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GEOMETRY OF DISCRIMINANTS AND DYNAMICS OF DIAGRAMS OF SMOOTH MAPPINGS

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This is a note on the various problems in the dynamical system aproach to diagrams of smooth mappings. The detaoled argument will appear elsewhere.

The *critical point set* of a C^∞-smooth map $f : N^n \to P^p$ is

$$\Sigma(f) = \{x \in N \mid \text{rank } df(x) \leq p\},$$

where we assume $p \leq n$. The *discriminant* of f is the image

$$D(f) = f(\Sigma(f)).$$

Clearly the singular point set is closed by definition, and the discriminant is closed if f is proper. From now on we assume all manifolds are compact. For generic $f \in C^\infty(N, P)$ in Whitney topology, $\Sigma(f)$ is locally homeomorphic to an real algebraic set of dimension $p-1$ and the singular point set represents a homology class Poincaré dual to the Stiefel-Whitney class $W_{n-p+1}(TN - f^*TP)$. Here $TN - f^*TP$ is the virtual vector bundle over N with the fiber $T_x f^{-1} f(x)$ at $x \in N$ (rank $=$ $n-p$ for generic f). It is known that for any (contact invariant) singularity class I, the Poincaré dual of the singular locus $\Sigma^I(f)$ is witten as a polynomial of Stiefel-Whitney class of $TN - f^*TP$, which is called Thom polynomial.

Generic f admits an A_f- and B-regular stratification. If f is real analytic, the stratification is subanalytic and the direct image is a constractible function. The fibre $f^{-1}(y)$ of generic $y \in P$ has a unique Z_2-euler number. The *image* $[\text{Im}(f)]$ is the homology class defined by

$$[\text{Im}(f)] \cdot [y] \; = \; \chi(f^{-1}(y)) \; \in Z_2.$$

The discriminant set $D(f)$ defines the cohomology class $[D(f)] \in H_1(P, Z_2)$, which assigns to a singular chain $c : S^1 \to P$ the Z_2-euler number

$$c \; \in \; H_1(P, Z_2) \longmapsto \chi(N \underset{f=c}{\times} S^1) \; \in \; Z_2.$$

To study systematically introduce the *direct image* f_*Z of the constant sheaf Z over N which assigns to an open $U \subset P$,

$$f_*Z(U) = H_*(f^{-1}(U), Z).$$

The discriminant is interpreted as

$$[D(f)] \cdot [c] = \int_C f_*Z = \int_C f_*1$$

where $\int$ means the integration of the direct image sheaf f_*Z_2.

Macpherson defined for a complex constractible function α on a variety V a homology class $c_*(\alpha) \in H_*(V, Z)$ which satisfies the following properties.

(1) $f_*c_*(\alpha) = c_*f_*(\alpha)$

(2) $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$

(3) $c_*(1) = \text{Dual } c(v)$,

where f is a holomorphic map, $c(v)$ is the Chern class of V and $f_*\alpha$ is the direct image of the constructible function α defined by

$$f_*\alpha(y) = \sum_W m_W \ \chi(f_{-1}(y) \cap W), \alpha = \sum_W m_W \ 1_W.$$

To construct $c_*(f_*1)$ he defined a decomposition of the image (constructible function) by V_i in the manner of

$$f_*1 = \sum_i eu(V_i),$$

where $eu(V_i)$ is the constructible function defined by

$$eu(V_i)(y) = \text{Euler obstruction of } V_i \text{ at } y.$$

The union of those V_i with positive codimension is the discriminant of f.

Two maps $f : N \to P, g : M \to P$ are *bordant* if there exists a smooth map $h : W \to P$ such that

(1) $\partial W = N + M$,

(2) f, g are restrictions of h.

The bordism class $[f]$ is determned by the Stiefel number, which is the collection of

$$F(w_1, \ldots, w_n) \cdot [f^{-1}(y)],$$

evaluated at $y \in P$ for all polynomial of Stiefel-Whitney classes of weighted degree $n - p$. The discriminant class $[D(f)]$ is determined by the bordism class of $[f]$. The *sum* of f, g are defined by

$$f + g : N \cup M \to P.$$

The *product* is defined by the fiber product

$$f \times g = N \underset{f=g}{\times} M \to P.$$

If f, g are transversal, the fiber product $N \underset{f=g}{\times} M$ is smooth. By the transversality theorem, f, g attain the transversality after slight perturbation and the fiber product is well defined as a bordism class. Clearly $f \times g = g \times f$, $f \times (g \times h) = (f \times g) \times h$ and $f \times (g + h) = f \times g \ + \ f \times h$. The *bordism ring* $\Omega(P)$ is generated by all bordism classes and the sum $+$ and product $\times$.

Theorem.
(1) $Im(f \times g) = Im(f) \ \times \ Im(g)$,
(2) $D(f \times g) = D(f) \times Im(g) + Im(f) \times D(g)$

Proof. The first statement is simple interpretation of the formula $\chi(X \times Y) = \chi(X) \times \chi(Y)$. The second statement is seen by "integrating" the function $Im(f \times g)$ over cycles of P.

The above theorem should be interpreted and generalized as formulas of constructible functions rather than homology classes or sheaves. So it seems impotant to generalize Macpherson's result for real smooth mappings.

Problem. *Generalize Macpherson's result for smooth mappings to understand the image of the various singular loci.*

To extract the singularities of mappings to P we may define the *reduced bordism ring* $\tilde{\Omega}(P)$ introducing the *quotient* $f/g = (f : g)$ and the equivalence relation $\sim$ in the following manner. We denote

$$(f : g) \sim (f' : g') \qquad \text{if} \qquad f \times g' = f' \times g$$

and

$$(f : id_P) \sim (id_P : id_P) = 1 \qquad \text{if } f \text{ is a locally trivial fiber bundle}$$

and define the product $\times$ by

$$(f : g) \times (f' : g') = (f \times f' : g \times g').$$

By definition

$$(f : g) \times (g : f) = 1 \qquad .$$

The image and discriminant homology classes are naturally defined for the reduced bordism classes.

Consider the divergent diagram of smooth maps

$$\mathbb{R}^1 \xleftarrow{\lambda} N \xrightarrow{f} P,$$

$p \leq n$, and regard in two ways as the families of the restrictions

$$f_t = f : \lambda^{-1}(t) \to P, \qquad t \in \mathbb{R},$$
$$\lambda_y = \lambda : f^{-1}(y) \to \mathbb{R}, \qquad y \in P.$$

The discriminant sets of the restrictions f_t , $t \in \mathbb{R}$ constitute complete solutions of a certain first order PDE with the 1st integral t. We call a discriminant a *solution*. Here a PDE on P is a subvariety V of the projective cotangent bundle PT^*P with the canonical contact form ω. We say a PDE is nonsingular if V is nonsingular. Assume dim $V = p$ (holonomic). A 1st integral is a smooth function λ on V such that $d\lambda \wedge \omega$ vanishes identically on V.

Theorem. *All germs of nonsingular 1st order PDE with nonsingular 1st integrals are obtained by divergent diagrams of smooth map germs.*

Problem. *Study nonsingular 1st order PDE of dim p, which admits local 1st integrals.*

In the global case the restrictions f_t are all bordant. So it would be interesting to ask

Problem. *Study p-fold product* $f_t \times \cdots \times f_t$ *in the bordism group.*

Furthermore we can discuss the fiber product of 1st order PDE's on P, in the manner of fiber product of diagrams.

Next we consider general problem of the divergent diagrams

$$Q \xleftarrow{g} N \xrightarrow{f} P.$$

The most important geometric structure is the families of the discriminants of the restrictions

$$\begin{aligned} f_z &= f : g^{-1}(z) \to P, \qquad z \in Q, \\ g_y &= g : f^{-1}(y) \to Q, \qquad y \in P. \end{aligned}$$

These restrictions seem to play conducting role in the Radon transformation of sheaves $\mathcal{E}, \mathcal{F}$ over P, Q defined as follows.

$$\begin{aligned} \mathcal{F} &\longrightarrow f_* g^* \mathcal{F} \\ g_* f^* \mathcal{E} &\longleftarrow \mathcal{E} \end{aligned}$$

Here we present the following theorem

Theorem. *For a constructible sheaf* $\mathcal{F}$ *on* N, *the direct image* $f_*\mathcal{F}$ *under generic real analytic mapping* f *is topologically stable, i.e. the direct image is constructible and the locally trivial stratification of* P *(along which the cohomology of the direct image is locally trivial) is topologically stable under deformation of* f.

The theorem suggests that generic Radon transformation is topologically (cohomologically) stable.

Problem. *Study the stability of* $\mathcal{E} \to f_* g^* g_* f^* \mathcal{E}$.

To observe the dynamical system-aspect of the divergent diagrams, consider the special case

$$\mathbb{P} \xleftarrow{g} C \xrightarrow{f} \mathbb{P},$$

where $\mathbb{P}$ is the complex projective line, C is a Riemann surface and f, g are holomorphic functions. The equivalence relation $\sim$ on C is generated by the relations

$$x \sim y \qquad \text{if} \qquad g(x) = g(y) \text{ or } f(x) = f(y).$$

The *orbit* $O(x)$ of an x is the equivalence class of x. The *basin* of an orbit $O(x)$ is the set of those $y \in C$ for which the closure of $O(y)$ contains $O(x)$. Let x be a common singular point of f, g and assume that the group of germs of holomorphic diffeomorphisms of C generated by the monodromy of f, g at x is noncommutative. Then the basin of $O(x)$ is open. The complement of those open basins seem to possess an interesting structure. For example assume C is defined by the polynomial

$$(x-y)(x^2+c-y)=\epsilon$$

and f, g are the projections onto the x- and y-lines respectively. The C is elliptic curve and the infinity (∞, ∞) is the unique isolated equivalence class. The complement of the basin of the infinity presents fractal structure by numerical experimantation. Clearly for the case $\epsilon = 0$, the complent is the filled-in Julia set. We call the complement the basin *generalized Julia set.* Finally we propose

Problem. *Prove the existence of the generalized Julia set.*

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