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A NOTE ON MILNOR AND THURSTON'S MONOTONICITY THEOREM

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The aim of this note is to give a simplified proof of the so-called Milnor and Thurston's monotonicity theorem. We begin with stating the theorem.

Let us consider a family of maps $Q_a(x) = a - x^2$ from $\mathbb{R}$ to itself. The kneading sequence for Q_a is an infinite sequence $K(a) = (e_1, e_2, \cdots)$ of three symbols L, C and R defined as

$$e_i = \begin{cases} L, & \text{if } Q_a^i(0) < 0; \\ C, & \text{if } Q_a^i(0) = 0; \\ R, & \text{if } Q_a^i(0) > 0, \end{cases}$$

On the set $\{L, C, R\}^{\mathbb{N}}$, the so-called signed lexicographical order $\prec$ is defined as follows: for three sequences with the same first n-entries,

$$I_L = (e_1, e_2, \cdots, e_n, L, \cdots),$$
$$I_C = (e_1, e_2, \cdots, e_n, C, \cdots),$$
$$I_R = (e_1, e_2, \cdots, e_n, R, \cdots),$$

we decide $I_L \prec I_C \prec I_R$ if the number of the symbol R in $\{e_1, e_2, \cdots, e_n\}$ is even, and $I_R \prec I_C \prec I_L$ otherwise. Milnor and Thurston's monotonicity theorem is

Theorem. *The correspondence $a \mapsto K(a)$ is monotone increasing.*

This surprisingly strong theorem was conjectured by Milnor and Thurston, and proved firstly by Duady, Hubberd and Sullivan. The proof we give here is a modification of the proof in [2].

The theorem follows from

Proposition. *If $K(a_0) = (e_1, e_2, \cdots, e_n, C, \cdots)$ and $e_i \neq C$ for $1 \leq i \leq n$, then*

$$\frac{\partial_a (Q_a^{n+1}(0))|_{a=a_0}}{DQ_{a_0}^n(Q_{a_0}(0))} > 0 \tag{1}$$

where $DQ_{a_0}^n$ denote the derivative of the n-times iteration of the map Q_{a_0} and $\partial_a(Q_a^{n+1}(0))$ denote the derivative of $Q_a^{n+1}(0)$ as a function of the parameter a.

In fact, suppose that a_0 satisfies the assumption of the proposition and that the number of R in $(e_1, e_2, \cdots, e_n)$ is even (resp. odd). Then the denominator in the left hand side of (1) is positive (resp. negative) and, from the proposition, so is the numerator. This implies that e_{n+1} varies as $L \to C \to R$ (resp. $R \to C \to L$) when the parameter a pass a_0 from the left to the right. Now consider the truncated kneading sequence $K^{(n)}(a) = (e_1, e_2, \cdots, e_n)$ for each n. For each parameter at which $K^{(n)}(a)$ changes, we can find the situation in the proposition. Thus the above observation shows that $K^{(n)}(a)$ depends on a monotonously. Letting $n \to \infty$, we get the theorem.

Let us denote $w_i = Q_a^i(0)$ for $i = 1, 2, \cdots, n$ and put $\omega = (w_1, w_2, \cdots, w_n) \in \mathbb{R}^n$. Consider the so-called Thurston map:

$$T(z_1, z_2, \cdots, z_n) = (\sigma_1\sqrt{z_1 - z_2}, \sigma_2\sqrt{z_1 - z_3}, \cdots, \sigma_{n-1}\sqrt{z_1 - z_n}, \sigma_n\sqrt{z_1})$$

where σ_i is the sign of w_i. Then $T(\omega) = \omega$ and T is defined on a neighborhood of ω. By easy calculations, we obtain

Lemma 1. $\dfrac{\partial_a(Q_a^{n+1}(0))|_{a=a_0}}{DQ_{a_0}^n(Q_{a_0}(0))} = \det(I_n - D_\omega T)$ *where I_n denotes the $n \times n$ unit matrix and $D_\omega T$ the derivative of T at ω.*

So we reduce the proposition to

Lemma 2. *No eigenvalue of $D_\omega T$ is contained in $[1, \infty)$.*

Let $X = \{(z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid 0 < |z_i| < 3, \text{ and } z_i \neq z_j \text{ if } i \neq j\}$ and

$$Y_\epsilon = \{(z_1, z_2, \cdots, z_n) \in X \mid |z_i| > 10^i\epsilon \text{ and } |z_i - z_j| > 10^{\min\{i,j\}}\epsilon \quad \text{if } i \neq j\}$$

for $\epsilon > 0$. Then the (multi-valued) complex extension of T,

$$T_{\mathbb{C}}(z_1, z_2, \cdots, z_n) = (\sqrt{z_1 - z_2}, \sqrt{z_1 - z_3}, \cdots, \sqrt{z_1}) : \mathbb{C}^n \to \mathbb{C}^n,$$

maps X into itself in the sense that, for every $x \in X$ and every branch of $T_{\mathbb{C}}$, the image belongs to X. Moreover, if ϵ is sufficiently small, $T_{\mathbb{C}}$ maps Y_ϵ into a compact subset of Y_ϵ in this sense. Take ϵ so small that $\omega \in Y_\epsilon$. Let $M_\mu : \mathbb{C} \to \mathbb{C}$ be a map defined by

$$M_\mu(z_1, \cdots, z_n) = (w_1 + \mu(z_1 - w_1), \cdots, w_n + \mu(z_n - w_n)) : \mathbb{C}^n \to \mathbb{C}^n.$$

We choose $\mu > 1$ so close to 1 that the composition $S := M_\mu \circ T_{\mathbb{C}}$ maps Y_ϵ into itself. Let $\pi : \tilde{Y}_\epsilon \to Y_\epsilon$ be the universal covering and let $\tilde{\omega} \in \tilde{Y}_\epsilon$ be a point such that $\pi(\tilde{\omega}) = \omega$. Then there is a (single valued) lift $\tilde{S} : \tilde{Y}_\epsilon \to \tilde{Y}_\epsilon$ of S such that $\tilde{S}(\tilde{\omega}) = \tilde{\omega}$.

Now consider the Kobayashi metric $|\cdot|_K$ on $\tilde{Y}_\epsilon$, which is defined as

$$|v|_K = [\sup\{r \geq 0 \mid \text{there is a holomorphic map } \phi : D_r \to \tilde{Y}_\epsilon \text{ s.t. } d\phi(e) = v.\}]^{-1}$$

for any tangent vector v where $D_r = \{z \in \mathbb{C} \mid |z| < r\}$ and e is the unit vector at $0 \in D_r$. (See [1] for generalities.) Then it is easy to check that $|\cdot|_K$ is equivalent to the Euclidean metric at each point. From the definition, we have $|d\Phi(v)|_K \leq |v|_K$ for any holomorphic map $\Phi : \tilde{Y}_\epsilon \to \tilde{Y}_\epsilon$ and any tangent vector v. So it follows that the spectral radius of $D_{\tilde{\omega}}\tilde{S}$ is not bigger than 1. Since $D_{\tilde{\omega}}\tilde{S} = \mu \cdot D_\omega T$ and $\mu > 1$, the spectral radius of $D_{\tilde{\omega}}\tilde{S}$ is smaller than 1. We have proved lemma 2 and so the main theorem.

References

1. S.Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Marcel Dekker, 1970.
2. J. Milnor and W. Thurston, *On iterated maps of the interval*, Lect. Notes in Math. **1342** (1989), 465–563.