Masakazu Kojima (小島　政和)

Tokyo Institute of Technology

Abstract. We present a variational condition for a global optimality to a class of quasiconvex minimization problems where vanishing gradients are not enough to the optimality. Using this condition we show that a quasiconvex minimization problem, whose local minima may not global ones, can be reduced to a generalized equation, and can, in principle, be solved by homotopy approaches.

Key Words. Generalized convexity, quasiconvexity, variational criterion, homotopy.

1 Introduction

Mathematical programming problems pioneered by the contributions of Dantzig in the 1940's became one of central areas in Economic Theory with broad applications. In convex programs we can use the first-order optimality conditions (Karush-Kuhn-Tucker conditions) by which we can obtain a big connection to other important problems such as complementarity, variational inequality, or fixed point problems (see, e.g., Refs.[1,4,9,12,16]). Many powerful computational methods can be transferred between these problems. For example, the recent interior point approach for mathematical programs can be applied to an wide class of complementarity problems (Ref.[10]). With appropriate relaxations of convexity assumptions one can extend the first-order optimality conditions to more general problems (see, e.g., Refs.[12,16]). A basic obstacle in quasiconvex minimization problems is the fact that the usual gradients are not able to characterize the increasing directions at critical points. In the other words, a vanishing gradient is not sufficient for the optimality.

In this paper, using a generalized quasisubdifferential we present a first-order optimality condition for a quasiconvex minimization problem. By the optimality condition we can reduce the optimization problem to a generalized variational inequality. Therefore we can, in principle, solve a quasiconvex minimization problem which is multi-extremal, by path-following methods, provided that quasisubdifferentials are computable. For an illustration we present a homotopy inclusion which defines a monotone curve from an interior feasible solution to the set of optimal solutions.

In Section 2 we present a generalized convexity and a connection to the previous results. In Section 3 we present an optimality condition and a reduction to a generalized variational inequality. In Section 4 we apply a homotopy approach to the variational inequality, and in Section 5 we draw concluding remarks.

2 A generalized convexity and the explicit quasiconvexity

Let $f : R^n \mapsto R \cup \{\pm\infty\}$ be a function. We call the set $\{x : f(x) < \sup_{x \in R^n} f(x)\}$ the domain of f and denote it by $dom f$:

$$dom f \;=\; \{x : f(x) < \sup_{x \in R^n} f(x)\}.$$

The domain of a quasiconvex function is a convex set. If f is a nonconstant convex function, then $\sup_{x \in R^n} f(x) = +\infty$ [15], hence $dom f$ has the usual meaning:

$$dom f \;=\; \{x : f(x) < +\infty\}.$$

Definition 2.1. A function f is explicitly quasiconvex iff f is quasiconvex and for every $x \in dom f$, $y \in dom f$, $f(x) < f(y)$, $\lambda \in [0,1)$ one has $f(x + \lambda(y - x)) < f(y)$.

The concept of explicit quasiconvexity (sometimes called **strict quasiconvexity,** or **functional quasiconvexity**) was used in many literatures [11,12,16]. It is straightforward to see that any local minimum of an explicitly quasiconvex function is global minimum [12,16]. Let f be a quasiconvex function. We consider the following properties of f:

(A1) Any local minimizer of f on an arbitrary convex set $D \subseteq dom f$ is a global minimizer of f on D;

(A2) Any local minimizer of f on an arbitrary segment $[x, y] \subseteq dom f$ is a global minimizer of f on $[x, y]$;

(A3) Any local minimizer of f on $dom f$ is a global minimizer.

Theorem 2.1. Let f be a quasiconvex function.
(i) Explicit quasiconvexity $\Leftrightarrow$ (A1) $\Leftrightarrow$ (A2) $\Rightarrow$ (A3);
(ii) If f is usc then
Explicit quasiconvexity $\Leftrightarrow$ (A1) $\Leftrightarrow$ (A2) $\Leftrightarrow$ (A3).

In order to prove the theorem we need the following lemma.

Lemma 2.1. If f is usc, quasiconvex and

$$f(0) = min\{f(x) : x \in R^n\},$$

then the property (A3) is equivalent to one of the following assertions:
(A3') f is explicitly quasiconvex;
(A3") For every x such that $\sup_{y \in R^n} f(y) > f(x) > f(0)$, one has

$$f(x) > f(\lambda x) \quad \forall \lambda \in (0,1).$$

Proof. (A3') $\Rightarrow$ (A3): see [12];

(A3) $\Rightarrow$ (A3"): Let x be a vector such that $\sup_{y \in R^n} f(y) > f(x) > f(0)$. Since f satisfies (A3) and $f(x) > f(0)$, x is not a local minimizer, hence

$$x \in cl\{y : f(y) < f(x)\}. \tag{1}$$

Since f is usc, the set $\{y : f(y) < f(x)\}$ is open. From (1) this implies that [15]

$$[0, x) \subseteq int\ cl\{y : f(y) < f(x)\} = \{y : f(y) < f(x)\}.$$

Therefore, $f(x) > f(\lambda x) \quad \forall \lambda \in [0, 1)$.

(A3") $\Rightarrow$ (A3'): Let x and y be vectors in $dom f$ such that:

$$f(x) < f(y) < sup\{f(x) : x \in R^n\}.$$

Suppose that there is $\lambda \in (0, 1)$ such that $f((1-\lambda)x + \lambda y) = f(y)$.Then

$$f((1-\theta)x + \theta y) = f(y) \quad \forall \theta \in [\lambda, 1].$$

Since $\{z : f(z) < f(y)\}$ is open, convex and the intersection of $\{z : f(z) < f(y)\}$ and the segment $[(1-\lambda)x + \lambda y, y]$ is empty, there is a linear function $\langle v, . \rangle$ strictly separating $\{z : f(z) < f(y)\}$ and $[(1-\lambda)x + \lambda y, y]$:

$$\langle v, z \rangle < 1 \quad \forall z : f(z) < f(y) \tag{2}$$

$$\langle v, (1-\theta)x + \theta y \rangle \geq 1 \quad \forall \theta \in [\lambda, 1]. \tag{3}$$

Since $f((1-\theta)x + \theta y) = f(y) > f(x) \geq f(0)$, from (A3") one has

$$f(t((1-\theta)x + \theta y)) < f((1-\theta)x + \theta y) \quad \forall t \in (0, 1)\ \forall \theta \in [\lambda, 1]$$

$$\Rightarrow \langle v, t((1-\theta)x + \theta y) \rangle < 1 \quad \forall t \in (0, 1)\ \forall \theta \in [\lambda, 1].$$

This together with (3) implies that

$$\langle v, (1-\theta)x + \theta y \rangle = 1 \quad \forall \theta \in [\lambda, 1].$$

Therefore the hyperplane $\{z : \langle v, z\rangle = 1\}$ contains the line $\{(1-t)x + ty : t \in R\}$, hence $\langle v, x\rangle = 1$. We arrive at a contradiction with (2) and the fact that $f(x) < f(y)$.□

The proof of Theorem 2.1.

The proof of assertion (i) can be found in [12]. In order to prove assertion (ii) it remains to prove that if f is usc then the property (A3) implies the explicit quasiconvexity. Let f be an usc quasiconvex function satisfying (A3). We shall prove that f is explicitly quasiconvex. Let $x, y \in dom f$ such that $f(x) < f(y)$. Setting

$$g(z) := max\{f(x), f(x+z)\},$$

we have an usc quasiconvex function g satisfying (A3) and

$$g(0) = min\{g(z) : z \in R^n\} = f(x) < f(y) = g(y-x).$$

By Lemma 2.1 this implies that

$$\begin{aligned} & g(y-x) > g(\lambda(y-x)) \quad \forall\lambda \in (0,1) \\ & \Rightarrow f(y) > f(x + \lambda(y-x)) \quad \forall\lambda \in (0,1). \end{aligned}$$

This completes the proof.□

A quasiconvex function satisfying property (A3) will be called **essentially quasiconvex.** The essential quasiconvexity is slightly weaker than the explicit quasiconvexity, and in the class of usc quasiconvex functions they are equivalent.

3 A generalized Karush-Kuhn-Tucker condition

Let f be a subdifferentiable closed convex function and D be a closed convex set. It is well known that the following criterion

$$0 \in \partial f(x) + N(x, D) \tag{4}$$

is necessary and sufficient for the optimality of x. In a quasiconvex minimization, where f is quasiconvex and $\partial f(\cdot)$ stands for Clarke's generalized subdifferentials, this criterion is not sufficient for the optimality. However we shall see that if we replace the set of subdifferentials by the set of the so-called quasisubdifferentials, then the condition (4) is sufficient for the optimality even in a general quasiconvex minimization.

Definition 3.1. **let $f : R^n \mapsto R \cup \{\pm\infty\}$ be a function. A vector v is called a quasisubdifferential of f at x if**

$$\langle v, x\rangle = 1 \quad \text{and} \quad f(x) \leq f(y) \ \ \forall y : \langle v, y\rangle \geq 1.$$

Denote by $\partial^H f(x)$ the set of quasisubdifferentials of f at x. If $\partial^H f(x) \neq \emptyset$ then f is quasisubdifferentiable at x, and if $\partial^H f(x) \neq \emptyset \ \ \forall x \in X$ then f is quasisubdifferentiable on X.

It is obvious that

$$\begin{aligned} & v \in \partial^H f(x) \\ \Leftrightarrow \ & \langle v, x\rangle = 1 \ \ and \ \ f(x) = inf\{f(y) : \langle v, y\rangle \geq 1\}. \end{aligned}$$

There are close relationships between the usual subdifferentials and the quasisubdifferentials. If f is a finite convex function, then for any $x : f(x) > f(0)$ the set $\partial^H f(x)$ is nonempty, compact, convex and [20]

$$\partial^H f(x) = \left\{ \frac{v}{\langle v, x\rangle} : v \in \partial f(x) \right\}. \tag{5}$$

Example 3.1. Let $h_i : R \to R \cup \{\pm\infty\}$ $(i = 1, ..., n)$ be usc increasing functions. Consider the function $f : R^n \to R \cup \{\pm\infty\}$ defined as follows

$$f(x) = max\{h_i(0), h_i(x_i) : i = 1, ..., n\} \quad \forall x = (x_1, ...x_n) \in R^n.$$

It is easy to see that f is usc, essentially quasiconvex. For any x : $\infty > f(x) > f(0)$ denote by $I(x)$ the set of indices i such that $f(x) = h_i(x_i)$. Then,

$$\partial^H f(x) = \left\{ \sum_{i \in I(x)} \frac{\theta_i}{x_i} e_i : \sum_{i \in I(x)}^{n} \theta_i = 1, \theta_i \geq 0 \; i \in I(x) \right\},$$

where e_i is the i-th unit vector in R^n.

We consider now the following program

$$min\{f(x) : x \in D\}, \tag{6}$$

where f is quasiconvex, and D is a closed convex set. We suppose a technical assumption that

$$f(0) < \min\{f(x) : x \in D\} < \sup\{f(x) : x \in R^n\} \tag{7}$$

and denote

$$kerf = \{x : f(x) \leq f(0)\}.$$

If f is usc then f is quasisubdifferentiable on $R^n \setminus kerf$, hence on D, and furthermore $\partial^H f(x)$ is convex, compact for every $x \in R^n \setminus kerf$ [20].

Theorem 3.1. If f is usc, essentially quasiconvex then a vector $x \in D$ is optimal to problem (9) if and only if

$$0 \in \partial^H f(x) + N(x, D). \tag{8}$$

If f is a general quasiconvex function, criterion (8) is still sufficient for the global optimality (although it is not of a second-order type). With a coercivity condition and the lower semi-continuity of f, program (6) has an optimal solution, hence criterion (8) is satisfied at at least an optimal solution $x \in D$ [20]. But the above theorem show further that if f is essentially quasiconvex then condition (8) is satisfied at all optimal solutions.

The proof of Theorem 3.1.

Since criterion (8) is sufficient for the optimality in a general case (see Theorem 4.1, [20]), it remains to prove that it is necessary. Suppose that $x \in D$ is an optimal solution. Then, $\{y : f(y) < f(x)\} \cap D = \emptyset$. There is a linear function $\langle v, . \rangle$ separating $\{y : f(y) < f(x)\}$ and D:

$$\langle v, y \rangle < 1 \quad \forall y : f(y) < f(x) \tag{9}$$

$$\langle v, y \rangle \geq 1 \quad \forall y \in D. \tag{10}$$

Since $x \in D$, one has $\langle v, x \rangle \geq 1$. On the other hand, $x \in cl\{y : f(y) < f(x)\}$, because f is essentially quasiconvex. So, $\langle v, x \rangle = 1$. This together with (9) follows that

$$f(x) = inf\{f(y) : \langle v, x \rangle \geq 1\}.$$

Therefore, $v \in \partial^H f(x)$. Furthermore, $v \in -N(x, D)$, because of (10) and $\langle v, x \rangle = 1$. This completes the proof.□

Suppose now that D is given by the following inequalities

$$D = \{x : f_i(x) \leq 0 \quad i = 1, ..., m\},$$

where $f_i : R^n \to R$ are convex functions. Assume that Slater's condition is satisfied, i.e.,

$$\exists x : \quad f_i(x) < 0 \ \ \forall i = 1, ..., m.$$

It is well known that for every $x \in D$ there are nonnegative numbers $\lambda_1, ..., \lambda_m$ such that [15]

$$N(x, D) = cone\{\sum_{i=1}^{m} \lambda_i \partial f_i(x)\}, \quad \lambda_i f_i(x) = 0 \quad i = 1, ..., m.$$

From Theorem 3.1 we can obtain a generalized Karush-Kuhn-Tucker condition

$$0 \in \partial^H f(x) + \sum_{i=1}^{m} \lambda_i \partial f_i(x) \tag{11}$$

$$\lambda_i f_i(x) = 0 \quad i = 1, ..., m \tag{12}$$

$$\lambda_i \geq 0 \ \ i = 1, ..., m, \quad f_i(x) \leq 0 \ \ i = 1, ..., m. \tag{13}$$

Example 3.2. Suppose that f is given as in Example 3.1 and f_i ($\forall i$) is differentiable. Then the condition (11)-(13) becomes the following equations and inequations:

$$0 = \sum_{i=1}^{n} \frac{\theta_i}{x_i} e_i + \sum_{i=1}^{m} \lambda_i \nabla f_i(x) \tag{14}$$

$$\lambda_i f_i(x) = 0, \quad f_i(x) \leq 0, \quad \lambda_i \geq 0 \quad i = 1, ..., m \tag{15}$$

$$\sum_{i=1}^{n} \theta_i = 1, \quad \theta_i \geq 0, \quad \theta_i(f(x) - h_i(x_i)) = 0 \quad i = 1, ..., n \tag{16}$$

where $\nabla f_i(x)$ denotes the gradient of f_i at x. Note that the system (14)-(16) is different from K-K-T condition even when functions h_i are differentiable. Indeed, K-K-T condition gives us the system, where (14) is replaced by

$$0 = \sum_{i=1}^{n} \theta_i \nabla h_i(x_i) + \sum_{i=1}^{m} \lambda_i \nabla f_i(x).$$

At a "stationary" point x such that $\nabla h_i(x_i) = 0$ ($\forall i = 1, ..., n$), K-K-T condition is satisfied, but x may not be a minimum, because h_i may be nonconvex. Therefore the system (14)-(16) excludes any stationary point, which is not a minimum.

In order to convert the condition (11)-(13) to a generalized equation we can use a simple reduction in [9]. By setting $\lambda_i^+ = max(0, \lambda_i)$, $\lambda_i^- = max(0, -\lambda_i)$ $i = 1, ..., m$ the system (11)-(13) is transformed into the following generalized equation

$$0 \in H(x, \lambda_1, ..., \lambda_m), \tag{17}$$

where H is a point-to-set mapping from R^{n+m} to R^{n+m} defined as follows

$$H(x, \lambda_1, ..., \lambda_m) = \{(v, \lambda_1^- - f_1(x), ..., \lambda_m^- - f_m(x)) : v \in \partial^H f(x) + \sum_{i=1}^{m} \lambda_i^+ \partial f_i(x)\}.$$

Using the generalized K-K-T condition (8) we present now a generalized variational inequality.

Theorem 3.2. **If $f \in \Psi$ is usc, essentially quasiconvex, then a vector $x \in D$ is optimal to problem (6) if and only if x is a solution of a generalized**

variational inequality

$$\min_{u \in \partial^H f(x)} \langle u, x-y \rangle \leq 0 \quad \forall y \in D. \tag{18}$$

Proof. One has

$$\begin{aligned}
& x \ \ solves \ \ (6) \\
\Leftrightarrow \ & 0 \in \partial^H f(x) + N(x,D) \quad (by\ Theorem 3.1) \\
\Leftrightarrow \ & \partial^H f(x) \cap -N(x,D) \neq \emptyset \\
\Leftrightarrow \ & \min_{u \in \partial^H f(x)} \sup_{y \in D} \langle u, x-y \rangle \leq 0 \quad (since\ \ \partial^H f(x)\ \ is\ \ compact) \\
\Leftrightarrow \ & \sup_{y \in D} \min_{u \in \partial^H f(x)} \langle u, x-y \rangle \leq 0 \quad (since\ \ \partial^H f(x)\ \ is\ \ convex) \\
\Leftrightarrow \ & \min_{u \in \partial^H f(x)} \langle u, x-y \rangle \leq 0 \quad \forall y \in D. \square
\end{aligned}$$

If D is the cone $\{x : x \geq x_0\}$, where x_0 is a vector in R^n, and the quasisubdifferential mapping $x \mapsto \partial^H f(x)$ is singleton on D, then it is easy to check that the variational inequality (18) becomes

$$\langle \partial^H f(x), x - x_0 \rangle = 0, \quad \partial^H f(x) \geq 0,$$

hence the set of optimal solutions is exactly the set of solutions of the following complementarity problem

$$\langle \partial^H f(x), x - x_0 \rangle = 0, \quad \partial^H f(x) \geq 0, \quad x \geq x_0.$$

Note that the quasisubdifferential mapping $x \mapsto \partial^H f(x)$ is not necessarily monotone in this complementarity problem.

4 Homotopy inclusion

In the previous sections we show that if f is usc, then a quasiconvex minimization (6) can be reduced to the variational inequality (18). In this section we present a homotopy inclusion to show that this variational inequality can be solved by path-following methods.

Assume additionally throughout this section that D is bounded. Denote by $\pi(y)$ the projection of vector $y \in R^n$ on D:

$$\pi(y) = argmin\{\| x - y \| : x \in D\}.$$

The variational inequality (18) is equivalent to the following inclusion [4]

$$u \in \partial^H f(\pi(u)) + \pi(u). \tag{19}$$

In the sequel we shall prove that $\partial^H f(x)$ ($\forall x \in D$) is contained in a compact set K (not depending on x) and the quasisubdifferential mapping $x \mapsto \partial^H f(x)$ is hemi-continuous.

From (7) it follows that there is α such that

$$\inf\{f(x) : x \in D\} > \alpha > f(0).$$

Set $K := \{y : f(y) < \alpha\}^0$. Since $0 \in int\{y : f(y) < \alpha\}$, K is compact.

Theorem 4.1. If f is usc, then $\partial^H f(x) \subseteq K$ for all $x \in D$.

Proof. Let $x \in D$. Since f is usc, one has [21]

$$\{v : f^H(v) \leq -f(x)\} = \{y : f(y) < f(x)\}^0.$$

So,

$$\begin{aligned} \partial^H f(x) &= \{v : \langle v, x\rangle = 1,\ f^H(v) = -f(x)\} \\ &\subseteq \{v : f^H(v) \leq -f(x)\} \subseteq \{y : f(y) < f(x)\}^0. \end{aligned}$$

Since $\alpha < f(x)$, one has $\{y : f(y) < \alpha\} \subseteq \{y : f(y) < f(x)\}$, hence $\{y : f(y) < f(x)\}^0 \subseteq K$. Therefore, $\partial^H f(x) \subseteq K$.$\square$

Theorem 4.2. If f is continuous on D, then $\partial^H f(.)$ is hemi-continuous on D.

Proof. Let $x \in D$ and suppose that $\{x_n\} \to x$, $\{v_n\} \to v$ and $v_n \in \partial^H f(x_n)$. One has

$$\begin{aligned} & v_n \in \partial^H f(x_n) \\ \Leftrightarrow \quad & \langle v_n, x_n \rangle = 1, \quad f^H(v_n) \leq -f(x_n). \end{aligned}$$

Since f is usc, f^H is lsc [19]. This together with the continuity of f implies

$$\langle v, x \rangle = 1, \ f(v) \leq \liminf_{n \to \infty} f^H(v_n) \leq \lim_{n \to \infty} -f(x_n) = -f(x).$$

This means $v \in \partial^H f(x)$. Therefore $\partial^H f(.)$ is hemi-continuous.□

Now we discuss on a homotopy of the inclusion (19) when f is essentially quasiconvex. If the function f is convex, or in the other words program (12) is a convex program, then we can use the usual subdifferentials in the inclusion (19):

$$u - \pi(u) \in \partial f(\pi(u)).$$

Denote by $U(t)$ ($t \in [0,1]$) the set of solutions of the linear homotopy inclusion

$$u - \pi(u) \in t\partial f(\pi(u)) + (1-t)(\pi(u) - x_0),$$

where $x_0 \in D$. Then the path $\{\pi(U(t)), t \in [0,1]\}$ from x_0 to the set of optimal solutions is monotone, i.e., $\pi(U(t))$ is singleton for any $t \in [0,1)$, because $\pi(U(t))$ contains a unique minimizer of the strictly convex function $tf(x)+(1-t) \parallel x-x_0 \parallel^2 /2$ on D. However if f is essentially quasiconvex, but no longer convex, then the function $tf(x) + (1-t) \parallel x - x_0 \parallel^2 /2$ may not be quasiconvex. Therefore we have to use another homotopy to obtain a monotone path in the case where f is essentially quasiconvex. The homotopy will be constructed based on a retraction. A general retraction was proposed by Yamamoto[25] to solve an equation $q(x) = 0$. For a family $\{D_t : t \in [0,1]\}$ of compact convex subsets of R^n a continuous mapping $r_t : R^n \to R^n$ is a retraction onto D_t if $r_t(R^n) \subseteq D_t$ and $r_t(x) = x \ \ \forall x \in D_t$ [25]. The projection mapping $\pi_t(.)$ onto D_t is, of course, a retraction. We shall use this type of retractions and a family $\{D_t : t \in [0,1]\}$ which is a continuation from the set of a singleton vector $x_0 \in intD$ to D. Suppose that

$$D = \{x : d(x) \leq 0\},$$

where $d(.)$ is a finite convex function and the constraint qualification

$$\exists x : \quad d(x) < 0$$

is satisfied. Let x_0 be an arbitrary vector in $dom f \cap intD$. For every $t \in [0,1]$, set

$$D_t = \{x : (1-t) \parallel x - x_0 \parallel^2 + td(x) \leq 0\}.$$

It is obvious that

$$\begin{aligned} &D_0 = \{x_0\} \\ &D_1 = D \\ &D_0 \subseteq D_t \subseteq D_{t'} \subseteq D \ \ \forall 0 \leq t \leq t' \leq 1. \end{aligned}$$

Since $(1-t) \parallel x - x_0 \parallel^2 + td(x)$ is strictly convex for all $t \in [0,1)$, the set D_t $(t \in [0,1))$ is strictly convex, i.e.,

$$x_1 \in D_t, \ x_2 \in D_t, \ x_1 \neq x_2, \ \lambda \in (0,1) \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in intD_t.$$

Now we prove the uniqueness of a minimizer of an essentially quasiconvex function on a strictly convex set.

Theorem 4.3. Let $f \in \Psi$ be an usc function. Then f is essentially quasiconvex if and only if for every strictly convex set C such that

$$f(0) < \inf\{f(x) : x \in C\} < \sup\{f(x) : x \in R^n\}. \tag{20}$$

f **has at most a minimizer on** C.

Proof. Suppose that f is essentially quasiconvex. If there are a strictly convex set C satisfying (20) and two minimizer $x_1, x_2, x_1 \neq x_2$ of f on C then $(x_1 + x_2)/2$ is a minimizer on C as well. From (20) it follows that

$$\sup_{x \in R^n} f(x) > f((x_1 + x_2)/2) > f(0).$$

Therefore by lemma 2.1 one has

$$f((x_1 + x_2)/2) > f(\lambda(x_1 + x_2)/2) \;\; \forall \lambda \in [0, 1).$$

We arrive at a contradiction with the fact that $(x_1 + x_2)/2 \in intC$ and $(x_1 + x_2)/2$ is a minimizer on C. Now suppose conversely that f is not essentially quasiconvex. By definition there is a local minimizer, x, of f on $dom f$ which is not a global minimizer. So there is a ball C centered at x such that $C \cap \{y : f(y) < f(x)\} = \emptyset$. This implies that x is a minimizer of f on the strictly convex set C. Since $f(\lambda x) \leq f(x) \;\; \forall \lambda \in [0, 1]$, this implies that f has more than one minimizer on the strictly convex set C.$\Box$

Let π_t be the projection mapping on D_t. We consider the following homotopy of the inclusion (19):

$$u \in E_t(u) := \partial^H f(\pi_t(u)) + \pi_t(u). \tag{21}$$

Since $x_0 \in intD$, the mapping $\pi_t(u)$ is hemi-continuous w.r.t. $(t, u) \in [0, 1] \times R^n$. Therefore, if f is continuous on D then by Theorem 4.2 $\partial^H f(.)$ is hemi-continuous w.r.t. x on D and hence $E_t(u)$ is hemi-continuous w.r.t. (t, u). Set

$$X(t) := \pi_t(U),$$

where U is the set of solutions of (21). If f is essentially quasiconvex then $X(t)$ is exactly the set of minimizer of f on D_t. Therefore, by Theorem 4.3 $X(t)$ is singleton for every $t \in [0, 1)$ because D_t satisfies (20):

$$x_0 \in D_t \Rightarrow D_t \cap dom f \neq \emptyset$$

$$D_t \subseteq D \Rightarrow D_t \cap ker f = \emptyset.$$

Thus, $\{X(t), t \in [0,1]\}$ forms a monotone path from $X(0) = x_0$ to the set of optimal solutions to program (6).

5 Conclusions

Based on a generalized convexity (the essential quasiconvexity) we presented a connection between a mathematical program and a generalized variational inequality. It is understandable that this generalized convexity is a composition of the quasiconvexity and the property "a local minimum is global" (briefly "L=G"). Without property "L=G" we can only convert the mathematical program into a variational inequality, but not vice versa. We also present a homotopy inclusion which defines a monotone curve from a feasible interior solution to the set of optimal solutions.

Similarly as in convex minimization the computation problem of quasisubdifferentials is very important. In certain cases we can relatively easily compute a quasisubdifferential, but in the most general case we have to solve a quasiconvex minimization if we want to compute a quasisubdifferential of a quasiconvex function.

There are relationships between optimality conditions and duality. In principle if we have a K-K-T-type optimal condition, then we can obtain a convex-type duality. An optimal K-K-T vector in a primal problem is an optimal solution in the dual problem and an optimal solution in the primal is an optimal K-K-T vector in the dual.

References

[1] Cottle R.W., Pang J.-S., and Venkateswaran V.,(1989) **Sufficient matrices and the linear complementarity problem**, Linear Algebra and Its Applications, 114/115, 231-249.

[2] Crouzeix J.-P.,(1977) **Conjugacy in Quasi-convex analysis,** Convex Analysis and Its Applications, Lecture Notes in Economics and Math. Systems, Springer-Verlag, Berlin 144, 66-99.

[3] Eaves B.C.,(1971) **Computing Kakutani fixed points,** SIAM J. Appl. Math. 21, 236-244.

[4] Garcia C.B., and Zangwill W.I.,(1981) **Pathways to solution fixed points and equilibria,** Prentice-Hall, Inc. Englewood Cliffs.

[5] Hanson M.A.,(1964) **Bounds for functionally convex optimal control problems,** J. Math. Anal. Appl. 8, 84-89.

[6] Hartman P., and Stampacchia G.,(1966) **On some nonlinear differential-functional equations,**Acta Mathematica 115, 271-310.

[7] Karamardian S.,(1971) **The complementarity problem,**Math. Prog. 2, 107-129.

[8] Kojima M.,(1979) **A complementarity pivoting approach to parametric nonlinear programming,**Math. Oper. Res. 4, 464-477.

[9] Kojima M.,(1980) **Strongly stable stationary solutions in nonlinear programs,** Analysis and computation of fixed points, ed. Robinson S.M., Academic Press, New York, 93-138.

[10] Kojima M., Megiddo N., Noma T., and Yosishe A.,(1991) **A unified approach to interior point algorithms for linear complementarity problems,** Lecture Notes in Computer Science 538, Springer-Verlag.

[11] Mangasarian O.L.,(1965) **Pseudo-convex functions,** SIAM J. Control 3, 281-190.

[12] Mangasarian O.L.,(1969) **Nonlinear programming,** Mc Graw-Hill, New York.

[13] Penot J.-P., and Volle M.,(1990) **On quasi-convex duality,** Math. Oper. Res. 4, 597-625.

[14] Robinson S.M.,(1979) **Generalized equations and their solutions, Part I: basic Theory,** Math. Prog. Studies 10, 128-141.

[15] Rockafellar R.T.,(1970) **Convex analysis,** Princeton, Princeton Academic Press.

[16] Schaible S., and Ziemba W.T.(Editors), (1981) **General concavity in optimization and economics,** Academic Press, New York.

[17] Stoer J., and Witzgall C.,(1970) **Convexity and optimization in finite dimensions I,** Springer-Verlag, Berlin.

[18] Tanaka Y., Fukushima M., and Ibaraki T.,(1989) **On generalized pseudoconvex functions,** J. Math. Anal. Appl. 144, 342-355.

[19] Thach P.T.,(1991) **Quasiconjugates of functions, duality relationship between quasiconvex minimization under a reverse convex constraint and quasiconvex maximization under a convex constraint, and applications,** J. Math. Anal. Appl. 159, 299-322.

[20] Thach P.T.,(1991) **A nonconvex duality with a zero gap and and applications,** (to appear in SIAM on Optimization).

[21] Thach P.T.,(1993) **Global optimality criteria and a duality with a zero gap in nonconvex optimization problems,** SIAM on Math. Anal. 24, 1537-1556.

[22] Tuy H.,(1964) **Sur les inegalites lineaires,** Colloquium Mathematicum 13, 107-123.

[23] Yamamoto Y.,(1983) **A new variable dimension algorithm for the fixed point problem,** Math. Prog. 25, 329-342.

[24] Yamamoto Y.,(1984) **A variable dimension fixed point algorithm and the orientation of simplices,** Math. Prog. 30, 301-312.

[25] Yamamoto Y.,(1984) **A unifying model based on retraction for fixed point algorithms,** Math. Prog. 28, 192-197.