

## Minimax Theorems of Vector-Valued Functions\*

弘前大学理学部<sup>†</sup> 田中 環 (Tamaki Tanaka)<sup>‡</sup>

**Abstract.** In this paper, we consider multicriteria games, in which we have the following particular questions: If we give reasonable definitions for minimax values and maximin values of a vector-valued function in an ordered vector space, what minimax equation or inequality holds? Also, if we give a suitable definition for saddle points of the vector-valued function, under what conditions do there exist such saddle points? Moreover, what relationship holds among such minimax values and maximin values and saddle values? These questions are called minimax problems for vector-valued functions.

Then, we will give interesting answers to such open questions and will show three types of minimax theorems for vector-valued functions.

**Key Words.** Minimax theorems, multicriteria games, vector optimization, multi-objective programming, multiple criteria decision making.

### 1. Introduction and Preliminaries

It is well-known in game theory that scalar games have the following result: a real-valued payoff function possesses a saddle point if and only if the minimax value and the maximin value of the function are coincident. One of unsolved problems in game theory is whether games with multiple noncomparable criteria have an acceptable theory similar to standard results for scalar games. This kind of game is called “a multicriteria game,” and the payoff takes its values in a vector space. Such a game has been researched in the past as found in [2], [25], [6], [11], and [12]. On the other hand, minimax theorems for a vector-valued function and some generalizations of a saddle point concept have been explored actively from the mathematical aspect; see [23], [8], [9], and [10]. Also, the author have separately researched such minimax problems in general setting and proved minimax theorems, existence theorems for saddle points, and saddle point theorems in [28], [29], [30], [32], and [34]; all of such researches are contained in the author’s doctoral thesis [33].

These papers give interesting answers to such open questions above. It is, however, unsatisfactory that few papers of them are devoted to the multicriteria games. As a matter of fact, minimax theorems and saddle point problems for a real-valued function are closely connected with (scalar) game theory. Therefore, it is great important to reexamine the results as shown in the above papers from the multicriteria game’s angle.

---

\*AMS 1990 Subject Classifications. Primary 49J35, Secondary 90C29.

<sup>†</sup>Address: 〒036 青森県弘前市文京町3番地 (Department of Information Science, Faculty of Science, Hirosaki University, Hirosaki 036, JAPAN) Email: tanaka@si.hirosaki-u.ac.jp

<sup>‡</sup>The author is very grateful to Professor K. Tanaka of Niigata University and Professor S. Iwamoto of Kyushu University for their useful suggestions and encouragement on this research.

The aim of this research is to clarify the structure of the multicriteria two-person games and gives an acceptable theory to them. To this end, we will define and characterize several strategies for players of the game. Also, we will present some modifications of author's minimax theorems in [33] for a vector-valued function. Some elementary results and tools in this paper are based on [33] and [34].

The organization of the paper is as follows. In Section 2, we shall give an formulation for a multicriteria two-person zero-sum game, and define an optimal response strategy. In Section 3, we shall give some generalization of the saddle point concept, which are called "cone saddle points," and prove some existence theorems for such saddle points. In Section 4, we shall state a saddle point theorem and three types of minimax theorems for a vector-valued function.

Now, we give the preliminary terminology used throughout the paper. To begin with, the main spaces with mathematical structures on which our results work are a real topological vector space (t.v.s. for short) or a real locally convex space (l.c.s. for short) as a domain of functions and an ordered real topological vector space (ordered t.v.s. for short) as a range space of functions. We assume that the topologies are Hausdorff; one of the reasons why we work on a Hausdorff l.c.s. is the purpose of applying Browder's coincidence theorem; see [3] and [26]. (The coincidence theorem is a cyclical version of Fan-Glicksberg type's fixed-point theorems; see [7] and [13].)

If  $C$  is a convex cone of a real vector space  $S$ , the relation  $\leq_C$  defined below is a (partial) vector ordering of  $S$ : for  $x, y \in S$

$$x \leq_C y \iff y - x \in C. \quad (1.1)$$

Conversely, let  $S$  be a real ordered vector space with a vector ordering  $\leq$ , and let  $C := \{x \in S \mid 0 \leq x\}$ . Then  $C$  is a convex cone of  $S$ , and its ordering  $\leq_C$  is coincident with  $\leq$ ; see page 2 in [19]. Thus, there is a one-to-one correspondence between vector orderings of a real ordered vector space  $S$  and convex cones in  $S$ , and hence we assume that such real ordered vector space [resp. ordered t.v.s.] has a convex cone  $C$  and that the ordering is defined by (1.1).

Throughout this paper, let  $Z$  be an ordered t.v.s. with an ordering defined by a convex cone  $C$ . The convex cone  $C$  is assumed to be pointed, i.e.,  $C \cap (-C) = \{0\}$ , and hence the ordering is antisymmetric and  $C \ni 0$ . Moreover, for the convenience, the convex cone  $C$  is assumed to be solid, i.e., its (topological) interior  $\text{int}C$  is nonempty, and hence  $C^0 := (\text{int}C) \cup \{0\}$  is a pointed convex cone and induces another (antisymmetric) vector ordering  $\leq_{C^0}$  weaker than  $\leq_C$  in  $Z$ . Also, we remark that the orderings  $\leq_C$  and  $\leq_{C^0}$  are two directed antisymmetric partial orderings; "an ordering  $\leq$  is directed" means that given  $x, y \in Z$ , there exists  $z \in Z$  such that  $x \leq z$  and  $y \leq z$ .

Next, with respect to each of the orderings  $\leq_C$  and  $\leq_{C^0}$ , we shall define minimal elements and maximal elements of a subset  $A$  of  $Z$ . As the concept, we will adopt "cone extreme point," the concept of which was proposed by P.L.Yu in [35]. An element  $z_0$  of a subset  $A$  of  $Z$  is said to be a  $C$ -minimal point of  $A$  if  $\{z \in A \mid z \leq_C z_0, z \neq z_0\} = \emptyset$ , and a  $C$ -maximal point of  $A$  if  $\{z \in A \mid z_0 \leq_C z, z \neq z_0\} = \emptyset$ ; which are equivalent to  $A \cap (z_0 - C) = \{z_0\}$  and  $A \cap (z_0 + C) = \{z_0\}$ , respectively. We denote the set of such all  $C$ -minimal [resp.  $C$ -maximal] points of  $A$  by  $\text{Min}A$  [resp.  $\text{Max}A$ ]. Also,  $C^0$ -minimal and  $C^0$ -maximal points of  $A$  are defined similarly, and denoted by  $\text{Min}_w A$  and  $\text{Max}_w A$ , respectively. These  $C^0$ -minimality and  $C^0$ -maximality are weaker concepts than  $C$ -minimality and  $C$ -maximality, respectively: it should be remarked that  $\text{Min}A \subset \text{Min}_w A \subset A$  and  $\text{Max}A \subset$

$\text{Max}_w A \subset A$ . Moreover,  $\text{Min} \alpha A = \alpha \text{Min} A$ ,  $\text{Max} \alpha A = \alpha \text{Max} A$ ,  $\text{Min}_w \alpha A = \alpha \text{Min}_w A$ ,  $\text{Max}_w \alpha A = \alpha \text{Max}_w A$  for any  $\alpha > 0$ ; and  $\text{Min}(A + a) = \text{Min} A + a$ ,  $\text{Max}(A + a) = \text{Max} A + a$ ,  $\text{Min}_w(A + a) = \text{Min}_w A + a$ ,  $\text{Max}_w(A + a) = \text{Max}_w A + a$  for any  $a \in Z$ .

## 2. A Multicriteria Two-Person Zero-Sum Game

A multicriteria two-person zero-sum game, or a two-person vector-valued zero-sum game, is a 4-tuple  $\Gamma := (X, Y, -f, f)$ , where  $X$  and  $Y$  are nonempty sets and  $f$  is a mapping  $f : X \times Y \rightarrow Z$ , and  $Z$  is an ordered t.v.s. with an ordering defined by a convex cone  $C$ . The set  $X$  (resp.  $Y$ ) is the set of strategies of player 1 (resp. player 2) and the mapping  $-f$  (resp.  $f$ ) is the payoff function of this player. (Possibly such strategies are mixed strategies and such payoff functions are the expected payoffs.) When player 1 and player 2 choose a strategy  $x \in X$  and a strategy  $y \in Y$ , respectively, the payoffs with respect to player 1 and player 2 are given by vectors  $-f(x, y)$  and  $f(x, y)$ , respectively. Each player chooses a strategy in order to increase his payoff and wants to find a strategy maximizing his payoff. Thus, player 1 is regarded as minimizer and player 2 as maximizer.

Under this game, there are two approaches for finding equilibrium strategies. One is to find optimal response strategy pairs, and the other is to find optimal security strategy pairs. These concepts are coincident when  $Z = \mathbf{R}$ , i.e., the payoff function  $f$  is scalar-valued. The concepts of optimal response strategy and optimal security strategy are given in [11].

In this paper, we will adopt the approach of finding optimal response strategy pairs. To this end, we generalize the concept of optimal response strategy to more general type.

**Definition 2.1.** A strategy  $x_0 \in X$  is said to be an optimal response strategy for player 1 against a strategy  $y \in Y$  of player 2 if  $f(x_0, y) \in \text{Min} f(X, y)$ . Similarly, a strategy  $y_0 \in Y$  is said to be an optimal response strategy for player 2 against a strategy  $x \in X$  of player 1 if  $f(x, y_0) \in \text{Max} f(x, Y)$ . The sets of all optimal response strategies for each player against an opponent's given strategy ( $y \in Y$  and  $x \in X$ ) are defined by  $R_1(y)$  and  $R_2(x)$ , respectively.

**Definition 2.2.** A strategy  $x_0 \in X$  is said to be a weak optimal response strategy for player 1 against a strategy  $y \in Y$  of player 2 if  $f(x_0, y) \in \text{Min}_w f(X, y)$ . Similarly, a strategy  $y_0 \in Y$  is said to be a weak optimal response strategy for player 2 against a strategy  $x \in X$  of player 1 if  $f(x, y_0) \in \text{Max}_w f(x, Y)$ . The sets of all weak optimal response strategies for each player against an opponent's given strategy ( $y \in Y$  and  $x \in X$ ) are  $R_1^w(y)$  and  $R_2^w(x)$ , respectively.

For the convenience, we denote each optimal response set as follows:

$$D_1 := \{(x, y) \in X \times Y \mid y \in R_2(x), x \in X\}, \quad (2.1)$$

$$D_2 := \{(x, y) \in X \times Y \mid x \in R_1(y), y \in Y\}. \quad (2.2)$$

Similarly, weak optimal response sets are denoted by  $D_1^w$  and  $D_2^w$ . It should be remarked that for any  $y \in Y$  and  $x \in X$ ,

$$R_1(y) \subset R_1^w(y), \quad R_2(x) \subset R_2^w(x) \quad (2.3)$$

and that

$$D_1 \subset D_1^w, \quad D_2 \subset D_2^w. \quad (2.4)$$

Then, we find the following idea of equilibrium strategies.

**Definition 2.3.** A point  $(x_0, y_0)$  is said to be an equilibrium optimal response strategy pair [resp. equilibrium weak optimal response strategy pair] of the game if  $x_0 \in R_1(y_0)$  and  $y_0 \in R_2(x_0)$  [resp.  $x_0 \in R_1^w(y_0)$  and  $y_0 \in R_2^w(x_0)$ ]. The set of all equilibrium optimal response strategy pairs (resp. equilibrium optimal response strategy pairs) is given by  $D_1 \cap D_2$  (resp.  $D_1^w \cap D_2^w$ ).

These concepts coincide with ones of a  $C$ -saddle point and a weak  $C$ -saddle point of  $f$ , respectively;

$$f(x_0, y_0) \in \text{Max}f(x_0, Y) \cap \text{Min}f(X, y_0), \quad (2.5)$$

$$f(x_0, y_0) \in \text{Max}_w f(x_0, Y) \cap \text{Min}_w f(X, y_0). \quad (2.6)$$

Such concepts of generalized saddle point are defined in Section 3. We present the following example, which is given in [6].

**Example 2.1.** Consider a multicriteria two person (matrix) game  $\Gamma = (X, Y, -f, f)$  as follows:

$$X = Y = \left\{ x \in \mathbf{R}^2 \mid x = (x_1, x_2), \sum_{i=1}^2 x_i = 1, x_1, x_2 \geq 0 \right\},$$

$$Z = \mathbf{R}^2, C = \mathbf{R}_+^2 = \left\{ z \in \mathbf{R}^2 \mid z = (z_1, z_2), z_1 \geq 0, z_2 \geq 0 \right\},$$

$$f(x, y) = (x^T A_1 y, x^T A_2 y), \quad \text{where } A_1 = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}.$$

Then, we have

$$R_1(y) = \begin{cases} X & 0 \leq y_1 < \frac{1}{3}, \frac{2}{3} < y_1 \leq 1; \\ \{(0, 1)\} & \text{otherwise,} \end{cases}$$

$$R_2(x) = \begin{cases} Y & 0 \leq x_1 < \frac{1}{3}, \frac{2}{3} < x_1 \leq 1; \\ \{(0, 1)\} & \text{otherwise,} \end{cases}$$

for  $x = (x_1, x_2) \in X$  and  $y = (y_1, y_2) \in Y$ . Also, the corresponding optimal response sets  $D_1$  and  $D_2$  transformed by the substitution  $x_2 = 1 - x_1$  and  $y_2 = 1 - y_1$  are shown in Figure 2.1.

In this example, when player 1 and player 2 choose their strategies  $x^* = (x_1, x_2)$  and  $y^* = (y_1, y_2)$  satisfying  $(x^*, y) \in D_1$  and  $(x, y^*) \in D_2$  for some  $y \in Y$  and  $x \in X$ , respectively, the pair  $(x^*, y^*)$  is not always in  $D_1 \cap D_2$ , and hence the set of all equilibrium optimal response strategy pairs does not always have interchangeability, which holds for scalar games. Therefore, we proceed to the next usual concept of strategy.

We called the following subsets of  $Z$

$$\text{Min} \bigcup_{x \in X} \text{Max}_w f(x, Y), \quad \text{Max} \bigcup_{y \in Y} \text{Min}_w f(X, y) \quad (2.7)$$

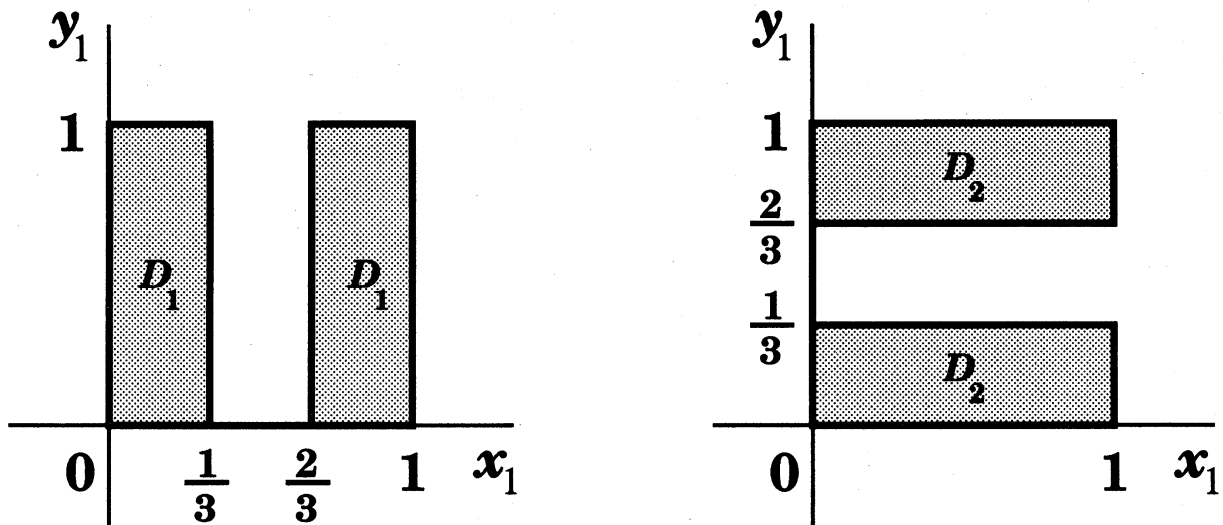


Figure 2.1: The optimal response sets  $D_1$  and  $D_2$  in Example 2.1.

the set of all minimax values of  $f$  and the set of all maximin values of  $f$ , respectively (see [34] and [33]). Also, we call a strategy  $x^*$  [resp.  $y^*$ ] attaining a minimax value [resp. a maximin value] above, i.e., satisfying  $f(x^*, y) \in \text{Min}f(D_1^w)$  for some  $y \in R_2^w(x^*)$  [resp.  $f(x, y^*) \in \text{Max}f(D_2^w)$  for some  $x \in R_1^w(y^*)$ ] a minimax strategy [resp. a maximin strategy] of player 1 [resp. player 2], written  $M_{\text{ax}}^{\text{in}}(f)$  [resp.  $M_{\text{in}}^{\text{ax}}(f)$ ]. In order to show the existence of such minimax strategies and maximin strategies, we will review some of the fundamental properties of cone extreme points.

Let  $A$  be a nonempty subset of an ordered t.v.s.  $Z$  with an ordering defined by a (solid) pointed convex cone  $C$ . We say that the set  $A$  has the “domination property” (e.g., see page 697 in [21] and page 53 in [22]) if

$$\text{Min}A \neq \emptyset \quad [\text{resp. } \text{Max}A \neq \emptyset], \quad (2.8)$$

and

$$A \subset \text{Min}A + C \quad [\text{resp. } A \subset \text{Max}A - C]. \quad (2.9)$$

In particular, to produce conditions ensuring the condition (2.8) is one of the most important questions of vector optimization theory; e.g., see [14], [16], [20], [21], and [27]. In this paper, we need the following lemmas; see Lemmas 5.2 and 5.4 in [34] or Lemmas 1.2.3 and 1.2.5 in [33]:

**Lemma 2.1.** *Let  $Z$  be an ordered t.v.s. with an ordering defined by a (solid) pointed convex cone  $C$ , and  $A$  a subset of  $Z$ . If the convex cone  $C$  of  $Z$  satisfies the condition*

$$\text{cl}C + (C \setminus \{0\}) \subset C \quad (2.10)$$

*and if  $A$  is nonempty and compact, then  $\text{Min}A \neq \emptyset$ ,  $A \subset \text{Min}A + C$  and  $\text{Max}A \neq \emptyset$ ,  $A \subset \text{Max}A - C$ .*

**Lemma 2.2.** *Let  $Z$  be an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$ , and  $A$  a subset of  $Z$ . If  $A$  is nonempty and compact, then  $\text{Min}_w A \neq \emptyset$ ,  $A \subset \text{Min}_w A + C^0$  and  $\text{Max}_w A \neq \emptyset$ ,  $A \subset \text{Max}_w A - C^0$ . Moreover,  $\text{Min}_w A$  and  $\text{Max}_w A$  are compact sets.*

Moreover, we need the following concept and related results given by Luc:

**Definition 2.4.** A set  $A$  is said to be  $C$ -complete if there are no covers of the form  $\{(z_\alpha - \text{cl}C)^c\}$  where  $\{z_\alpha\}$  is a net in  $A$  such that  $z_\beta \leq_C z_\alpha$  and  $z_\alpha \not\leq_C z_\beta$  for each  $\alpha, \beta$  with  $\alpha < \beta$ .

**Lemma 2.3.** (See Theorem 2.6 and Corollary 2.12 in [21].) *If a convex cone  $C$  of a t.v.s. is correct, and if  $A$  is nonempty and  $C$ -complete, then the conditions (2.8) and (2.9) hold.*

**Remark 2.1.** We recall that a set  $A$  is said to be  $C$ -compact if  $(z - \text{cl}C) \cap A$  is compact for each  $z \in A$ ; and that a set  $A$  is said to be  $C$ -semicompact if any cover of the form  $\{(z_\lambda - \text{cl}C)^c : z_\lambda \in A, \lambda \in \Lambda\}$  has a finite subcover; see [5]. Then, any compact set is  $C$ -compact, hence  $C$ -semicompact, and hence  $C$ -complete whatever the convex cone  $C$  is; see Lemma 2.2 in [21]. The last general condition is rather "far" from compactness but still guarantees the condition (2.8).

Let  $S_1$  and  $S_2$  be two topological spaces, respectively. A mapping  $F$  from  $S_1$  into  $S_2$  is said to be upper semicontinuous at  $x \in S_1$ , if for any open neighborhood  $V$  of  $F(x)$ , there exists a neighborhood  $U$  of  $x$  such that  $F(y) \subset V$  for all  $y \in U$ . We say that  $F$  is upper semicontinuous (u.s.c. for short) if it is so at every  $x \in S_1$ ; see Definition 1 in page 41 of [1]. If  $S$  is a compact set in  $S_1$  and  $F$  is an u.s.c. compact-valued mapping from  $S$  into  $S_2$ , then the image  $F(S)$  under  $F$  of  $S$  is compact; see Proposition 3 in page 42 of [1]. Based on this fact, we have the following theorem:

**Theorem 2.1.** (See Lemma 5.5 in [34].) *Let  $X$  and  $Y$  be nonempty compact sets in two topological spaces, respectively, and  $Z$  an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$ . If a vector-valued function  $f : X \times Y \rightarrow Z$  is continuous, and if  $C$  satisfies the condition (2.10), then*

$$\left[ \text{Min} \bigcup_{x \in X} \text{Max}_w f(x, Y) \right] + C \supset \text{Max}_w f(x', Y) \neq \emptyset, \quad (2.11)$$

$$\left[ \text{Max} \bigcup_{y \in Y} \text{Min}_w f(X, y) \right] - C \supset \text{Min}_w f(X, y') \neq \emptyset \quad (2.12)$$

for each  $x' \in X$  and  $y' \in Y$ .

The theorem above shows that under the conditions, there exists at least one optimal response strategy for each player against an opponent's given strategy, and that there exist a minimax strategy and a maximin strategy, too.

When  $f(x_0, y_0) \in \text{Max}_w f(x_0, Y) \cap \text{Min}_w f(X, y_0)$ , we say that  $f$  has a weak  $C$ -saddle point  $(x_0, y_0)$  as defined in the next section. If  $f$  has such saddle points, we can see by the theorem that a certain minimax inequality holds, which is called a saddle point theorem of a vector-valued function, located in Section 4, as a corollary of the theorem above. This can be interpreted in the following way: Minimax values and maximin values are lower bounds and upper bounds of saddle values, respectively, in the sense of  $\leq_C$ .  $x^* \in M_{\text{ax}}^{\text{in}}(f) \cap R_1^w(y^*)$  In order to illustrate these results, we give the following example:

**Example 2.2.** Consider the multicriteria two person game  $\Gamma = (X, Y, -f, f)$  in Example 2.1. Let

$$\arg \operatorname{minimax} f := \left\{ (x, y) \in X \times Y \mid f(x, y) \in \operatorname{Min} \bigcup_{z \in X} \operatorname{Max}_w f(x, Y) \right\}$$

and

$$\arg \operatorname{maximin} f := \left\{ (x, y) \in X \times Y \mid f(x, y) \in \operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w f(X, y) \right\}.$$

Then, we have

$$\operatorname{Min} \bigcup_{z \in X} \operatorname{Max}_w f(x, Y) = \left\{ z \in \mathbf{R}^2 \mid z = (z_1, z_2), z_2 = -2z_1, 0 \leq z_1 \leq 1 \right\},$$

$$\arg \operatorname{minimax} f = (\{(0, 1)\} \times Y) \cup \left( \left\{ (\alpha, 1 - \alpha) \mid 0 \leq \alpha \leq \frac{1}{3}, \frac{2}{3} \leq \alpha \leq 1 \right\} \times \{(1, 0)\} \right),$$

$$\operatorname{Min}_{\text{ax}}^{\text{in}}(f) = \left\{ x \in \mathbf{R}^2 \mid x = (\alpha, 1 - \alpha), 0 \leq \alpha \leq \frac{1}{3}, \frac{2}{3} \leq \alpha \leq 1 \right\},$$

$$\operatorname{Max} \bigcup_{y \in Y} \operatorname{Min}_w f(X, y) = \left\{ z \in \mathbf{R}^2 \mid z = (z_1, z_2), z_1 = -2z_2, 0 \leq z_1 \leq 2 \right\},$$

$$\arg \operatorname{maximin} f = (X \times \{(0, 1)\}) \cup \left( \{(1, 0)\} \times \left\{ (\alpha, 1 - \alpha) \mid 0 \leq \alpha \leq \frac{1}{3}, \frac{2}{3} \leq \alpha \leq 1 \right\} \right),$$

$$\operatorname{Max}_{\text{in}}^{\text{ax}}(f) = \left\{ y \in \mathbf{R}^2 \mid y = (\alpha, 1 - \alpha), 0 \leq \alpha \leq \frac{1}{3}, \frac{2}{3} \leq \alpha \leq 1 \right\},$$

and

$$D_1^w \cap D_2^w = (X \times \{(0, 1)\}) \cup (\{(0, 1)\} \times Y) \cup (E(\Gamma) \times E(\Gamma))$$

where

$$E(\Gamma) = \left\{ x \in \mathbf{R}^2 \mid x = (\alpha, 1 - \alpha), 0 \leq \alpha \leq \frac{1}{3}, \frac{2}{3} \leq \alpha \leq 1 \right\}.$$

The decision space and objective space transformed by the substitution  $x_2 = 1 - x_1$  and  $y_2 = 1 - y_1$  are shown in Figure 2.2. Moreover, we have  $E(\Gamma) = \operatorname{Min}_{\text{ax}}^{\text{in}}(f) = \operatorname{Max}_{\text{in}}^{\text{ax}}(f)$  and  $E(\Gamma) \times E(\Gamma) \subset D_1^w \cap D_2^w$ .

Now, we shall investigate the existence of elements of  $D_1^w \cap D_2^w$  in the next section.

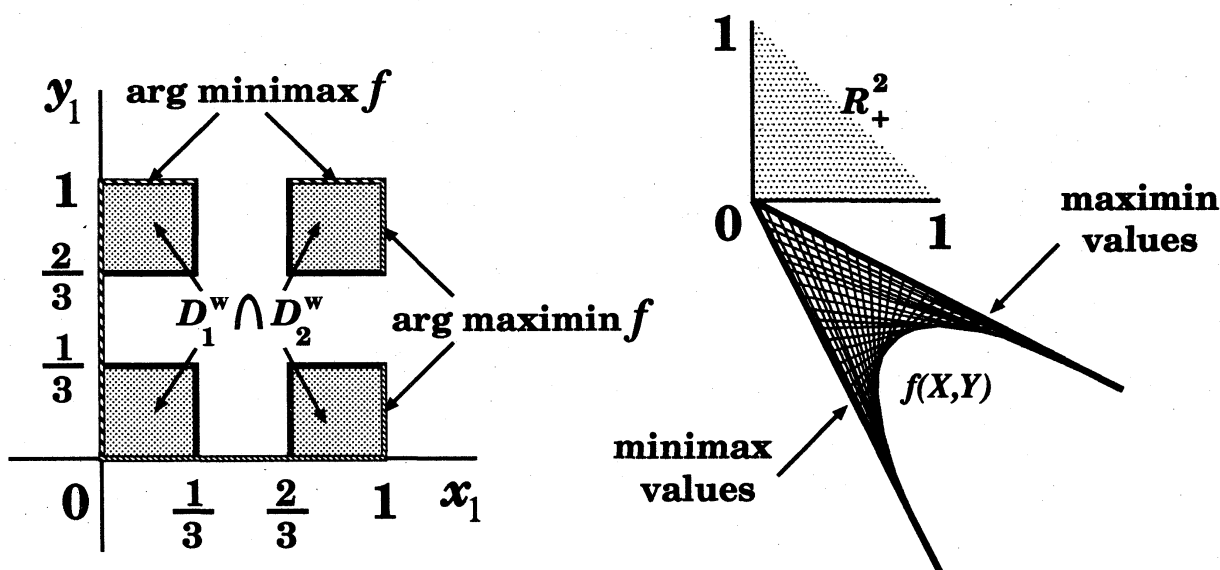


Figure 2.2: The decision space and objective space in Example 2.2.

### 3. Existence of Generalized Saddle Points

Under the previous notation given in Section 1, we will give reasonable definitions for saddle point and saddle value of a vector-valued function. It follows that the definitions are reasonable from the facts that, as shown in Section 2, any equilibrium optimal response strategy pair [resp. equilibrium optimal response strategy pair]  $(x_0, y_0)$  satisfies the condition (2.5) [resp. (2.6)]. Let  $Z$  be an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$  and  $f : X \times Y \rightarrow Z$  a vector-valued function, respectively.

**Definition 3.1.** (i) A point  $(x_0, y_0)$  is said to be a  $C$ -saddle point of  $f$  with respect to  $X \times Y$ , if  $f(x_0, y_0) \in \text{Max}f(x_0, Y) \cap \text{Min}f(X, y_0)$ ;  
(ii) A point  $(x_0, y_0)$  is said to be a weak  $C$ -saddle point of  $f$  with respect to  $X \times Y$ , if  $f(x_0, y_0) \in \text{Max}_w f(x_0, Y) \cap \text{Min}_w f(X, y_0)$ .

For the convenience, we denote the set of all  $C$ -saddle points [resp. weak  $C$ -saddle points] of  $f$  by  $\text{SP}(f)$  [resp.  $\text{SP}_w(f)$ ] and the set of all  $C$ -saddle values [resp. weak  $C$ -saddle values] of  $f$  by  $\text{SV}(f)$  [resp.  $\text{SV}_w(f)$ ].

We note that any  $C$ -saddle point of  $f$  is a weak  $C$ -saddle point of  $f$  obviously. Also, in the case  $C^0 = C$ , the two concepts are coincident.

Now, we give the definition of  $C$ -semicontinuous; see Definition 2.4 in [5].

**Definition 3.2.** Let  $X$  be a topological space and  $Z$  an ordered t.v.s. with an ordering defined by a pointed convex cone  $C$ . A vector-valued function  $f : X \rightarrow Z$  is said to be  $C$ -semicontinuous if  $f^{-1}(z - \text{cl}C)$  is closed in  $X$  for each  $z \in Z$ .

First of all, we state the first existence theorem for generalized saddle points, which is a generalization of Theorem 3.2.1 in [33].

**Theorem 3.1.** Let  $X$  and  $Y$  be nonempty compact sets in two topological spaces, respectively, and  $Z$  an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$ .



A vector-valued function  $f : X \times Y \rightarrow Z$  has at least one weak  $C$ -saddle point if one of the following conditions holds:

- (i)  $f$  is of the type  $f(x, y) = u(x) + v(y)$  where  $u$  and  $-v$  are  $C$ -semicontinuous;
- (ii)  $f$  is of the type  $f(x, y) = u(x) + \beta(x)v(y)$  where  $u$  is continuous,  $-v$  is  $C$ -semicontinuous, and  $\beta : X \rightarrow \mathbf{R}_+$  is continuous.

If, in addition,  $C$  satisfies the condition (2.10), then  $f$  has at least one  $C$ -saddle point.

*Proof.* (i) By Corollary 3.1 in [5], the two sets  $u(X)$  and  $v(Y)$  are  $C$ -semicompact and  $-C$ -semicompact, respectively. Then, from Lemma 2.3 and Remark 2.1, it follows that there exist  $x_0 \in X$  and  $y_0 \in Y$  such that

$$u(x_0) \in \text{Min}_w u(X) \quad \text{and} \quad v(y_0) \in \text{Max}_w v(Y).$$

Hence, we have

$$\begin{aligned} f(x_0, y_0) &= u(x_0) + v(y_0) \\ &\in (u(x_0) + \text{Max}_w v(Y)) \cap (\text{Min}_w u(X) + v(y_0)) \\ &= \text{Max}_w (u(x_0) + v(Y)) \cap \text{Min}_w (u(X) + v(y_0)) \\ &= \text{Max}_w f(x_0, Y) \cap \text{Min}_w f(X, y_0), \end{aligned}$$

which shows that the point  $(x_0, y_0)$  is a weak  $C$ -saddle point of  $f$ .

(ii) Similarly, we can choose  $y_0 \in Y$  such that  $v(y_0) \in \text{Max}_w v(Y)$ . By the continuity of  $u$  and  $\beta$ ,  $x \mapsto u(x) + \beta(x)v(y_0)$  is also continuous, and so we can choose  $x_0 \in X$ , by Lemma 2.2, such that

$$\begin{aligned} u(x_0) + \beta(x_0)v(y_0) &\in \text{Min}_w \left( \bigcup_{x \in X} \{u(x) + \beta(x)v(y_0)\} \right) \\ &= \text{Min}_w f(X, y_0), \end{aligned}$$

which shows that  $f(x_0, y_0) \in \text{Min}_w f(X, y_0)$ . On the other hand, since  $\beta(x) > 0$  for any  $x \in X$ , we can easily verify that

$$\beta(x_0)\text{Max}_w v(Y) = \text{Max}_w (\beta(x_0)v(Y)),$$

which shows that  $f(x_0, y_0) \in \text{Max}_w f(x_0, Y)$ . Thus,  $f$  has a weak  $C$ -saddle point. The remainder of the proof follows immediately from Lemma 2.1.  $\square$

**Remark 3.1.** Since every continuous function is  $C$ -semicontinuous obviously, the theorem above holds for continuous functions  $u$  and  $v$ . Hence, the theorem is a generalization of Lemma 3.4 in [32].

To obtain another generalized version of Lemma 3.4 in [32], we introduce new concept of another weak continuity of vector-valued functions.

**Definition 3.3.** Let  $X$  be a topological space and  $Z$  an ordered t.v.s. with an ordering defined by a pointed convex cone  $C$ . A vector-valued function  $f : X \rightarrow Z$  is said to be  $C$ -lower semicontinuous on  $X$  if it satisfies the three equivalent conditions:

- (i) For all  $a \in Z$ ,  $f^{-1}(a + \text{int}C)$  is open;
- (ii) For each  $x_0 \in X$  and any open neighborhood  $V$  of  $f(x_0)$ , there exists an open neighborhood  $U$  of  $x_0$  such that  $f(x) \in V + C$  for all  $x \in U$ ;
- (iii) For each  $x_0 \in X$  and any  $d \in \text{int}C$ , there exists an open neighborhood  $U$  of  $x_0$  such that  $f(x) \in f(x_0) - d + \text{int}C$  for all  $x \in U$ .

Also, it is said to be  $C$ -upper semicontinuous on  $X$  if  $-f$  is  $C$ -lower semicontinuous on  $X$ .

**Proposition 3.1.** *Let  $X$  be a topological space and  $Z$  an ordered t.v.s. with an ordering defined by a pointed convex cone  $C$ . The conditions (i), (ii), (iii) of the above definition are equivalent to each other.*

*Proof.* We show that (i) and (iii) are equivalent to each other. Suppose that for all  $a \in Z$ ,  $f^{-1}(a + \text{int}C)$  is open. For each  $x_0 \in X$  and any  $d \in \text{int}C$ , we have  $x_0 \in f^{-1}(\text{int}C + (f(x_0) - d))$ , which is open. Hence, there exists an open neighborhood  $U$  of  $x_0$  such that  $f(x) \in f(x_0) - d + \text{int}C$  for all  $x \in U$ . Conversely, let  $x_0 \in f^{-1}(a + \text{int}C)$  and  $d := f(x_0) - a$ . Since  $d \in \text{int}C$ , there exists an open neighborhood  $U$  of  $x_0$  such that  $f(x) \in f(x_0) - d + \text{int}C$  for all  $x \in U$ , and hence  $f(x) \in a + \text{int}C$ . This implies that  $f^{-1}(a + \text{int}C)$  is open.

Next, we show that (ii) and (iii) are equivalent to each other. Let  $x_0 \in X$  and  $d \in \text{int}C$ . Then, there is an open neighborhood  $W$  of  $d$  such that  $W \subset \text{int}C$ , which implies that  $f(x_0) \in f(x_0) - d + W \subset f(x_0) - d + \text{int}C$ . Let  $V := f(x_0) - d + W$ , then there exists an open neighborhood  $U$  of  $x_0$  such that  $f(x) \in W + C \subset (f(x_0) - d + \text{int}C) + C$  for all  $x \in U$ . Since  $\text{int}C + C = \text{int}C$ , we have  $f(x) \in f(x_0) - d + \text{int}C$  for all  $x \in U$ . Conversely, let  $x_0 \in X$  and  $V$  an open neighborhood of  $f(x_0)$ . There is  $d \in \text{int}C$  such that  $f(x_0) - d \in V$ . By assumption, there exists an open neighborhood  $U$  of  $x_0$  such that  $f(x) \in f(x_0) - d + \text{int}C$  for all  $x \in U$ . Since  $V$  is open and so  $V + \text{int}C = V + C$ , we have  $f(x) \in V + C$  for all  $x \in U$ .  $\square$

Then, we can easily prove the following lemma.

**Lemma 3.1.** *Let  $X$  be a topological space and  $Z$  an ordered t.v.s. with an ordering defined by a pointed convex cone  $C$ . If  $f$  and  $g$  are  $C$ -lower semicontinuous functions from  $X$  to  $Z$ , then*

- (i)  $f + g$  is  $C$ -lower semicontinuous;
- (ii)  $\alpha f$  is  $C$ -lower semicontinuous for each  $\alpha > 0$ ;

Moreover, if  $\beta : X \rightarrow \mathbf{R}_+$  is lower semicontinuous, then for each  $v \in C$ ,

- (i)  $x \mapsto \beta(x)v$  is  $C$ -lower semicontinuous;
- (ii)  $x \mapsto f(x) + \beta(x)v$  is  $C$ -lower semicontinuous.

**Theorem 3.2.** *Let  $X$  and  $Y$  be nonempty compact sets in two topological spaces, respectively, and  $Z$  an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$ . A vector-valued function  $f : X \times Y \rightarrow Z$  has at least one weak  $C$ -saddle point if  $f$  is of the type  $f(x, y) = u(x) + \beta(x)v(y)$  where  $u$  is  $C$ -lower semicontinuous,  $-v$  is  $C$ -upper semicontinuous, and  $\beta : X \rightarrow \mathbf{R}_+$  is lower semicontinuous,  $C \cap \text{Max}_w v(Y) \neq \emptyset$ . If, in addition,  $C$  satisfies the condition (2.10), then  $f$  has at least one  $C$ -saddle point.*

*Proof.* In a similar way of Lemma 3.2 in [21], we can show that the set  $v(Y)$  is  $C$ -complete. Then, we can choose  $y_0 \in Y$  such that  $v(y_0) \in C \cap \text{Max}_w v(Y)$ . From Lemma 3.1, it follows that  $x \mapsto u(x) + \beta(x)v(y_0)$  is  $C$ -lower semicontinuous, and hence the image  $\bigcup_{x \in X} \{u(x) + \beta(x)v(y_0)\}$  is  $C$ -complete in a similar way of Lemma 3.2 in [21] again. Therefore, we can choose  $x_0 \in X$ , by Lemma 2.3, such that

$$\begin{aligned} u(x_0) + \beta(x_0)v(y_0) &\in \text{Min}_w \left( \bigcup_{x \in X} \{u(x) + \beta(x)v(y_0)\} \right) \\ &= \text{Min}_w f(X, y_0), \end{aligned}$$

which shows that  $f(x_0, y_0) \in \text{Min}_w f(X, y_0)$ . In the same way of the proof of Theorem 3.1, we can verify that  $f$  has a weak  $C$ -saddle point  $(x_0, y_0)$ .  $\square$

Second, to present the second existence theorem for generalized saddle points, we introduce new concepts of convexity and continuity of vector-valued functions.

**Definition 3.4.** Let  $X$  be a convex set in a real vector space and  $Z$  an ordered t.v.s. with an ordering defined by a (solid) pointed convex cone  $C$ . A vector-valued function  $f : X \rightarrow Z$  is said to be naturally quasi  $C$ -convex on  $X$  if

$$f(\lambda x_1 + (1 - \lambda)x_2) \in \text{co}\{f(x_1), f(x_2)\} - C \quad (3.1)$$

for every  $x_1, x_2 \in X$  and  $\lambda \in [0, 1]$ , where  $\text{co}A$  denotes the convex hull of the set  $A$ . The condition (3.1) is equivalent to the following condition: there exists  $\mu \in [0, 1]$  such that  $f(\lambda x_1 + (1 - \lambda)x_2) \leq_C \mu f(x_1) + (1 - \mu)f(x_2)$ . Also, it is said to be naturally quasi  $C$ -concave on  $X$  if  $-f$  is naturally quasi  $C$ -convex on  $X$ .

**Remark 3.2.** In [31] and [34], we mentioned the relationship among various types of the convexity generalized to vector-valued functions: we note, in particular, that every  $C$ -convex function is naturally quasi  $C$ -convex, and that every properly quasi  $C$ -convex function is naturally quasi  $C$ -convex. (Let  $X$  be a convex set in a real vector space. A vector-valued function  $f : X \rightarrow Z$  is said to be (i)  $C$ -convex on  $X$  if  $f(\lambda x_1 + (1 - \lambda)x_2) \leq_C \lambda f(x_1) + (1 - \lambda)f(x_2)$  for every  $x_1, x_2 \in X$  and  $\lambda \in [0, 1]$ ; (ii) properly quasi  $C$ -convex on  $X$  if either  $f(\lambda x_1 + (1 - \lambda)x_2) \leq_C f(x_1)$  or  $f(\lambda x_1 + (1 - \lambda)x_2) \leq_C f(x_2)$  for every  $x_1, x_2 \in X$  and  $\lambda \in [0, 1]$ .)

**Lemma 3.2.** Let  $X$  be a convex set in a real vector space and  $Z$  an ordered t.v.s. with an ordering defined by a (solid) pointed convex cone  $C$ , and we denote the set of all continuous linear functionals on  $Z$  by  $Z^*$ . If a mapping  $f : X \rightarrow Z$  is naturally quasi  $C$ -convex on  $X$  then for each  $\varphi \in Z^*$ , the composite mapping  $\varphi \circ f$  is a (ordinary) quasi convex function.

**Definition 3.5.** Let  $X$  be a topological space and  $Z$  another topological space. A mapping  $f : X \rightarrow Z$  is said to be demicontinuous on  $X$  if  $f^{-1}(M) := \{x \in X \mid f(x) \in M\}$  is closed in  $X$  for each closed half-space  $M \subset Z$ .

**Remark 3.3.** Every continuous mapping is demicontinuous obviously. Also, various relationship among continuity,  $C$ -lower semicontinuity,  $C$ -semicontinuity, demicontinuity of vector-valued functions is illustrated in Figure 3.3.

**Lemma 3.3.** *Let  $X$  be a topological space and  $Z$  a t.v.s. If a mapping  $f : X \rightarrow Z$  is demicontinuous on  $X$ , then for each  $\varphi \in Z^*$ , the composite mapping  $\varphi \circ f$  is continuous.*

Then, we have the second existence theorem of weak  $C$ -saddle points, which generalizes Lemma 3.3 in [32] and Theorem 3.1 in [34], and the proof is based on Hartung's minimax theorem; see [15].

**Theorem 3.3.** *Let  $X$  and  $Y$  be nonempty compact convex sets in two t.v.s.'s, respectively, and  $Z$  an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$ . If a vector-valued function  $f : X \times Y \rightarrow Z$  satisfies that*

- (i)  $x \mapsto f(x, y)$  is demicontinuous and naturally quasi  $C$ -convex on  $X$  for every  $y \in Y$ ;
- (ii)  $y \mapsto f(x, y)$  is demicontinuous and naturally quasi  $C$ -concave on  $Y$  for every  $x \in X$ ,

then the vector-valued function  $f$  has at least one weak  $C$ -saddle point.

*Proof.* Since the pointed convex cone  $C$  is solid, it follows from page 18 of [17] that there exist nonzero functionals  $\varphi_1, \varphi_2 \in C^* \setminus \{0\}$  (possibly  $\varphi_1 = \varphi_2$ ). With these functionals we associate the following sets:

$$A_\alpha(x; \varphi) := \{y \in Y \mid \varphi(f(x, y)) \geq \alpha\}, \quad (3.2)$$

$$B_\beta(y; \varphi) := \{x \in X \mid \varphi(f(x, y)) \leq \beta\}, \quad (3.3)$$

for each  $x \in X$ ,  $y \in Y$ , and  $\alpha, \beta \in \mathbf{R}$ . By Lemmas 3.2 and 3.3, the sets above are closed convex subsets in compact convex sets, respectively. Thus, the proof follows from Theorem 1 in [15] and Theorem 2.4 in [30].  $\square$

Consequently, the following corollary, which modifies Lemma 3.3 in [32] into l.c.s. version, is proved immediately by Remark 3.2 and the theorem above.

**Corollary 3.1.** *Let  $X$  and  $Y$  be nonempty compact convex sets in two t.v.s.'s, respectively, and  $Z$  an ordered t.v.s. with a solid pointed convex cone  $C$ . If a vector-valued function  $f : X \times Y \rightarrow Z$  satisfies one of the following conditions:*

- (i)  $x \mapsto f(x, y)$  is continuous and properly quasi  $C$ -convex on  $X$  for every  $y \in Y$ ,  
 $y \mapsto f(x, y)$  is continuous and properly quasi  $C$ -concave on  $Y$  for every  $x \in X$ ;
- (ii)  $x \mapsto f(x, y)$  is continuous and properly quasi  $C$ -convex on  $X$  for every  $y \in Y$ ,  
 $y \mapsto f(x, y)$  is continuous and  $C$ -concave on  $Y$  for every  $x \in X$ ;
- (iii)  $x \mapsto f(x, y)$  is continuous and  $C$ -convex on  $X$  for every  $y \in Y$ ,  
 $y \mapsto f(x, y)$  is continuous and properly quasi  $C$ -concave on  $Y$  for every  $x \in X$ ;
- (iv)  $x \mapsto f(x, y)$  is continuous and  $C$ -convex on  $X$  for every  $y \in Y$ ,  
 $y \mapsto f(x, y)$  is continuous and  $C$ -concave on  $Y$  for every  $x \in X$ ;

then the vector-valued function  $f$  has at least one weak  $C$ -saddle point.

At last, we shall give the third existence theorem for generalized saddle points.

**Theorem 3.4.** (See Theorem 4.1 in [28] and Theorem 3.1 in [29].) *Let  $X$  and  $Y$  be nonempty compact convex sets in two l.c.s.'s, respectively, and  $Z$  an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$ . If a vector-valued function  $f : X \times Y \rightarrow Z$  is continuous and if the following sets*

$$T(y) := \{x \in X \mid f(x, y) \in \text{Min}_w f(X, y)\}, \quad (3.4)$$

$$U(x) := \{y \in Y \mid f(x, y) \in \text{Max}_w f(x, Y)\} \quad (3.5)$$

*are convex for every  $y \in Y$  and  $x \in X$ , respectively, then the vector-valued function  $f$  has at least one weak  $C$ -saddle point.*

#### 4. Minimax Theorems for Vector-Valued Functions

In a few of the author's papers, he has proposed some minimax theorems for vector-valued functions. In this section, we shall present some of most general versions of such minimax theorems. To this end, we need the following saddle point theorem of a vector-valued function, which is a corollary of Theorem 2.1.

**Theorem 4.1. (Saddle Point Theorem)** *Let  $X$  and  $Y$  be nonempty compact sets in two topological spaces, respectively, and  $Z$  an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$ . If a vector-valued function  $f : X \times Y \rightarrow Z$  is continuous and if  $C$  satisfies the condition (2.10), then*

$$\left[ \text{Min} \bigcup_{x \in X} \text{Max}_w f(x, Y) \right] + C \supset \text{SV}_w(f),$$

$$\left[ \text{Max} \bigcup_{y \in Y} \text{Min}_w f(X, y) \right] - C \supset \text{SV}_w(f).$$

*Hence, if  $f$  has a weak  $C$ -saddle point  $(x_0, y_0) \in X \times Y$ , then there exist*

$$z_1 \in \text{Min} \bigcup_{x \in X} \text{Max}_w f(x, Y) \quad \text{and} \quad z_2 \in \text{Max} \bigcup_{y \in Y} \text{Min}_w f(X, y)$$

*such that  $z_1 \leq_C f(x_0, y_0)$  and  $f(x_0, y_0) \leq_C z_2$ .*

We will say that the minimax inequality holds if there exist

$$z_1 \in \text{Min} \bigcup_{x \in X} \text{Max}_w f(x, Y) \quad \text{and} \quad z_2 \in \text{Max} \bigcup_{y \in Y} \text{Min}_w f(X, y)$$

such that  $z_1 \leq_C z_2$ .

**Theorem 4.2. (Minimax Theorem I)** *Let  $X$  and  $Y$  be nonempty compact sets in two topological spaces, respectively, and  $Z$  an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$ . If a vector-valued function  $f : X \times Y \rightarrow Z$  is of the type  $f(x, y) = u(x) + \beta(x)v(y)$  where  $u$  and  $v$  are continuous and  $\beta$  is continuous into  $\mathbf{R}_+$ , and if  $C$  satisfies the condition (2.10), then the minimax inequality holds. If, in particular, the*

vector-valued function  $f$  is of the type  $f(x, y) = u(x) + v(y)$  where  $u$  and  $v$  are continuous, then

$$\text{Max} \bigcup_{y \in Y} \text{Min}_w f(X, y) \subset \left[ \text{Min} \bigcup_{x \in X} \text{Max}_w f(x, Y) \right] + C$$

and

$$\text{Min} \bigcup_{x \in X} \text{Max}_w f(x, Y) \subset \left[ \text{Max} \bigcup_{y \in Y} \text{Min}_w f(X, y) \right] - C.$$

*Proof.* The first part of the proof follows immediately from Theorems 3.1 and 4.1. The second part of the proof can be done in the same way as in the proof of Theorem 3.2 in [28].  $\square$

**Theorem 4.3. (Minimax Theorem II)** *Let  $X$  and  $Y$  be nonempty compact convex sets in two t.v.s.'s, respectively, and  $Z$  an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$ . If a vector-valued function  $f : X \times Y \rightarrow Z$  is continuous and satisfies:*

- (i)  $x \mapsto f(x, y)$  is naturally quasi  $C$ -convex on  $X$  for every  $y \in Y$ ;
- (ii)  $y \mapsto f(x, y)$  is naturally quasi  $C$ -concave on  $Y$  for every  $x \in X$ ,

and if  $C$  satisfies the condition (2.10), then the minimax inequality holds.

*Proof.* The proof follows immediately from Theorems 3.3 and 4.1.  $\square$

**Theorem 4.4. (Minimax Theorem III)** *Let  $X$  and  $Y$  be nonempty compact convex sets in two l.c.s.'s, respectively, and  $Z$  an ordered t.v.s. with an ordering defined by a solid pointed convex cone  $C$ . Assume that a vector-valued function  $f : X \times Y \rightarrow Z$  is continuous and that  $C$  satisfies the condition (2.10). If the sets of (3.4) and (3.5) are convex for every  $y \in Y$  and  $x \in X$ , respectively, then the minimax inequality holds.*

*Proof.* The proof follows immediately from Theorem 3.4 and Theorem 4.1.  $\square$

## 5. Conclusions

Minimax theorems for vector-valued functions have a similar statement to the ordinary minimax theorems for real-valued functions. For a real-valued function  $F$ , the following inequality

$$\min_{x \in A} \sup_{y \in B} F(x, y) \leq F(x_0, y_0) \leq \max_{y \in B} \inf_{x \in A} F(x, y) \quad (5.1)$$

holds under suitable conditions which are sufficient for the function  $F$  to possess a saddle point  $(x_0, y_0) \in A \times B$ . Replacing the usual total ordering  $\leq$  by a general partial ordering  $\leq_C$ , defined by a pointed convex cone  $C$ , we may expect to get a similar minimax inequality to (5.1). This has come true in Section 4, and minimax theorems presented in Section 4 tell us that there exist some minimax strategy and maximin strategy of  $f$  such that their

values are ordered by  $\leq_C$  and dominated each other whenever  $f$  has a weak  $C$ -saddle point. This is illustrated in Figure 2.2 of Example 2.2.

Unfortunately, minimax strategies [resp. maximin strategies] do not always give equilibrium values, but may give equilibrium values in the sense of security levels in the case that minimax strategies and maximin strategies satisfy some conditions. Thus, we note that the minimax values and maximin values are lower bounds and upper bounds of such equilibrium values, respectively. This is also illustrated in Figure 2.2 of Example 2.2.

## References

- [1] Aubin JP, Cellina A (1984) *Differential Inclusions*. Springer-Verlag, Berlin
- [2] Blackwell D (1956) An analog of the minimax theorem for vector payoffs. *Pacific J Math* 6:1–8
- [3] Browder FE (1984) Coincidence theorems, Minimax Theorems, and Variational Inequalities. *Contemporary Mathematics* 26:67–80
- [4] Chen GY (1991) A generalized section theorem and a minimax inequality for a vector-valued mapping. *Optimization* 22:745–754
- [5] Corley HW (1980) An existence result for maximizations with respect to cones. *J Optim Theory Appl* 31:277–281
- [6] Corley HW (1985) Games with vector payoffs. *J Optim Theory Appl* 47:491–498
- [7] Fan K (1952) Fixed-point and minimax theorems in locally convex topological linear spaces. *Proc Nat Acad Sci U.S.A.* 38:121–126
- [8] Ferro F (1982) Minimax type theorems for  $n$ -valued functions. *Annali di Matematica Pura ed Applicata*, 32:113–130
- [9] Ferro F (1989) A minimax theorem for vector-Valued functions. *J Optim Theory Appl* 60:19–31
- [10] Ferro F (1991) A minimax theorem for vector-valued functions, part 2. *J Optim Theory Appl* 68:35–48
- [11] Ghose D, Prasad UR (1989) Solution concepts in two-person multicriteria games. *J Optim Theory Appl* 63:167–189
- [12] Ghose D (1991) A necessary and sufficient condition for pareto-optimal security strategies in multicriteria matrix games. *J Optim Theory Appl* 68:463–481
- [13] Glicksberg IL (1952) A further generalization of the kakutani fixed point theorems, with application to nash equilibrium points. *Proc Amer Math Soc* 3:170–174
- [14] Hartley R (1978) On cone-efficiency, cone-convexity, and cone-compactness. *SIAM J Appl Math* 34:211–222
- [15] Hartung J (1982) An extension of sion's minimax theorem with an application to a method for constrained games. *Pacific J Math* 103:401–408
- [16] Henig MI (1982) Existence and characterization of efficient decisions with respect to cones. *Math Programming* 23:111–116

- [17] Holmes RB (1975) Geometric Functional Analysis and Its Applications. Springer-Verlag, New York
- [18] Jahn J (1984) Scalarization in vector optimization. Math Programming 29:203–218
- [19] Jameson G (1970) Ordered Linear Spaces. Lecture Notes in Mathematics, Vol.141, Springer-Verlag, Berlin
- [20] Karwat AS (1986) On existence of cone-maximal points in real topological linear spaces. Israel J Math 54:33–41
- [21] Luc DT (1989) An existence theorem in vector optimization. Math Oper Res 14:693–699
- [22] Luc DT (1989) Theory of Vector Optimization. Lecture Notes in Economics and Mathematical Systems, Vol.319, Springer-Verlag, Berlin
- [23] Nieuwenhuis JW (1983) Some minimax theorems in vector-valued functions. J Optim Theory Appl 40:463–475
- [24] Rödder W (1977) A generalized saddle-point theory: its application to duality theory for linear vector optimum problems. European J Oper Res 1:55–59
- [25] Shapley LS (1959) Equilibrium points in games with vector payoffs. Naval Research Logistics Quarterly 6:57–61
- [26] Simons S (1986) Cyclical coincidences of multivalued maps. J Math Soc Japan 38:515–525
- [27] Tanaka T (1987) On cone-extreme points in  $R^n$ . Science Reports of Niigata University 23:13–24
- [28] Tanaka T (1988) Some minimax problems of vector-valued functions. J Optim Theory Appl 59:505–524
- [29] Tanaka T (1989) Existence theorems for cone saddle points of vector-valued functions in infinite-dimensional spaces. J Optim Theory Appl 62:127–138
- [30] Tanaka T (1990) A characterization of generalized saddle points for vector-valued functions via scalarization. Nihonkai Mathematical Journal 1:209–227
- [31] Tanaka T (1990) Cone-convexity of vector-valued functions. The Science Reports of the Hirosaki University 37:170–177
- [32] Tanaka T (1991) Two types of minimax theorems for vector-valued functions. J Optim Theory Appl 68:321–334
- [33] Tanaka T (1992) Minimax Theorems in Vector Optimization. Ph.D. thesis (Doctor of Science), Niigata University, Niigata
- [34] Tanaka T (1994) Generalized quasiconvexities, cone saddle points, and a minimax theorem for vector-valued functions. to appear in J Optim Theory Appl 81, No.2, May
- [35] Yu PL (1974) Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjectives. J Optim Theory Appl 14:319–377



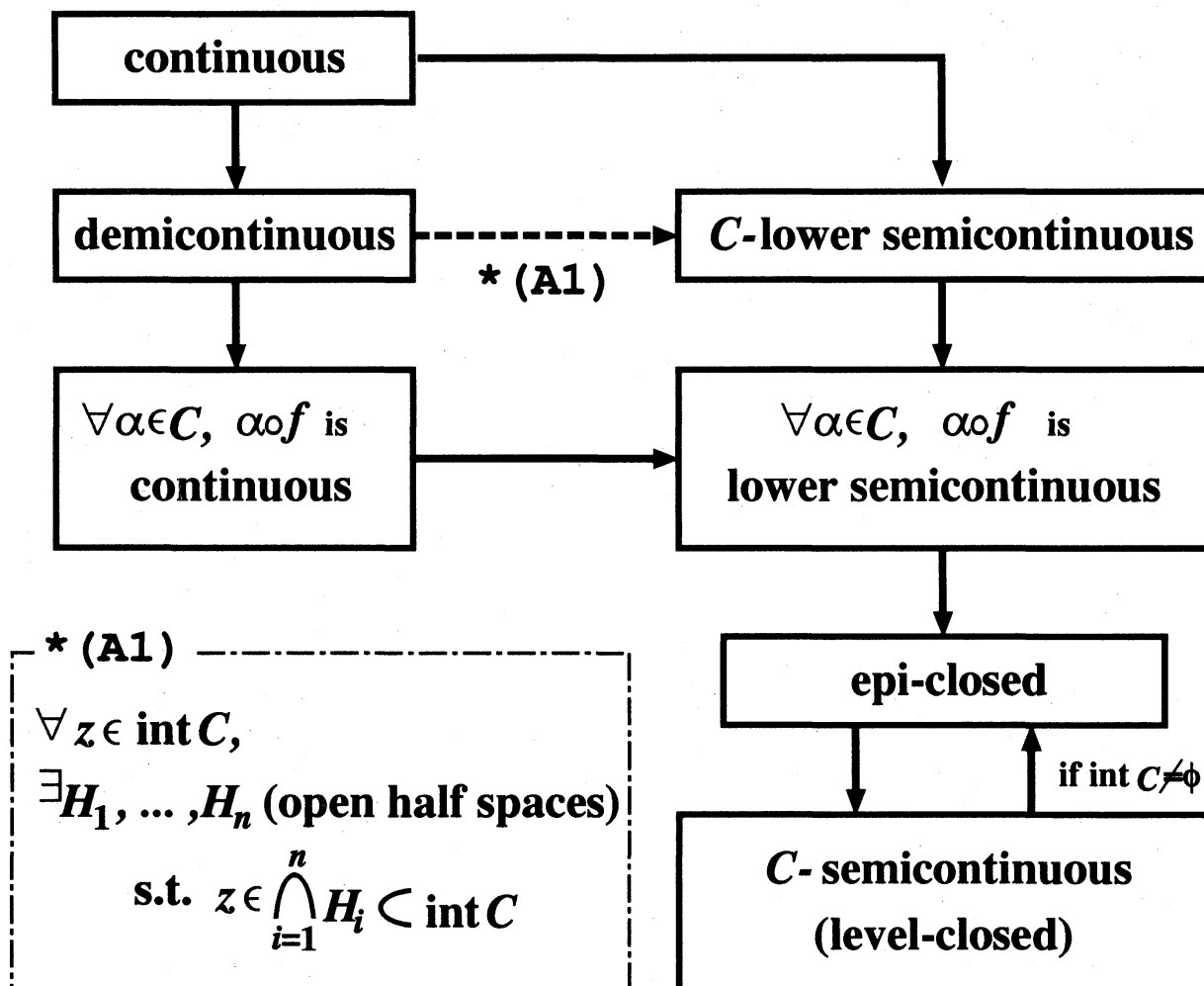


Figure 3.3: Relationship among various semicontinuities.