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Periodic Stability of Nonlinear Flexible Systems

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1. Introduction

Let Ω be a bounded domain in a finite dimensional Euclidean space and we consider the class of the flexible systems that can be described by the following second order damped evolution equation in $X := L^2(\Omega)$ with a nonlinear forcing term under a periodic perturbation:

$$\frac{d^2u(t)}{dt^2} + 2\alpha A\frac{du(t)}{dt} + Au(t) = F(u(t)) + w(t), \quad t > 0, \tag{1.1}$$

$$u(0) = u_0, \quad u_t(0) = u_1, \quad w(t+T) = w(t). \tag{1.2}$$

We assume that A is a selfadjoint positive definite operator with dense domain $D(A)$ in $L^2(\Omega)$, and that A^{-1} exists and is compact. Then it is well known that there exist eigenvalues λ_i and corresponding eigenfunctions $\varphi_{i,j}(x)$ of the operator A satisfying the following conditions:

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_i < \cdots, \quad \lim_{i\to\infty} \lambda_i = \infty,$$
$$A\varphi_{ij} = \lambda_i\varphi_{ij}, \quad j = 1, \cdots, m_i, \quad i = 1, 2, \cdots,$$
$$\{\varphi_{ij}(\cdot)\} \quad \text{forms a complete orthonormal system in } L^2(\Omega).$$

For each constant $0 \leq \sigma \leq 1$, the domain $D(A^\sigma)$ of the fractional power A^σ, denoted by X_σ, is topologized by the norm:

$$|x|_\sigma^2 := |A^\sigma x|_0^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} \lambda_i^{2\sigma}|(x, \varphi_{ij})|^2, \quad x \in X_\sigma \tag{1.3}$$

where $|\cdot|_0$ denotes the norm of X.

The formulation (1.1) includes vibrations in mechanically flexible systems, e.g. flexible arms of industrial robots or flexible structure such as antennas of space crafts (cf. [1], [10], [11], [12] in a linear system: $F \equiv 0$). In this paper we treat the case with nonlinear forcing, which is determined not only by the displacement $u(t, x)$, but also by the bending force $u_{xx}(t, x)$. Our main object is to show sufficient conditions for periodicity and stability of solutions under periodic perturbations $w(t)$. We describe some inequality relations by using system parameters, such as stability constants of the linear term, growth rates and (locally) Lipschitz constant of the nonlinear term. While it should be considerable that the first eigenvalue of the linear operator A essentially determines these relations, we find that the eigenvalues λ_h, λ_{h+1} which satisfies

$$0 < \lambda_1 < \cdots < \lambda_h < \frac{1}{\alpha^2} < \lambda_{h+1} < \cdots$$

have some significant properties for the stability of this system. If these values; $\lambda_1, \lambda_{k+1} - \frac{1}{\alpha^2}, \frac{1}{\alpha^2} - \lambda_h$ are sufficiently large, we can show the asymptotic behavior of solutions; the existence of a global attractor, and periodicity or asymptotically periodicity of solutions under periodic perturbations. Also we estimate some essential relations among these system parameters; $\lambda_1, \lambda_h, \lambda_{h+1}, \alpha,$ considering a system of one-dimensional nonlinear flexible beam.

Our formulation depends on the method by Sakawa [10] in linear flexible systems, using spectral properties of analytic semigroups (cf. [12]). To analyze nonlinear systems we apply a variation of the Gronwall inequality, which was introduced in [5] (see also [8]). As for the other methods to show periodic stability of nonlinear systems we can refer to [4], [6], [7], which mainly depends on the monotone operator theory.

In section 2 we give the formulation and prepare some Lemmas on analytic semigroups. In section 3, introducing inequality relations on system parameters, we show periodic stability of solutions. In section 4 we investigate these inequality relations in an actual case: one-dimensional nonlinear flexible beam system.

2. Formulation of Flexible System

First we introduce the formulation by Sakawa [10] in the linear case. Assume that

$$\alpha^2\lambda_i^2 - \lambda_i \neq 0, \quad i = 1, 2, \cdots$$

and that $\alpha > 0$ is so small:

$$\alpha\lambda_1 < \frac{1}{2\alpha}. \tag{2.1}$$

Define a complex valued function g by

$$g(\lambda) = \sqrt{\alpha^2\lambda^2 - \lambda},$$

then, since A is selfadjoint, one can define an operator $g(A)$ by

$$g(A)u = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} g(\lambda_i)(u, \varphi_{ij})\varphi_{ij},$$

$$D(g(A)) = \{u \in L^2(\Omega) : \sum_{i=1}^{\infty}\sum_{j=1}^{m_i} |g(\lambda_i)(u, \varphi_{ij})|^2 < \infty\}.$$

Note that $D(g(A)) = D(A)$ and define the following two operators by

$$A^+ := \alpha A - g(A), \quad A^- := \alpha A + g(A),$$

then for each $u \in D(A)$

$$A^{\pm}u = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}(\alpha\lambda_i \mp g(\lambda_i))(u, \varphi_{ij})\varphi_{ij}$$

and the eigenvalues and the eigenfunctions of $A^{\pm}$ are given by

$$\begin{aligned}&\nu_i = \alpha\lambda_i - g(\lambda_i), \quad \mu_i = \alpha\lambda_i + g(\lambda_i),\\ &A^+u = \nu_i\varphi_{ij}, \quad A^-u = \mu_i\varphi_{ij}, \quad j = 1, \cdots, m_i, \quad i = 1, 2, \cdots.\end{aligned}$$

From (2.1) it follows that there is an integer $h \geq 1$ such that

$$\alpha^2\lambda_h^2 - \lambda_h < 0, \quad \alpha^2\lambda_{h+1}^2 - \lambda_{h+1} > 0.$$

In this paper we can show that the following three eigenvalues $\lambda_1, \lambda_h, \lambda_{h+1}$ are the most essential parameters in the sufficient conditions for periodic stability.

Since $-A^+, -A^-$ generates analytic semigroups $S_1(t), S_2(t)$, respectively (cf. lemma 3.1 in [10]) and, especially, A^+ is a bounded operator, we can consider the following system of the semilinear equations:

$$\dot{\xi}(t) + A^+\xi(t) = g^{-1}(A)[F(\frac{\xi+\eta}{2}) + w(t)], \tag{2.2}$$

$$\dot{\eta}(t) + A^-\eta(t) = -g^{-1}(A)[F(\frac{\xi+\eta}{2}) + w(t)], \tag{2.3}$$

which can be described by

$$\dot{\zeta}(t) + \mathcal{A}\zeta(t) = \mathcal{F}(\zeta(t)) + \mathbf{w}(t), \tag{2.4}$$

where

$$\zeta(t) = \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A^+ & 0 \\ 0 & A^- \end{bmatrix},$$

$$\mathcal{F}(\zeta(t)) = \begin{bmatrix} g^{-1}(A)F(\frac{\xi+\eta}{2}) \\ -g^{-1}(A)F(\frac{\xi+\eta}{2}) \end{bmatrix}, \quad \mathbf{w}(t) = \begin{bmatrix} g^{-1}(A)w(t) \\ -g^{-1}(A)w(t) \end{bmatrix}$$

and $g^{-1}(A)$ is the inverse operator of $g(A)$, that is,

$$g^{-1}(A)u = \sum_{i=1}^{\infty}\sum_{j=1}^{m_i}(g(\lambda_i))^{-1}(u, \varphi_{ij})\varphi_{ij}$$

and $w \in C(0,T:X)$. Also, their mild forms are described as follows:

$$\xi(t) = S_1(t)\xi_0 + \int_0^t S_1(t-s)g^{-1}(A)[F(\frac{\xi+\eta}{2}(s)) + w(s)]ds, \qquad (2.5)$$

$$\eta(t) = S_2(t)\eta_0 - \int_0^t S_2(t-s)g^{-1}(A)[F(\frac{\xi+\eta}{2}(s)) + w(s)]ds \qquad (2.6)$$

We consider the following conditions on the nonlinear function F for a given fixed constant $\beta : 0 < \beta < 1$.

(F1) F is locally Lipschitz continuous from X_β to X: there exists a constant $k(c) > 0$ such that

$$|Fx - Fy|_0 \le k(c)|x-y|_\beta \quad \text{for } |x|_\beta, |y|_\beta \le c. \qquad (2.7)$$

(G1) There exists a constant K_0 :

$$|F(x)|_0 \le K_0(1+|x|_\beta), \quad x \in X_\beta$$

Under these conditions **(F1)**, **(G1)** for a fixed constant $0 < \beta < 1$, we can admit the mild solution (cf. [9]):

$$[\xi(t), \eta(t)] \in C(0,T:L^2(\Omega) \times D(A^\beta)) \cap C^1(0,T:L^2(\Omega) \times L^2(\Omega)) \qquad (2.8)$$

for each initial condition $[\xi_0, \eta_0] \in L^2(\Omega) \times D(A^\beta)$. Furthermore, we can estimate the regularity of the solutions as follows. If $[\xi_0, \eta_0] \in D(A) \times D(A^{1+\sigma})$ for some constant $\sigma : 0 < \beta \le \sigma < 1$, then by multipling λ_i^2 and $\lambda_i^{2(1+\sigma)}$ to the spectral expansion of (2.5) and (2.6), respectively, and applying the direct estimation such as (1.3) we have

$$\xi \in C(0,T:D(A)), \quad \eta \in C(0,T:D(A^{1+\sigma})). \qquad (2.9)$$

Then it follows from (2.2) and (2.3) that

$$\dot{\xi} \in C(0,T:D(A)), \quad \dot{\eta} \in C(0,T:D(A^\sigma)). \qquad (2.10)$$

Now, define the functions u, v by

$$u := \frac{\xi+\eta}{2}, \quad v := \frac{\xi-\eta}{2}, \qquad (2.11)$$

then from (2.9) and (2.10) it follows that

$$u, v \in C(0,T:D(A)) \cap C^1(0,T:D(A^\sigma)). \qquad (2.12)$$

Hereafter, we consider the case $\sigma = \beta$. From (2.2) and (2.3) we have

$$\dot{u} + \dot{v} + (\alpha A - g(A))(u+v) = g^{-1}(A)(F(u) + w),$$
$$\dot{u} - \dot{v} + (\alpha A + g(A))(u-v) = -g^{-1}(A)(F(u) + w)$$

and then the difference and the sum of the above equations give

$$\dot{u} = -\alpha A u + g(A)v, \tag{2.13}$$
$$\dot{v} - g(A)u + \alpha A v = g^{-1}(A)(F(u) + w). \tag{2.14}$$

By modifying the argument in [10] without the assumption $\dot{u} \in D(A)$ we have

$$\begin{aligned} \dot{v} &= g(A)u - \alpha A v + g^{-1}(A)(F(u)+w) \qquad (2.15)\\ &= g^2(A)g(A)^{-1}u - \alpha A g^{-1}(A)g(A)v + g^{-1}(A)(F(u)+w) \\ &= (\alpha^2 A^2 - A)g^{-1}(A)u - \alpha A g^{-1}(A)g(A)v + g^{-1}(A)(F(u)+w) \\ &= \alpha A(g^{-1}(A)\alpha A u - g^{-1}(A)g(A)v) - Ag^{-1}(A)u + g^{-1}(A)(F(u)+w). \end{aligned}$$

Thus, using (2.13), we have

$$\dot{v} = -\alpha A g^{-1}(A)\dot{u} - Ag^{-1}(A)u + g^{-1}(A)(F(u)+w). \tag{2.16}$$

Also we note that (2.14) and (2.16) give

$$\alpha A v - g(A)u = \alpha A g^{-1}(A)\dot{u} + Ag^{-1}(A)u \tag{2.17}$$

Obviously, by differentiating (2.13) under the assumption that $\dot{u} \in D(A)$ and using (2.16) we can obtain the evolution equation (1.1). On the other hand, without the assumption $\dot{u} \in D(A)$, consider an initial condition

$$[u(0), \dot{u}(0)] = [u_0, u_1] \in D(A) \times D(A^\beta),$$

then, since (2.13) yields

$$v(0) = g^{-1}(A)(u_1 + \alpha A u_0) \in D(A),$$

we have

$$\xi(0) = u(0) + v(0) \in D(A). \tag{2.18}$$

And also, since (2.13) yields

$$\begin{aligned} g(A)u - g(A)v &= -\dot{u} - (\alpha A u - g(A)u) \\ &= -\dot{u} - A^+ u, \end{aligned}$$

we have

$$\begin{aligned} \eta(0) &= u(0) - v(0) \\ &= g^{-1}(A)(-u_1 - A^+ u_0) \in D(A^{1+\beta}). \end{aligned} \tag{2.19}$$

Thus, by applying the previous argument with (2.18) and (2.19), we can admit the solution $u = (\xi + \eta)/2$ in the mild sense such that

$$[u, \dot{u}] \in C(0, T : D(A)) \times C(0, T : D(A^\beta)).$$

In order to show the periodic stability we need the estimate of the norm of $[u(t), \dot{u}(t)]$ by using the norm of $[\xi(t), \eta(t)]$. We can prepare the following Lemmas by applying fundamental calculations with respect to the spectral expansions.

Lemma 1. *Under the formulation above, let*

$$\xi \in C^1(0,T : D(A)) \cap C(0,T : D(A)),$$
$$\eta \in C^1(0,T : D(A^\beta)) \cap C(0,T : D(A^{1+\beta})),$$

then there exist constants $N_1, N_h > 0$ *such that*

$$N_1(|u(t)|_\beta + |\dot{u}(t)|_\beta) \le |A^+\xi(t)|_\beta + |A^-\eta(t)|_\beta \le N_h(|u(t)|_\beta + |\dot{u}(t)|_\beta). \tag{2.20}$$

Lemma 2. *For the operators* A, A^+, A^-, *the following inequalities hold:*

$$|Ax|_0 \ \ge\ \lambda_1^{1-\beta}|x|_\beta, \quad x \in X_1, \tag{2.21}$$
$$|A^+y|_\beta \ \ge\ \sqrt{\lambda_1}|y|_\beta, \quad y \in X_\beta, \tag{2.22}$$
$$|A^-z|_\beta \ \ge\ \sqrt{\lambda_1}|z|_\beta, \quad z \in X_{1+\beta}, \tag{2.23}$$
$$|A^-w|_\beta \ \ge\ \alpha\lambda_1^\beta|w|_1, \quad w \in X_{1+\beta}, \tag{2.24}$$
$$|A\xi|_0 \ \ge\ \alpha\lambda_1^{1-\beta}|A^+\xi|_\beta, \quad \xi \in X_1. \tag{2.25}$$

From Lemma1 and Lemma 2 we can derive the estimate of the norm $|u(t)|_1 + |\dot{u}(t)|_\beta$ by using the norms of ξ and η: $|A\xi|_0 + |A^-\eta|_\beta$.

Lemma 3. There exists a constant $K_p > 0$ such that

$$|u(t)|_1 + |\dot{u}(t)|_\beta \le K_p(|A\xi(t)|_0 + |A^-\eta(t)|_\beta) \tag{2.26}$$

where K_p is given by

$$K_p = \max\{N_1^{-1} + \frac{1}{2\alpha\lambda_1^\beta}, \frac{1}{2} + \frac{1}{\alpha\lambda_1^{1-\beta}N_1}\}.$$

Now we prepare the estimate, which corresponds to the well-known estimate of the operator norm of an analytic semigroup and its generator. We need the following notations:

$$\lambda(\beta) = \min\{\sqrt{\lambda_1}, \lambda_1^{1-\beta}\},$$
$$M_h = \max\{\frac{1}{\sqrt{1-\alpha^2\lambda_h}}, \sqrt{\frac{\alpha^2\lambda_{h+1}}{\alpha^2\lambda_{h+1}-1}} + 1\},$$
$$C_h = \max\{\sqrt{\frac{\lambda_h}{1-\alpha^2\lambda_h}}, \sqrt{\frac{\lambda_{h+1}}{\alpha^2\lambda_{h+1}-1}}\},$$
$$M_\beta = M_h(\lambda_1^\beta + \frac{1}{\alpha})\left(\frac{\beta}{\alpha\lambda_1 - \delta}\right)^\beta e^{-\beta} \tag{2.27}$$

where we fix a positive constant: $\delta < \alpha\lambda_1$. Since $\alpha^2\lambda_h < 1$, $\alpha C_h \leq M_h$ holds. Thus we have

$$(M_h\lambda_1^\beta + C_h)\left(\frac{\beta}{\alpha\lambda_1 - \delta}\right)^\beta e^{-\beta} \leq M_\beta. \tag{2.28}$$

Lemma 4. *For a constant $\delta : 0 < \delta < \alpha\lambda_1$, we have the following estimate:*

$$|AS_1(t)g^{-1}(A)y|_0 + |A^- A^\beta S_2(t)g^{-1}(A)y|_0 \leq M_\beta e^{-\delta t}t^{-\beta}|y|_0, \quad y \in L^2(\Omega). \tag{2.29}$$

3. Periodic Stability

In this section we show stability and also periodic stability of a flexible system by using the results of the previous section. First we show the existence of a global attractor for system (2.5)-(2.6).

Theorem 1. *Under Hypotheses* **(F1)**, **(G1)**, *let $[\xi_0, \eta_0] \in D(A) \times D(A^{1+\beta})$, $w \in BC(R^+ : X)$, the space of X-valued bounded continuous functions with the usual supremum norm $|\cdot|_\infty$, and assume that system parameters, $\delta, \alpha, \beta, \lambda_1, \lambda_h, \lambda_{h+1}, K_0$, satisfy the following inequality conditions: $0 < \delta < \alpha\lambda_1$, $0 < \beta \leq 1/2$ and*

$$\delta > \vartheta := \left(\frac{M_\beta K_0 \Gamma(\bar\beta)}{2\lambda(\beta)}\right)^{1/\bar\beta} \tag{3.1}$$

where $\bar\beta = 1 - \beta$. Then the following estimate holds

$$|A\xi(t)|_0 + |A^-\eta(t)|_\beta \leq K_1(t)(|A\xi_0|_0 + |A^-\eta_0|_\beta) + K_2|w|_\infty + K_3 \tag{3.2}$$

for some positive constants K_2, K_3 *and*

$$K_1(t) = \frac{e^{-(\vartheta-\delta)t}}{\bar\beta} + e^{-\alpha\lambda_1 t}\Gamma(\bar\beta). \tag{3.3}$$

Consequently, the solution $[u(t), \dot u(t)]$, *given by* $u = (\xi+\eta)/2$, *has a global attractor in* $X_1 \times X_\beta$: $\{[x,y] \in X_1 \times X_\beta : |x|_1 + |y|_\beta \le K_p(K_2|w|_\infty + K_3)\}$.

proof. From (2.5), (2.6) we have

$$\begin{aligned} |A\xi(t)|_0 &\le |S_1(t)A\xi_0|_0 + \int_0^t |AS_1(t-s)g^{-1}(A)\{F(\frac{\xi+\eta}{2}) + w(s)|_0\}ds, \\ |A^-\eta(t)|_\beta &\le |S_2(t)A^-\eta_0|_\beta + \int_0^t |A^-A^\beta S_2(t-s)g^{-1}(A)\{F(\frac{\xi+\eta}{2}) + w(s)\}|_0 ds. \end{aligned}$$

Summing up and using Lemma 2, Lemma 4 and **(G1)**, we obtain

$$\begin{aligned} &|A\xi(t)|_0 + |A^-\eta(t)|_\beta \\ &\quad \le e^{-\alpha\lambda_1 t}(|A\xi_0|_0 + |A^-\eta_0|_\beta) + \int_0^t M_\beta e^{-\delta(t-s)}(t-s)^{-\beta}(K_0 + |w(s)|_0)ds \\ &\qquad + \int_0^t M_\beta e^{-\delta(t-s)}(t-s)^{-\beta}\frac{K_0}{2\lambda(\beta)}(|A\xi(s)|_0 + |A^-\eta(s)|_\beta)ds. \end{aligned} \tag{3.4}$$

Multiplying each term by $e^{\delta t}$ and considering the estimate

$$\int_0^t M_\beta e^{\delta s}(t-s)^{-\beta}(K_0 + |w(s)|_0)ds \le M_\beta(K_0 + |w|_\infty)\int_0^t e^{\delta s}(t-s)^{-\beta}ds,$$

and putting

$$\begin{aligned} a(t) &:= e^{-(\alpha\lambda_1-\delta)t}(|A\xi_0|_0 + |A^-\eta_0|_\beta) + M_\beta(K_0 + |w|_\infty)\int_0^t e^{\delta\sigma}(t-\sigma)^{-\beta}d\sigma, \\ y(t) &:= e^{\delta t}(|A\xi(t)|_0 + |A^-\eta(t)|_\beta), \\ b &:= \frac{M_\beta K_0}{2\lambda(\beta)} \end{aligned}$$

in the Gronwall inequality, introduced in Appendix, we obtain the following estimate:

$$\begin{aligned} &|A\xi(t)|_0 + |A^-\eta(t)|_\beta \\ &\quad \le E(\vartheta t)e^{-\delta t}(|A\xi_0|_0 + |A^-\eta_0|_\beta) \\ &\qquad + \int_0^t E(\vartheta(t-s))\{M_\beta(K_0 + |w|_\infty)e^{-\delta t}[\delta e^{\delta s}\int_0^s \sigma^{-\beta}e^{-\delta\sigma}d\sigma + s^{-\beta}] \\ &\qquad - e^{-\delta t}(\alpha\lambda_1 - \delta)(|A\xi_0|_0 + |A^-\eta_0|_\beta)e^{-(\alpha\lambda_1-\delta)s}\}ds \end{aligned} \tag{3.5}$$

where, as we can see also in Appendix,

$$\vartheta = [b\Gamma(\bar\beta)]^{1/\bar\beta}, \quad E(z) := \sum_{n=0}^{\infty} \frac{z^{n\bar\beta}}{\Gamma(n\bar\beta+1)}, \quad E(z) \le \frac{e^z}{\bar\beta} + \Gamma(\bar\beta), \quad z \ge 0.$$

Thus, we have the following sequence of estimation:

$$\begin{aligned}
&|A\xi(t)|_0 + |A^-\eta(t)|_\beta \\
&\le (|A\xi_0|_0 + |A^-\eta_0|_\beta)e^{-\delta t}\{\frac{e^{\vartheta t}}{\bar\beta} + \Gamma(\bar\beta)\} \\
&\quad + M_\beta(K_0 + |w|_\infty)e^{-\delta t}\int_0^t (\frac{e^{\vartheta(t-s)}}{\bar\beta} + \Gamma(\bar\beta))(\delta e^{\delta s}\Gamma_1 + s^{-\beta})ds \\
&\quad - e^{-\delta t}(\alpha\lambda_1 - \delta)(|A\xi_0|_0 + |A^-\eta_0|_\beta)\int_0^t (\frac{e^{\vartheta(t-s)}}{\bar\beta} + \Gamma(\bar\beta))e^{-(\alpha\lambda_1-\delta)s}ds \\
&\le M_\beta(K_0 + |w|_\infty) \\
&\quad \times[\delta\Gamma_1\{\frac{1-e^{-(\delta-\vartheta)t}}{\bar\beta(\delta-\vartheta)} + \frac{\Gamma(\bar\beta)(1-e^{-\delta t})}{\delta}\} + e^{-\delta t}\frac{t^{1-\beta}}{1-\beta}\Gamma(\bar\beta) + e^{-(\delta-\vartheta)t}\frac{\Gamma_2}{\bar\beta}] \\
&\quad + (|A\xi_0|_0 + |A^-\eta_0|_\beta)e^{-\delta t}\{\frac{e^{\vartheta t}}{\bar\beta} + \Gamma(\bar\beta)\} - (\alpha\lambda_1 - \delta)(|A\xi_0|_0 + |A^-\eta_0|_\beta) \\
&\quad \times[\frac{e^{-(\delta-\vartheta)t}}{\bar\beta}\int_0^t e^{-(\vartheta+\alpha\lambda_1-\delta)s}ds + e^{-\delta t}\Gamma(\bar\beta)\int_0^t e^{-(\alpha\lambda_1-\delta)s}ds] \\
&\le M_\beta(K_0 + |w|_\infty) \\
&\quad \times[\delta\Gamma_1\{\frac{1-e^{-(\delta-\vartheta)t}}{\bar\beta(\delta-\vartheta)} + \frac{\Gamma(\bar\beta)(1-e^{-\delta t})}{\delta}\} + e^{-\delta t}\frac{t^{1-\beta}}{1-\beta}\Gamma(\bar\beta) + e^{-(\delta-\vartheta)t}\frac{\Gamma_2}{\bar\beta}] \\
&\quad + (|A\xi_0|_0 + |A^-\eta_0|_\beta)\{\frac{e^{-(\vartheta-\delta)t}}{\bar\beta}\left(\frac{\vartheta + (\alpha\lambda_1-\delta)e^{-(\vartheta+\alpha\lambda_1-\delta)t}}{\vartheta+\alpha\lambda_1-\delta}\right) \\
&\quad + \Gamma(\bar\beta)e^{-\alpha\lambda_1 t}\}
\end{aligned} \tag{3.6}$$

where

$$\Gamma_1 = \int_0^\infty s^{-\beta}e^{-\delta s}ds, \quad \Gamma_2 = \int_0^\infty s^{-\beta}e^{-\vartheta s}ds.$$

Hence, we have

$$|A\xi(t)| + |A^-\eta(t)|_\beta \le K_1(t)(|A\xi_0|_0 + |A^-\eta_0|_\beta) + K_2|w|_\infty + K_3, \tag{3.7}$$

for the following constants

$$\begin{aligned}
K_2 &= M_\beta \times [\delta\Gamma_1\{\frac{1}{\bar\beta(\delta-\vartheta)} + \frac{\Gamma(\bar\beta)}{\delta}\} + (\delta e)^{-(1-\beta)}(1-\beta)^{-\beta}\Gamma(\bar\beta) + \frac{\Gamma_2}{\bar\beta}], \\
K_3 &= K_2K_0. \quad \square
\end{aligned} \tag{3.8}$$

Remark. In case $1/2 < \beta < 1$ the assertion of Theorem 1 holds if one substitutes the constant $\Gamma(\bar{\beta})$ by $\Gamma'(\bar{\beta}) := \Gamma(\bar{\beta})/(\sin \bar{\beta}\pi)^{\bar{\beta}}$ (cf. Appendix).

Let $w \in BC(R^+ : X)$ be a periodic function: $w(t+T) = w(t)$ and then we consider periodicity of solution $[u(t), \dot{u}(t)]$. As in Theorem 1, we also use the pair of functions $[\xi(t), \eta(t)] \in X_1 \times X_{1+\beta}$. Let $[\xi_k(t), \eta_k(t)]$, $k = 1, 2$ be two pairs of solutions, starting with initial states $[\xi_k(0), \eta_k(0)]$, $k = 1, 2$, respectively, which are corresponding to the solutions $[u_k(t), \dot{u}_k(t)]$, $k = 1, 2$. Then, by following the proof of Lemma 1, we can show the same estimate:

$$\begin{aligned} &N_1(|u_1(t) - u_2(t)|_\beta + |\dot{u}_1(t) - \dot{u}_2(t)|_\beta) \\ &\quad \le |A^+(\xi_1(t) - \xi_2(t))|_\beta + |A^-(\eta_1(t) - \eta_2(t))|_\beta \\ &\quad \le N_h(|u_1(t) - u_2(t)|_\beta + |\dot{u}_1(t) - \dot{u}_2(t)|_\beta), \end{aligned} \tag{3.9}$$

which yields the equivalence between $[u(t), \dot{u}(t)]$ and $[\xi(t), \eta(t)]$ with respect to asymptotic periodicity and stability.

On the other hand, define the norm $|||[x, y]|||_{1,\beta}$ by

$$|||[x, y]|||_{1,\beta} := |Ax|_0 + |A^- y|_\beta,$$

which is equivalent to the $X_1 \times X_{1+\beta}$-norm. Then, from Lemma 3 and Theorem 1, we can show the periodicity or the asymptotic periodicity of $[u(t), \dot{u}(t)]$ in $X_1 \times X_\beta$ by estimating the norm $|||[\xi(t), \eta(t)]|||_{1,\beta}$.

Theorem 2. *Assume the same Hypotheses as theorem 1. For a given constant $r > 0$, let*

$$w \in \mathcal{W}_r := \{w \in BC(R^+ : X) : |w|_\infty \le r\}, \quad w(t) = w(t+T)$$

and take a constant $d > 0$:

$$d > K_2 r + K_3 \tag{3.10}$$

where K_2, K_3 are the constants, introduced in (3.8), and assume that

$$\delta > \vartheta' := \left(\frac{M_\beta k(d) \Gamma(\bar{\beta})}{2\lambda(\beta)} \right)^{1/\bar{\beta}}. \tag{3.11}$$

Then there exists a unique T-periodic solution $[\xi_\infty(t), \eta_\infty(t)]$ such that

$$|||[\xi_\infty(t), \eta_\infty(t)]|||_{1,\beta} \le d, \quad t \ge 0, \tag{3.12}$$

and consequently, there exists a unique T-periodic solution $[u_\infty, \dot{u}_\infty]$:

$$|u_\infty(t)|_1 + |\dot{u}_\infty(t)|_\beta \le K_p d, \quad t \ge 0.$$

proof. From Theorem 1 we can assume that there exists a solution $[\xi, \eta]$ with an initial condition $[\xi_0, \eta_0]$ in $X_1 \times X_{1+\beta}$:

$$\|[\xi(t), \eta(t)]\|_{1,\beta} \leq d, \quad t \geq 0.$$

Then we show that $[\xi_n(t), \eta_n(t)] := [\xi(t+nT), \eta(t+nT)]$ converges to a T-periodic solution $[\xi_\infty(t), \eta_\infty(t)]$ as $n \to \infty$.

By using (2.5) and (2.6), we have

$$\begin{aligned}
&|A(\xi_n - \xi_{n+m})(t)|_0 \\
&\quad \leq |S_1(nT)A(\xi - \xi_m)(t)|_0 \\
&\quad + \int_0^{nT} |AS_1(nT-s)g^{-1}(A)[F(\frac{\xi+\eta}{2}(s+t)) - F(\frac{\xi_m+\eta_m}{2}(s+t))|_0 ds, \\
&|A^-(\eta_n - \eta_{n+m})(t)|_\beta \\
&\quad \leq |S_2(nT)A^-(\eta - \eta_m)(t)|_\beta \\
&\quad + \int_0^{nT} |A^- A^\beta S_2(nT-s)g^{-1}(A)[F(\frac{\xi+\eta}{2}(s+t)) - F(\frac{\xi_m+\eta_m}{2}(s+t))|_0 ds.
\end{aligned}$$

It follows from Lemma 4, **(F1)**, (3.10) and Lemma 2 that

$$\begin{aligned}
&|A(\xi_n - \xi_{n+m})(t)|_0 + |A^-(\eta_n - \eta_{n+m})(t)|_\beta \\
&\quad \leq e^{-\alpha\lambda_1 nT}(|A(\xi - \xi_m)(t)|_0 + |A^-(\eta - \eta_m)(t)|_\beta) \\
&\quad + \int_0^{nT} M_\beta \frac{k(d)}{2\lambda(\beta)} e^{-\delta(nT-s)}(nT-s)^{-\beta}\{|A(\xi-\xi_m)(s+t)|_0 + |A^-(\eta-\eta_m)(s+t)|_\beta\} ds.
\end{aligned}$$

Thus the Gronwall inequality and the same argument as in Theorem 1 give

$$\begin{aligned}
&|A(\xi_n - \xi_{n+m})(t)|_0 + |A^-(\eta_n - \eta_{n+m})(t)|_\beta \\
&\quad \leq [\frac{e^{-(\delta-\vartheta')nT}}{\bar\beta} + \Gamma(\bar\beta)e^{-\alpha\lambda_1 nT}](|A(\xi - \xi_m)(t)|_0 + |A^-(\eta - \eta_m)(t)|_\beta) \\
&\quad \leq 2d[\frac{e^{-(\delta-\vartheta')nT}}{\bar\beta} + \Gamma(\bar\beta)e^{-\alpha\lambda_1 nT}]. \qquad (3.13)
\end{aligned}$$

It follows that the sequence $[\xi_n(t), \eta_n(t)]$ is a Cauchy sequence in $BC(R^+ : X_1 \times X_{1+\beta})$. Hence, there exists $[\xi_\infty(t), \eta_\infty(t)]$:

$$[\xi_n(t), \eta_n(t)] \to [\xi_\infty, \eta_\infty] \text{ in } BC(R^+ : X_1 \times X_{1+\beta}).$$

By taking the limit $n \to \infty$ of the mild formulas:

$$\begin{aligned}
\xi(t+nT) &= S_1(t)\xi(nT) + \int_0^t S_1(t-s)g^{-1}(A)[F(\frac{\xi+\eta}{2}(s+nT)) + w(s)]ds, \\
\eta(t+nT) &= S_2(t)\eta(nT) - \int_0^t S_2(t-s)g^{-1}(A)[F(\frac{\xi+\eta}{2}(s+nT)) + w(s)]ds
\end{aligned}$$

we can show that $[\xi_\infty(t), \eta_\infty(t)]$ satisfies the mild formulas (2.5), (2.6) with the initial state $[\xi_\infty(0), \eta_\infty(0)]$.

Furthermore, T-periodicity of $[\xi_\infty(t), \eta_\infty(t)]$ holds, since

$$\begin{aligned}
[\xi_\infty(t+T), \eta_\infty(t+T)] &= \lim_{n\to\infty}[\xi_\infty(t+T+nT), \eta_\infty(t+T+nT)] \\
&= \lim_{n\to\infty}[\xi_\infty(t+(n+1)T), \eta_\infty(t+(n+1)T)] \\
&= [\xi_\infty(t), \eta_\infty(t)].
\end{aligned}$$

Also, we can obtain the uniqueness of the T-periodic solution by using the following estimate for a sufficiently large N.

$$\begin{aligned}
&\|[\xi_\infty(t+NT), \eta_\infty(t+NT)] - [\xi'_\infty(t+NT), \eta'_\infty(t+NT)]\|_{1,\beta} \\
&= \|[\xi_\infty(t), \eta_\infty(t)] - [\xi'_\infty(t), \eta'_\infty(t)]\|_{1,\beta} \\
&\leq [\frac{e^{-(\delta-\vartheta')NT}}{\bar{\beta}} + \Gamma(\bar{\beta})e^{-\alpha\lambda_1 NT}]\|[\xi_\infty(t), \eta_\infty(t)] - [\xi'_\infty(t), \eta'_\infty(t)]\|_{1,\beta}. \quad \square
\end{aligned}$$

With respect to the stability of the periodic solutions, or aymptotical periodicity, we can prove the following theorem.

Theorem 3. *Assume the same Hypotheses as Theorem 2 and if $w \in \mathcal{W}_r$ and*

$$|w(t) - w_\infty(t)|_0 \to 0 \quad as\ t \to \infty$$

for some T-periodic function $w_\infty \in \mathcal{W}_r$, then the solution $[u(t:w), \dot{u}(t:w)]$, starting with any initial state $[u_0, u_1] \in X_1 \times X_\beta$, converges to the T-periodic solution $[u_\infty(t:w_\infty), \dot{u}_\infty(t:w_\infty)]$ under the periodic perturbation w_∞;

$$|u(t:w) - u_\infty(t:w_\infty)|_1 + |\dot{u}(t:w) - \dot{u}_\infty(t:w_\infty)|_\beta \to 0 \quad as\ t \to \infty. \tag{3.14}$$

proof. From Lemma 1 or from the previous remark, in stead of estimating the solution $u(\cdot : w)$, it is sufficient to show the convergence of the pair of functions $[\xi(t), \eta(t)]$ to a pair of T-periodic functions $[\xi_\infty, \eta_\infty]$ in $X_1 \times X_{1+\beta}$-norm. Here we can also assume that

$$\|[\xi(t), \eta(t)]\|_{1,\beta} \leq d, \quad t \geq 0.$$

Let N be a large integer which satisfies

$$\frac{e^{-(\delta-\vartheta')NT}}{\bar{\beta}} + \Gamma(\bar{\beta})e^{-\alpha\lambda_1 NT} < 1$$

and define the following sequences

$$
\begin{aligned}
w_m(t) &= w(t+mNT),\\
\xi_m(t) &= \xi(t+mNT),\\
\eta_m(t) &= \eta(t+mNT), \quad m=0,1,2,\cdots,
\end{aligned}
$$

then applying the same argument as that in the proof of Theorem 2 to the difference of the mild solutions,

$$[\xi(t+(m+1)NT),\eta(t+(m+1)NT)]-[\xi_\infty(t+(m+1)NT),\eta_\infty(t+(m+1)NT)],$$

starting with the initial values,

$$[\xi(t+mNT),\eta(t+mNT)],[\xi_\infty(t+mNT),\eta_\infty(t+mNT)],$$

respectively, we have

$$
\begin{aligned}
&|||[(\xi_{m+1}-\xi_\infty)(t),(\eta_{m+1}-\eta_\infty)(t)]|||_{1,\beta}\\
&\quad = |||[(\xi-\xi_\infty)(t+(m+1)NT),(\eta-\eta_\infty)(t+(m+1)NT)]|||_{1,\beta}\\
&\quad \le \{\frac{e^{-(\delta-\vartheta)NT}}{\bar\beta}+\Gamma(\bar\beta)e^{-\alpha\lambda_1 NT}\}|||[(\xi-\xi_\infty)(t+mNT),(\eta-\eta_\infty)(t+mNT)]|||_{1,\beta}\\
&\quad\quad +[\delta\Gamma_1\{\frac{1}{\bar\beta(\delta-\vartheta)}+\frac{\Gamma(\bar\beta)}{\delta}\}+(\delta e)^{-(1-\beta)}(1-\beta)^{-\beta}\Gamma(\bar\beta)+\frac{\Gamma_2}{\bar\beta}]\\
&\quad\quad \times M_\beta \sup\{|w(s)-w_\infty(s)|_0 : t+mNT\le s\le t+(m+1)NT\}.
\end{aligned}
$$

Put

$$
\begin{aligned}
\varphi_m &= |||[(\xi_m-\xi_\infty)(t),(\eta_m-\eta_\infty)(t)]|||_{1,\beta},\\
K &= [\frac{e^{-(\delta-\vartheta)NT}}{\bar\beta}+\Gamma(\bar\beta)e^{-\alpha\lambda_1 NT}]<1
\end{aligned}
$$

and for a small constant $\varepsilon>0$, take a large number $m_0 : m\ge m_0 \Longrightarrow$

$$
\begin{aligned}
&[\delta\Gamma_1\{\frac{1}{\bar\beta(\delta-\vartheta)}+\frac{\Gamma(\bar\beta)}{\delta}\}+(\delta e)^{-(1-\beta)}(1-\beta)^{-\beta}\Gamma(\bar\beta)+\frac{\Gamma_2}{\bar\beta}]\\
&\quad \times M_\beta \sup\{|w(s)-w_\infty(s)| : t+mNT\le s\le t+(m+1)NT\}<\varepsilon,
\end{aligned}
$$

then we have

$$\varphi_m \le K\varphi_{m-1}+\varepsilon\le\cdots\le K^m\varphi_0+\varepsilon\frac{1-K^m}{1-K}.$$

Since $\varphi_0 = |||(\xi-\xi_\infty)(t),(\eta-\eta_\infty)(t)]|||_{1,\beta}\le 2d$, we can conclude that for every small $\varepsilon>0$, there exists a large number $m_1 : m\ge m_1 \Longrightarrow$

$$|||[\xi(t+mNT),\eta(t+mNT)]-[\xi_\infty(t+mNT),\eta_\infty(t+mNT)]|||_{1,\beta}<\varepsilon$$

for every $t \in [0, NT]$, that is,

$$|||[\xi(t), \eta(t)] - [\xi_\infty(t), \eta_\infty(t)]|||_{1,\beta} < \varepsilon$$

for every $t \geq m_1 NT$. □

4. Flexible Beam

We consider the equation of motion of slender and flexible structures with internal viscous damping and with nonlinear forcing, determined by displacement $u(t,x)$ and bending force $u_{xx}(t,x)$, under a periodic perturbation:

$$\frac{\partial^2 u(t,x)}{\partial t^2} + 2\alpha \frac{\partial^5 u(t,x)}{\partial t \partial x^4} + \frac{\partial^4 u(t,x)}{\partial x^4} = f(x, u(t,x), \frac{\partial^2 u(t,x)}{\partial x^2}) + w(t,x), \tag{4.1}$$

where $0 < x < L$. The beam is clamped at one end, $x = 0$, and at the free end, $x = L$, the bending moment and the shearing force vanish. Then the boundary conditions and the initial conditions are given by

$$\begin{aligned} & u(t,0) = \frac{\partial u(t,0)}{\partial x} = 0, \\ & \frac{\partial^2 u(t,L)}{\partial x^2} + 2\alpha \frac{\partial^3 u(t,L)}{\partial x^2 \partial t} = 0, \\ & \frac{\partial^3 u(t,L)}{\partial x^3} + 2\alpha \frac{\partial^4 u(t,L)}{\partial x^3 \partial t} = 0, && (4.2) \\ & u(0,x) = u_0(x), \quad \frac{\partial u(0,x)}{\partial t} = u_1(x), && (4.3) \end{aligned}$$

and the periodic perturbation satisfies $w(t+T) = w(t)$. We define an operator A in $L^2(0,L)$ by

$$\begin{aligned} & D(A) = \{u \in H^4(0,L) : u(0) = u_x(0) = 0, \quad u_{xx}(L) = u_{xxx}(L) = 0\}, \\ & Au = \frac{\partial^4 u}{\partial x^4}. && (4.4) \end{aligned}$$

Let γ_i are the solutions of

$$\cosh\gamma \cos\gamma + 1 = 0$$

such that $0 < \gamma_1 < \gamma_2 < \cdots$, then the eigenvalues of A are given by

$$\lambda_i = \left(\frac{\gamma_i}{L}\right)^4, \quad i = 1, 2, \cdots$$

(cf. [10]).

We assume that the nonlinear function $f(x,u,v) : R \times R \times R \to R$ satisfies the growth condition

$$|f(x,u,v)| \le k_0(|u| + |v|) \quad \text{for some } k_0 > 0 \tag{4.5}$$

and the following Lipschitz and locally Lipschitz continuity: there exists positive constants $k_0(c), k$ such that

$$|f(x,u,v) - f(x,u',v')| \le k_0(c)|u-u'| + k|v-v'| \quad \text{for } |u|, |u'| \le c, \quad v, v' \in R. \tag{4.6}$$

Define a nonlinear mapping $F : D(A^{\frac{1}{2}}) \to L^2(0,L)$ by

$$F(u)(x) = f(x, u(x), u_{xx}(x)),$$

then, since the following injections

$$D(A^{\frac{1}{2}}) \hookrightarrow H^2(0,L) \hookrightarrow C(0,L)$$

are continuous, the condtions **(F1)**, **(G1)** hold for the constant $\beta = 1/2$ and some constants $K_0, k(c)$.

Now we investigate the inequality conditions in case where $\beta = 1/2$. Our purpose is to find some relations among the constants: $\lambda_1, \lambda_h, \lambda_{h+1}, \alpha, K_0, k(d)$ where $d > K_2 r + K_3$ and K_2, K_3 are given in (3.8). Let $\delta := \alpha\lambda_1/2$, then we can describe (2.27) by

$$M_{\frac{1}{2}} = M_h(\sqrt{\lambda_1} + \frac{1}{\alpha})(\frac{1}{\alpha\lambda_1})^{\frac{1}{2}} e^{-\frac{1}{2}}$$

and it follows that

$$\begin{aligned} \vartheta &= (M_{\frac{1}{2}} K_0 \frac{\sqrt{\pi}}{2\sqrt{\lambda_1}})^2 \\ &= M_h^2(\sqrt{\lambda_1} + \frac{1}{\alpha})^2 \frac{1}{\alpha\lambda_1^2} \frac{K_0^2 \pi}{4e}. \end{aligned}$$

Thus the condition $\alpha\lambda_1/2 > \vartheta$ can be described by

$$\alpha^2\lambda_1^2 > 2M_h^2(1 + \frac{1}{\alpha\sqrt{\lambda_1}})^2 \frac{K_0^2\pi}{4e}.$$

Taking the square root of each side above yields

$$\alpha(\sqrt{\lambda_1})^3 - K_0\sqrt{\frac{\pi}{2e}} M_h \sqrt{\lambda_1} - K_0\sqrt{\frac{\pi}{2e}} M_h \frac{1}{\alpha} > 0. \tag{4.7}$$

Hence we can admit the first eigenvalue $\lambda_1 : 0 < \lambda_1 < 1/(2\alpha^2)$, which satisfies the condition $\delta > \vartheta$, if

$$\alpha(\frac{1}{\sqrt{2\alpha}})^3 - K_0\sqrt{\frac{\pi}{2e}} M_h \frac{1}{\sqrt{2\alpha}} - K_0\sqrt{\frac{\pi}{2e}} M_h \frac{1}{\alpha} > 0. \tag{4.8}$$

It follows that

$$\alpha M_h < \frac{1}{(\sqrt{2}+1)K_0\sqrt{\frac{2\pi}{e}}}. \tag{4.9}$$

Hereafter, we use the notations

$$\begin{aligned} \kappa &= (\sqrt{2}+1)K_0\sqrt{\frac{2\pi}{e}}, \\ \kappa' &= (\sqrt{2}+1)k(d)\sqrt{\frac{2\pi}{e}}. \end{aligned}$$

Considering the definition of M_h and (4.9), we have

$$\frac{\alpha}{\sqrt{1-\alpha^2\lambda_h}} < \frac{1}{\kappa}, \tag{4.10}$$

$$\alpha\left(\sqrt{\frac{\alpha^2\lambda_{h+1}}{\alpha^2\lambda_{h+1}-1}}+1\right) < \frac{1}{\kappa}. \tag{4.11}$$

From (4.10) we can derive the conditions on $\lambda_h, \alpha, \kappa$:

$$\lambda_h < \frac{1}{\alpha^2} - \kappa^2, \quad \alpha\kappa < 1 \tag{4.12}$$

and, assuming $\alpha\kappa < 1/2$, from (4.11) we obtain

$$\lambda_{h+1} > \frac{1}{\alpha^2} + \frac{\kappa^2}{1-2\alpha\kappa}. \tag{4.13}$$

We note that, as the values

$$\frac{1}{\alpha^2} - \lambda_h, \quad \lambda_{h+1} - \frac{1}{\alpha^2}$$

become sufficiently large, $M_h \downarrow 2$.

When (4.12) and (4.13) are satisfied, the first eigenvalue λ_1 can be estimated as follows. The third order algebraic inequality

$$ax^3 - bx - \frac{b}{a} > 0, \quad a, b > 0 \tag{4.14}$$

admits a solution

$$x > \frac{b}{3a\left(\frac{b}{2a^2}+\sqrt{-\frac{b^3}{27a^3}+\frac{b^2}{4a^4}}\right)^{\frac{1}{3}}} + \left(\frac{b}{2a^2}+\sqrt{-\frac{b^3}{27a^3}+\frac{b^2}{4a^4}}\right)^{\frac{1}{3}}$$

in the positive real $x > 0$. For sufficiently small $a > 0$, a rough estimating,

$$\begin{aligned}\sqrt{-\frac{b^3}{27a^3} + \frac{b^2}{4a^4}} &= \frac{b}{2a^2}\sqrt{1 - \frac{4ab}{27}} \\ &\simeq \frac{b}{2a^2},\end{aligned}$$

gives a sufficient condition for (4.14)

$$x > \frac{b^{\frac{2}{3}}}{3a^{\frac{1}{3}}} + \frac{b^{\frac{1}{3}}}{a^{\frac{2}{3}}}$$

where we take a positive real value of each fractional power 1/3. Considering the case

$$a = \alpha, \quad b = \sqrt{\frac{\pi}{2e}} K_0 M_h$$

and α is sufficiently small, then we can estimate λ_1, which satisfies our inequality conditions, as follows:

$$\left(1 + \frac{1}{3}(K_0 M_h \alpha)^{\frac{1}{3}} (\frac{\pi}{2e})^{\frac{1}{6}}\right)^2 (K_0 M_h)^{\frac{2}{3}} \left(\frac{\pi}{2e}\right)^{\frac{1}{3}} \left(\frac{1}{\alpha}\right)^{\frac{4}{3}} < \lambda_1 < \frac{1}{2\alpha^2}. \tag{4.15}$$

For sufficiently small $\alpha > 0$, it follows that

$$C_1 (\frac{K_0 M_h}{\alpha^2})^{\frac{2}{3}} < \lambda_1 < \frac{1}{2\alpha^2}$$

for some constant $C_1 > 0$. Furthermore, if the values $1/\alpha^2 - \lambda_h$ and $\lambda_{h+1} - 1/\alpha^2$ are sufficiently large, for instance,

$$\begin{aligned}\frac{1}{\alpha^2} - \lambda_h &\simeq \frac{K}{\alpha^2}, \\ \lambda_{h+1} - \frac{1}{\alpha^2} &\simeq \frac{K}{\alpha^2},\end{aligned}$$

for some large $K > 0$, we have

$$M_h \simeq 1 + \sqrt{1 + \frac{1}{K}}.$$

Then we can estimate the stability condition for the first eigenvalue:

$$C_2 (\frac{K_0}{\alpha^2})^{\frac{2}{3}} < \lambda_1 < \frac{1}{2\alpha^2} \tag{4.16}$$

for some constant $C_2 > 0$.

For the condition $\delta > \vartheta'$, we can derive the same estimate as above, substituting K_0 by $k(d)$ and κ by κ'. Here, using a sufficiently small constant α, we can estimate the condition on the first eigenvalue λ_1 as follows. Assume that $\delta - \vartheta \simeq \delta \simeq \alpha\lambda_1$, then it follows from (3.8) that we can estimate the order of the constants $K_2, K_3 \simeq M_{\frac{1}{2}}\delta^{-1/2}$. When the perturbation of w is comparatively small: $r << \alpha^{-1}$, we can consider that $d \simeq M_{\frac{1}{2}}\delta^{-1/2}$ Since we can roughly estimate $M_{\frac{1}{2}} \simeq \alpha^{-3/2}\lambda_1^{-1/2}$, it follows that $d \simeq \alpha^{-2}\lambda_1^{-1}$. Assume that $k(d) = kd$, then under the same conditions for the other parameters as those, which gives (4.16), we have

$$\lambda_1 > C_3\alpha^{-\frac{4}{3}}[\alpha^{-2}\lambda_1^{-1}]^{\frac{2}{3}}, \quad C_3 > 0.$$

It follows that the periodic stability condition is possiblly satisfied if the value λ_1 has the order between $\alpha^{-\frac{8}{5}}$ and α^{-2}.

5. Appendix

In [5] the following Gronwall's inequality was proved:

Suppose $b \geq 0, \bar\beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < +\infty$, and suppose that $y(t)$ is nonnegative and locally integrable on $0 \leq t < +\infty$ with

$$y(t) \leq a(t) + b\int_0^t (t-s)^{\bar\beta-1}y(s)ds$$

on this interval, then

$$y(t) \leq a(t) + \vartheta\int_0^t E'(\vartheta(t-s))a(s)ds,$$

where

$$\vartheta = [b\Gamma(\bar\beta)]^{1/\bar\beta}, \quad E(z) = \sum_{n=0}^{\infty}\frac{z^{n\bar\beta}}{\Gamma(n\bar\beta+1)}, \quad E'(z) = \frac{dE(z)}{dz}.$$

If $a(t)$ is differentiable, we note that, since

$$\frac{dE(\vartheta(t-s))}{ds} = E'(\vartheta(t-s))\cdot(-\vartheta),$$

integration by parts gives

$$\begin{aligned} y(t) &\leq a(t) - \int_0^t \frac{dE(\vartheta(t-s))}{ds}a(s)ds \\ &= a(t) - [E(\vartheta(t-s))a(s)]_0^t + \int_0^t E(\vartheta(t-s))a'(s)ds \\ &= E(\vartheta t)a(0) + \int_0^t E(\vartheta(t-s))a'(s)ds. \end{aligned} \tag{5.1}$$

Here we consider the estimate of the entire function $E(z)$. If $0 < z < 1$, we can estimate

$$E(z) \leq 1 + \frac{z^{\bar\beta}}{\Gamma_0(1 - z^{\bar\beta})}, \tag{5.2}$$

where $\Gamma_0 := \inf_{1<x<2} \Gamma(x) \approx 0.8$.

On the other hand, for a constant $\alpha : 0 < \alpha < 1$, it is known [2] that, if $z \geq \alpha$,

$$E(z) \leq \frac{e^z}{\bar\beta} + |\frac{1}{2\pi i} \int_l \frac{u^{\bar\beta - 1} e^u}{u^{\bar\beta} - z^{\bar\beta}} du|$$

where the contour $l : (-\infty - 0i, 0, -\infty + 0i)$ is the negative real axis described twice. Since elementary calculations give

$$\inf\{|u^{\bar\beta} - z^{\bar\beta}| : \ u \in (-\infty, 0), \ z > \alpha\} \geq \begin{cases} \alpha^{\bar\beta} & \text{if } \frac{1}{2} \leq \bar\beta < 1, \\ (\alpha \sin \bar\beta\pi)^{\bar\beta} & \text{if } 0 < \bar\beta < \frac{1}{2}, \end{cases}$$

we have

$$E(z) \leq \frac{e^z}{\bar\beta} + \frac{\Gamma(\bar\beta)}{\pi \alpha^{\bar\beta}}$$

if $/2 \leq \bar\beta < 1$ and

$$E(z) \leq \frac{e^z}{\bar\beta} + \frac{\Gamma(\bar\beta)}{\pi(\alpha \sin \bar\beta\pi)^{\bar\beta}}$$

if $0 < \bar\beta < 1/2$. It follows from (5.2) that for every constant $\alpha_0 : 0 < \alpha_0 < 1$, which satisfies

$$1 + \frac{\alpha_0^{\bar\beta}}{\Gamma_0(1 - \alpha_0^{\bar\beta})} \leq \frac{e^{\alpha_0}}{\bar\beta} + \frac{\Gamma(\bar\beta)}{\pi \alpha_0^{\bar\beta}},$$

$$[1 + \frac{\alpha_0^{\bar\beta}}{\Gamma_0(1 - \alpha_0^{\bar\beta})} \leq \frac{e^{\alpha_0}}{\bar\beta} + \frac{\Gamma(\bar\beta)}{\pi(\alpha_0 \sin \bar\beta\pi)^{\bar\beta}},]$$

the following estimate holds

$$E(z) \leq \frac{e^z}{\bar\beta} + \frac{\Gamma(\bar\beta)}{\pi \alpha_0^{\bar\beta}}$$

$$[E(z) \leq \frac{e^z}{\bar\beta} + \frac{\Gamma(\bar\beta)}{\pi(\alpha_0 \sin \bar\beta\pi)^{\bar\beta}}]$$

for every $z \geq 0$ if $\frac{1}{2} \leq \bar\beta < 1$ $[0 < \bar\beta < \frac{1}{2}]$. For instance, taking $\alpha_0 = \pi^{-1/\bar\beta}$, we have

$$E(z) \leq \frac{e^z}{\bar\beta} + \Gamma(\bar\beta) \ \ [E(z) \leq \frac{e^z}{\bar\beta} + \frac{\Gamma(\bar\beta)}{(\sin \bar\beta\pi)^{\bar\beta}}]$$

for every $z \geq 0$ if $\frac{1}{2} \leq \bar{\beta} < 1$ $[0 < \bar{\beta} < \frac{1}{2}]$, since

$$1 + \frac{\alpha_0^{\bar{\beta}}}{\Gamma_0(1-\alpha_0^{\bar{\beta}})} \approx 1.58,$$

$$\frac{e^{\alpha_0}}{\bar{\beta}} + \frac{\Gamma(\bar{\beta})}{(\sin \bar{\beta}\pi)^{\bar{\beta}}} \geq \frac{e^{\alpha_0}}{\bar{\beta}} + \frac{\Gamma(\bar{\beta})}{\pi \alpha_0^{\bar{\beta}}} > 1.8.$$

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