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順序を保存する写像に対する固有値問題とその応用

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1. Introduction

The Perron-Frobenius theorem, which is concerned with the properties of eigenvalues and eigenvectors of square matrices whose components are nonnegative, has been extended and applied in various ways. It has been generalized to positive linear operators on a Banach space in [2], [7], [10], [13], [20]. From the point of view of applications to mathematical economics, extensions of the theory to nonlinear mappings have also been obtained in [11], [12], [17], [18], [19]. They are, however, concerned only with problems in a finite dimensional Euclidean space.

In this paper, we extend these results to nonlinear mappings on an infinite dimensional space. We consider the eigenvalue problem of an order-preserving mapping defined on a positive cone of an ordered Banach space. We prove the existence of the positive eigenvalue and discuss other properties of eigenvalues and eigenvectors. The notion of indecomposability for nonlinear mappings that we introduce in an infinite dimensional setting will play a key role in our argument. In Section 5 we apply our results to boundary value problems for a class of partial differential equations. First, we generalize the Fujita lemma, which is concerned with the properties of solutions of the equation $\Delta u + f(u) = 0$

with strictly convex function f, to the case where f is a convex function. The second example is a bifurcation problem for the semilinear elliptic equation of the form $\Delta u + \lambda f(u) = 0$ under the Dirichlet boundary conditions. We discuss properties of a bifurcation branch of solutions.

2. Notations and assumptions

Let E be an ordered Banach space, that is, a real Banach space provided with an *order cone* E_+ (a closed convex cone with vertex at 0 such that $E_+ \cap (-E_+) = \{0\}$). We assume that the interior of E_+, denoted by $(E_+)^i$, is nonempty. Such a space is called a *strongly ordered* Banach space. We also assume that $\dim E \geq 2$.

For $x, y \in E$ we write $x \gg y$ if $x - y \in (E_+)^i$, $x > y$ if $x - y \in E_+ \setminus \{0\}$, and $x \geq y$ if $x - y \in E_+$. For $x \in E$ we say that x is *strongly positive*, *positive*, *nonnegative* if and only if $x \gg 0, x > 0, x \geq 0$, respectively.

We assume that the norm on E is monotone, namely,

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq \|y\|. \tag{2.1}$$

For $x \geq 0$, we denote

$$E_x = \{y \geq 0 \mid y \leq \lambda x \text{ for some } \lambda > 0\}.$$

Note that $E_x = \{0\}$ if and only if $x = 0$, and $E_x = E_+$ if and only if $x \gg 0$.

Let T be a mapping from E_+ into itself. We will impose on T the following conditions:

A1(compactness): T is continuous and the image of a bounded set by T is relatively compact, A2(subhomogeneity):

$$T(\lambda x) \leq \lambda T x \quad \text{for any } \lambda > 1, x \geq 0,$$

A3(order-preserving property):

$$x \leq y \quad \text{implies} \quad Tx \leq Ty,$$

A4(indecomposability):

$$\{0\} \subsetneq E_{x-y} \subsetneq E_{+} \quad \text{implies} \quad Tx - Ty \notin E_{x-y}.$$

The condition A4 is an infinite dimensional extension of that for a mapping on an n-dimensional Euclidean space [11], and is also a non-linear extension of that for a linear operator [9]. It is often useful to express the indecomposability condition in the following form:

Lemma 1 *Assume* A4 *and let* $x \geq y$. *Then there exists a constant* $\lambda > 0$ *such that*

$$Tx - Ty \leq \lambda(x - y)$$

if and only if either $x = y$ *or* $x \gg y$.

We define

$$\begin{aligned} VP(T) &= \{\lambda \mid \lambda \text{ is an eigenvalue of } T\} \\ &= \{\lambda \mid Tx = \lambda x \text{ for some } x > 0\} \end{aligned}$$

and denote the set of eigenvectors corresponding to λ by W_λ. We then set

$$W = \bigcup_{\lambda \in VP(T)} W_\lambda.$$

For each $\rho > 0$ we denote $S_\rho = \{x \geq 0 \mid \|x\| = \rho\}$ and

$$\lambda_\rho(T) = \begin{cases} \sup VP(T|_{S_\rho}) & \text{if } VP(T|_{S_\rho}) \neq \emptyset, \\ -\infty & \text{if } VP(T|_{S_\rho}) = \emptyset, \end{cases}$$

where $T|_{S_\rho}$ means the restriction of T on S_ρ. We also set

$$W(\rho) = W_{\lambda_\rho(T)} \cap S_\rho.$$

This defines a multivalued mapping $W\colon (0,\infty) \to 2^{E_+}$.

3. Eigenvalue problem

We obtain the following theorem:

Theorem 2 *Let T satisfy the assumptions* A1–A4. *Then T has an eigenvalue and*

$$VP(T) = \{\lambda_\rho(T) \mid \rho > 0\} \not\ni 0, \quad W = \cup_{\rho>0} W(\rho).$$

Further the following properties hold:

(i) $\lambda_\rho(T) > 0$ *for each* $\rho > 0$,

(ii) $\lambda_\rho(T)$ *is continuous and nonincreasing with respect to* $\rho > 0$,

(iii) $W(\rho)$ *is a singleton for each* $\rho > 0$,

(iv) $0 \ll W(\rho) \ll W(\rho')$ *when* $0 < \rho < \rho'$,

(v) $\rho \mapsto W(\rho)$ *is a continuous mapping from* $(0,\infty)$ *to* E_+.

Remark 3 If the inequality in A2 holds strictly, that is,

$$T(\lambda x) < \lambda T x$$

for any $\lambda > 1$ and $x \gg 0$, then $\lambda_\rho(T)$ is strictly decreasing with respect to $\rho > 0$.

Remark 4 The assumption A4 can be relaxed somewhat. To be more precise, instead of assuming that T satisfies A4, assume simply that $W \neq \emptyset$ and that $T|_{W\cup\{0\}}$ satisfies A4, that is,

$A4'$: $\{0\} \subsetneq E_{x-y} \subsetneq E_+$ implies $Tx-Ty \notin E_{x-y}$ for $x, y \in W\cup\{0\}$.

Then the same statements as those of Theorem 2 hold.

As the space is limited, we omit the proof of our theorem. See the forthcoming paper [14] for details.

4. Eigenvalue problem for the case where the positive cone has empty interior

In this section we deal with the case where the ordered Banach space E is not necessarily a strongly ordered one; in other words, E_+ may have empty interior. The results in this section make our theory applicable to mappings on such spaces as L^p and the Sobolev spaces.

We assume that

$$0 \leq x \leq y \quad \text{implies} \quad \|x\| \leq \|y\| \tag{4.1}$$

and that there exists some strongly ordered Banach space $V(\dim V \geq 2)$, embedded continuously into E, with a positive cone $V_+ = E_+ \cap V$ such that $TE_+ \subset V_+$. We do not assume that $0 \leq x \leq y$ implies $\|x\|_V \leq \|y\|_V$ for x, $y \in V$, where $\|\cdot\|_V$ denotes the norm on V. For x, $y \in E$, we write $x \gg y$ if and only if x, $y \in V$ and $x - y \in (V_+)^i$. We say $x \in E$ is strongly positive if $x \gg 0$.

We replace some of the assumptions given in Section 2 by the following:

B1(compactness): $T: (E_+, \|\cdot\|) \to (V_+, \|\cdot\|_V)$ is a compact mapping,
B4(indecomposability): for $x,\ y \in V$,

$$\{0\} \subsetneq E_{x-y} \cap V \subsetneq V_+ \quad \text{implies} \quad Tx - Ty \notin E_{x-y}.$$

Replacing the assumption A1 by B1 and A4 by B4, we can prove the same statements as those of Theorem 2 and Remarks 3, 4.

5. Applications

Example 1 (Generalized Fujita lemma)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the Dirichlet boundary value problem:

$$\text{(5.1)} \qquad \begin{cases} \Delta u + f(x,u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where φ is a continuous function on $\partial\Omega$. Here $f(x,u): \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is locally Hölder continuous in x, u, and locally uniformly Lipschitz continuous in u, that is, for any bounded closed interval $[a,b] \subset \mathbb{R}$, there exists some constant $C > 0$ such that

$$\sup_{x \in \overline{\Omega}} \sup_{\substack{u,v \in [a,b] \\ u \neq v}} \frac{|f(x,u) - f(x,v)|}{|u - v|} \leq C.$$

Hereafter we consider only classical solutions. (It is easily shown that any bounded weak solution is classical.) For two solutions u and v, we write $u \leq v$ if $v(x) - u(x) \geq 0$ $(x \in \overline{\Omega})$, $u < v$ if $u \leq v$ and $u \neq v$, and $u \ll v$ if $v(x) - u(x) > 0$ $(x \in \Omega)$ and $\partial v/\partial n(x) - \partial u/\partial n(x) < 0$ $(x \in \partial\Omega)$. Here $\partial/\partial n(x)$ denotes the outer normal derivative at $x \in \partial\Omega$.

We have the following:

Proposition 5 *Suppose that $u \mapsto f(x,u)$ is concave for each $x \in \Omega$. Let u_1, u_2, u_3 be solutions of* (5.1) *satisfying $u_1 < u_2$ and $u_1 < u_3$. Then either* (i), (ii) *or* (iii) *holds.*

(i) $u_1 \ll u_2 = u_3$.

(ii) $u_1 \ll u_2 \ll u_3$ *and* $u_2 = \overline{r}u_1 + (1-\overline{r})u_3$ *for some* $\overline{r} \in (0,1)$. *Furthermore, for any* $r \in [0,1]$,

$$f(x, ru_1(x) + (1-r)u_3(x)) = rf(x, u_1(x)) + (1-r)f(x, u_3(x))$$

(*hence* $ru_1(x) + (1-r)u_3(x)$ *is a solution of* (5.1)).

(iii) $u_1 \ll u_3 \ll u_2$ *and* $u_3 = \overline{r}u_1 + (1-\overline{r})u_2$ *for some* $\overline{r} \in (0,1)$. *The same statement as* (ii) *holds with u_2, u_3 exchanged each other.*

Remark 6 The statement of the proposition remains true with the relations $\leq$, $<$ and $\ll$ replaced by $\geq$, $>$ and $\gg$, respectively, if the concavity assumption on $u \mapsto f(x,u)$ is replaced by the convexity assumption. To see this, simply replace u by $-u$, $f(x,u)$ by $-f(x,-u)$.

Outline of the proof of Proposition 5 Put

$$g(x,w) = f(x, w + u_1(x)) - f(x, u_1(x))$$

and let us consider the following problem:

$$(5.2) \qquad \begin{cases} \Delta w + g(x,w) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Obviously u is a solution of (5.1) if and only if $u - u_1$ is a solution of (5.2).

Since u_1, u_2 and u_3 are continuous functions, there exists some constant $k > 0$ such that $g(x,w) + kw$ is strictly increasing in $w \in [0, \overline{w}(x)]$

with $\overline{w}(x) = \max\{u_2(x) - u_1(x), u_3(x) - u_1(x)\}$ for each $x \in \overline{\Omega}$. Clearly $\overline{w} \neq 0$.

Let $E = L^p(\Omega)$, $E_+ = L^p(\Omega)_+ = \{u \in L^p(\Omega) \mid u(x) \geq 0 \text{ a.e. } x \in \Omega\}$ and $V = C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$. Here $C_0(\overline{\Omega})$ denotes the space of continuous functions on $\overline{\Omega}$ vanishing on the boundary $\partial\Omega$. Note that the positive cones in the space $L^p(\Omega)$ or $C_0(\overline{\Omega})$ have empty interior, whereas $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ has a positive cone with nonempty interior. On the other hand, the norm in $L^p(\Omega)$ or $C_0(\overline{\Omega})$ has the monotonicity as defined in (4.1), while that of $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$ does not have such a property.

Set

$$\overline{g}(x,w) = \begin{cases} g(x,w) + kw & \text{if } w \leq \overline{w}(x) \\ g(x,\overline{w}(x)) + k\overline{w}(x) & \text{if } w > \overline{w}(x) \end{cases}$$

and define the mapping $T\colon E_+ \to V_+$ by

$$Tw = (-\Delta_D + k)^{-1}\overline{g}(x, w(x)).$$

Here Δ_D denotes the Laplace operator under the Dilichlet boundary conditions. Taking $p > n$ and using the L^p estimates of Agmon, Douglis and Nirenberg [1] and the Sobolev embedding theorem, we find that $T\colon E_+ \to V_+$ satisfies the hypotheses B1, A2. Furthermore, by the maximum principle and the boundary lemma of E. Hopf, we see that T also satisfies A3 and the property defined in Remark 4.

Note that w is a solution of (5.2) satisfying $0 < w \leq \overline{w}$ if and only if w is an eigenvector of T corresponding to 1. Applying the generalized version of Theorem 2 and Remarks 3, 4 to this case, we obtain the conclusion of the proposition. □

The following is an immediate consequence of Proposition 5 and Remark 6. This is a generalization of the lemma first established by Fujita under stronger assumptions [3], [4]. The original motivation of Fujita was to study the structure of the set of solutions of the so-called Emden-Fowler equation $\Delta u+\lambda e^u = 0$, but the lemma turns out to be exceedingly useful in various other problems.

Corollary 7 (a generalized Fujita lemma) *Suppose that $u \mapsto f(x,u)$ is concave for each $x \in \Omega$ or convex for each $x \in \Omega$. Let u_1, u_2, u_3 be solution of* (5.1) *satisfying $u_1 \le u_2 \le u_3$. Then either* (a), (b) *or* (c) *holds.*

(a) $u_1 = u_2 = u_3$.

(b) $u_1 = u_2 \ll u_3$ *or* $u_1 \ll u_2 = u_3$.

(c) $u_1 \ll u_2 \ll u_3$ *and statement* (ii) *of Proposition* 5 *holds.*

Example 2 (Bifurcation problem)

Next we consider the following problem:

$$(5.3)\qquad \begin{cases} \Delta u + \lambda f(v) = 0 & \text{in } \Omega, \\ \Delta v + \lambda g(u) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $f: \mathbb{R} \to \mathbb{R}$ is a locally Hölder continuous function satisfying

(F.1) $f(0) = 0$, $f_* := \lim_{\rho \searrow 0} \{f(\rho)/\rho\} > 0$,

(F.2) $0 \le f(\alpha u) \le \alpha f(u)$ for any $\alpha > 1$, $u > 0$,

(F.3) $f(u)$ is nondecreasing in $u > 0$.

We also assume that $g: \mathbb{R} \to \mathbb{R}$ satisfies precisely the same conditions as above and denote those conditions by (G.1), (G.2), (G.3).

Let $\tilde{E} = E \times E$, $\tilde{E}_+ = E_+ \times E_+$ and $\tilde{V} = V \times V$ with E, E_+, V defined in Example 1.

In what follows we consider the number λ in (5.3) to be an unspecified constant, therefore each solution of (5.3) will be written in the form of a pair $(\lambda,\ (u,v))$. Obviously $(\lambda,(0,0))$ is a solution of (5.3) for all $\lambda \in \mathbb{R}$, which we call a "trivial solution". We say $(\lambda,(u,v))$ is a "positive solution" if $(u,v) > 0$.

The pair $(\lambda,\ (u,v))$ is a positive solution of (5.3) if and only if $(u,v) > 0$, and satisfies

$$\lambda T(u,\ v) = (u,\ v),$$

where T is defined by

$$T(u,\ v) = ((-\Delta_D)^{-1} f(v),\ (-\Delta_D)^{-1} g(u)).$$

By the same way as that of Example 1, we find that $T\colon \tilde{E}_+ \to \tilde{V}_+$ satisfies the hypotheses B1, A2, A3 and the property defined in Remark 4. Applying the generalized version of Theorem 2 and Remarks 3, 4 to this case, we obtain the following:

Proposition 8 (i) *Positive solutions of* (5.3) *bifurcate at* $(\lambda_1^*,\ (0,0))$ *from the trivial solutions. Here* $\lambda_1^* = \lambda_1/\sqrt{f_* g_*}$, *where* λ_1 *is the smallest eigenvalue of* $-\Delta_D$. *There is no other bifurcation point. Moreover, there exist mappings* $\lambda\colon (0,\infty) \to \mathbb{R}_+$ *and* $(u,v)\colon (0,\infty) \to \tilde{E}_+$ *such that* $\{(\lambda(\rho),(u(\rho),v(\rho))) \mid \rho \in (0,\infty)\}$ *coincides with the set of all positive solutions of* (5.3). *Furthermore,* λ *is a nondecreasing, subhomogeneous and continuous function, while* (u,v) *is continuous and satisfies* $\|u(\rho)\|_{L^p} + \|v(\rho)\|_{L^p} = \rho$.

(ii) *In addition to condition* (F.2), *assume further that* f *satisfies*

(F.2$'$) there exists some $\delta > 0$ such that $0 \le f(\alpha u) < \alpha f(u)$ for any $\alpha > 1$, $u \in (0,\delta)$,

or that g *satisfies the same condition as above in addition to* (G.2). *Then*

$\lambda(\rho)$ is strictly increasing in ρ and hence (u, v) is parametrizable by λ.

Remark 9 Proposition 8 implies that positive solutions of (5.3) form a single bifurcation branch without a secondary bifurcation (Figure 1). Our method, of course, is also applicable to the single equation

$$(5.4) \qquad \begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which can be handled more easily than the system (5.3). The results for these problems are to some extent known, particularly those for (5.4). But our proof has an advantage in that it requires weaker regularity, monotonicity and subhomogeneity assumptions than those results found in the literature (such as [8], [9]).

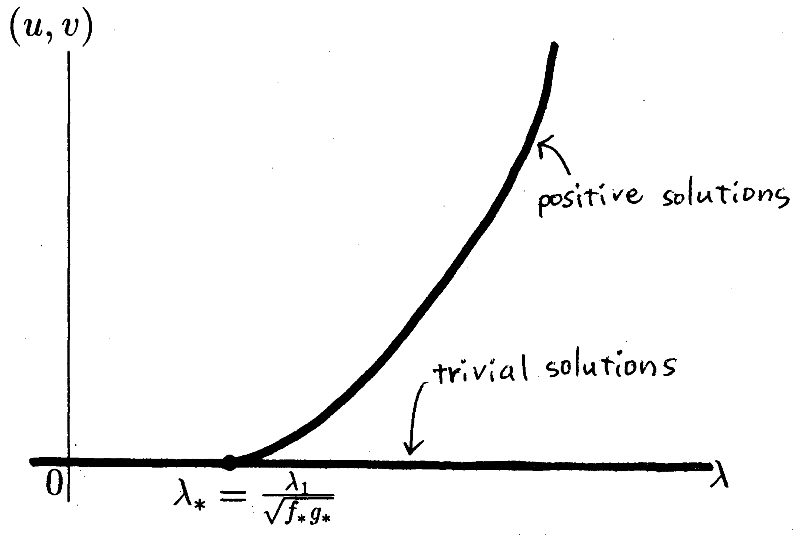

Figure 1: **Example 2**

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