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ON THE RATE OF CONVERGENCE OF A SIMPLE MARKOV CHAIN ON A HALF-LINE

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1. Introduction. Let $\{X_k\}$ be a Markov chain on $[0, \infty)$ defined by

(1.1) $\quad X_k = h(X_{k-1} + Y_k), \qquad k = 1, 2, \ldots$

where $h(x)$ is a continuous function on R^1 which satisfies that $h(x) = 0$ on $(-\infty, a]$, $h(x)$ is strictly increasing on $[a, \infty)$ for some $a \geq 0$ and that $\lim_{x\to\infty} h(x) = \infty$. Here, $\{Y_k\}$ is a sequence of independent and identically distributed random variables. This form of Markov chain has been considered as the model of dams and storage systems (see, for example, Bather [1], Glynn [3], Moran [7] and Prabhu [10]). As a typical example of (1.1), we have a simple random walk on $[0, \infty)$

(1.2) $\quad X_k = (X_{k-1} + Y_k)^+, \qquad k = 1, 2, \ldots$

where $x^+ = \max(x, 0)$. We shall denote by Y the generic variable with the distribution of any of the Y_k and assume that $E(|Y|) < \infty$. In the present paper, we consider sufficient conditions for (Harris) ergodicity and various rates of convergence of the Markov chain $\{X_k\}$ defined by (1.1). In particular, we are mainly concerned with sub-geometric rate of convergence results for various rate functions ψ as in [13] and [15].

It is well known that finiteness of appropriate moments of hitting times on small sets (e.g. relatively compact sets under standard continuity conditions on the transition probabilities) implies various kinds of ergodicity of Markov chains (see Theorem 1 of [15] and also [8], [9]). Tweedie [15] presented convenient criteria (see Theorem 3 of [15]) which imply the finiteness of these moments, and applied the criteria to the random walk (1.2) in order to obtain several reasonable sufficient conditons for ergodicity and rate of convergence results. Thorisson improved Tweedie's conditions for sub-geometric rate of convergence of (1.2) in [13]. In the present paper, we generalize their results to the Markov chain (1.1).

2. Preliminaries. We suppose that $\{X_k\}$ is a temporally homogeneous Markov chain on state space $(S, \mathcal{F})$; the σ-field $\mathcal{F}$ of subsets of S is assumed to be countably generated. We write

(2.1) $$P^k(x, A) = \mathrm{P}(X_k \in A | X_0 = x), \qquad x \in S, \quad A \in \mathcal{F}$$

for the k-step transition probabilities of $\{X_k\}$. We assume that $\{X_k\}$ is ϕ-irreducible; that is, for some measure ϕ on $\mathcal{F}$, we have $\sum_{k=1}^{\infty} P^k(x, A) > 0$ for every $x \in S$ and every $A \in \mathcal{F}$ with $\phi(A) > 0$. We also assume for convenience that $\{X_k\}$ is aperiodic.

We define $\{X_k\}$ to be (*Harris*) *ergodic* if, for some probability measure π on $\mathcal{F}$, and every $x \in S$,

(2.2) $$\|P^k(x, \cdot) - \pi\| \longrightarrow 0, \qquad k \longrightarrow \infty,$$

where $\| \ \|$ denotes total variation of signed measures on $\mathcal{F}$. We define $\{X_k\}$ to be *geometrically ergodic* if there exists a $\rho < 1$ such that

(2.3) $$\rho^{-k}\|P^k(x, \cdot) - \pi\| \longrightarrow 0, \qquad k \longrightarrow \infty,$$

for every $x \in S$.

Let Λ_0 denote the class of sequences $\psi : \mathrm{N} \longrightarrow \mathrm{R}_+$ which satisfy

(i) ψ is non-decreasing, with $\psi(j) \geq 2$ for all $j \geq 0$.

(ii) $\{\log \psi(j)\}/j$ is non-increasing and tends to 0 as $j \longrightarrow \infty$.

We denote by Λ the class of sequences $\psi : \mathrm{N} \longrightarrow \mathrm{R}_+$ for which there exists some $\psi_0 \in \Lambda_0$ such that

$$\liminf_{j\to\infty} \psi(j)/\psi_0(j) > 0, \qquad \limsup_{j\to\infty} \psi(j)/\psi_0(j) < \infty.$$

Examples of sequences in Λ are

(2.4) $$\psi(j) = j^{\alpha}(\log j)^{\beta}\exp(\gamma j^{\delta})$$

for $0 < \delta < 1$ and either $\gamma > 0$ or $\gamma = 0$ and $\alpha > 0$. See details on the classes Λ_0 and Λ in [5], [9], [12] and [15]. For $\psi \in \Lambda$, we call $\{X_k\}$ *ergodic of order* ψ if

(2.5) $$\psi(k)\|P^k(x, \cdot) - \pi\| \longrightarrow 0, \qquad k \longrightarrow \infty,$$

for every $x \in S$ (see [15] and also [9]).

Let $h^*(x)$ $(x \geq 0)$ be the inverse function of the strictly increasing function $h(x)$ on $[a, \infty)$. We denote the hitting time on a set B by $\tau_B = \inf(k \geq 1: X_k \in B)$, and write E_x and P_x respectively for expectation and probability conditional on $X_0 = x$.

3. Ergodicity and geometric ergodicity. First of all, we employ the following standard assumption (of negative drift) for ergodicity.

Assumption 1. $\mathrm{E}(Y) < 0$.

We also employ the following assumption, which is obviously satisfied by the random walk (1.2). Under both assumptions, we obtain ergodicity of the Markov chain (1.1) as follows.

Assumption 2. There exists $q_0 > 0$ such that

(3.1) $$x\,\mathrm{P}(Y \geq -x) \leq h^*(x), \qquad x \geq q_0 .$$

Theorem 1. *Suppose that Assumption 1 holds.*

(1) *If Assumption 2 also holds, then* $\{X_k\}$ *is ergodic.*

(2) *If it also holds that*

$$\text{(3.2)} \qquad \limsup_{x\to\infty} \; x\,\mathrm{P}(Y \ge -x)/h^*(x) \; < \; 1,$$

then $\{X_k\}$ *is geometrically ergodic.*

Corollary 1. *Suppose that Assumption 1 holds.*

(1) *If there exists* $q_0 > 0$ *such that*

$$\text{(3.3)} \qquad h^*(x) \ge x, \qquad x \ge q_0 \qquad \left(\text{i.e., } h(x) \le x, \qquad x \ge h^*(q_0)\right),$$

then $\{X_k\}$ is ergodic.

(2) *If it holds that*

$$\text{(3.4)} \qquad \limsup_{x\to\infty} \; x/h^*(x) \; < \; 1 \qquad \left(\text{i.e., } \limsup_{x\to\infty} \; h(x)/x \; < \; 1\right),$$

then $\{X_k\}$ *is geometrically ergodic.*

Remark 1. (1) It is easy to see that (3.1) implies

$$\text{(3.5)} \qquad \limsup_{x\to\infty} \; x/h^*(x) \; \le \; 1 \qquad \left(\text{i.e., } \limsup_{x\to\infty} \; h(x)/x \; \le \; 1\right).$$

Hence, there exists $r = r(u) \in (0, 1)$ such that

$$\text{(3.6)} \qquad h^*(x) \ge rx, \qquad x \ge u, \qquad \left(\text{i.e., } h(x) \le r^{-1}x, \qquad x \ge h^*(u)\right),$$

and $\lim_{u\to\infty} r(u) = 1$. (2) It will be ssen that (3.5) is not a sufficient condition for (Harris) ergodicity under Assumption 1 (see Example 2 in Section 4).

Example 1. Let $h^*(x)$ and $\mathrm{P}(Y < -x)$ have the expressions

$$\text{(3.7)} \qquad h^*(x) = x - x^{-\alpha}\xi(x), \qquad x \ge q,$$

$$\text{(3.8)} \qquad \mathrm{P}(Y < -x) = x^{-\beta}\eta(x), \qquad x \ge q,$$

respectively, where $q \in (0, q_0)$, $\eta(x) \ge 0$ and $\eta(x)$ satisfies

$$\text{(3.9)} \qquad \int_q^\infty x^{-\beta}\eta(x)\,dx \; < \; \infty.$$

We also assume that $\xi(x)$ and $\eta(x)$ are bounded or slowly varying functions. Then, we have $\beta \geq 1$ due to (3.9). In this case, (3.1) is equivalent to

$$x^{-\alpha}\xi(x) \leq x^{1-\beta}\eta(x), \qquad x \geq q_0 .$$

Hence, (3.1) holds if either of the following conditions holds:

(i) $\alpha > \beta-1$,

(ii) $\alpha = \beta-1$ and there exists $q' > 0$ such that $\xi(x) \leq \eta(x)$ for any $x \geq q'$.

Remark 2. The integrability condition (3.9) is equivalent to $E[|Y|\cdot I(Y < 0)] < \infty$, where $I(A)$ denotes the indicator function of a set A.

We put

$$(3.10) \qquad h^*(x) = x - d(x).$$

Then, due to (3.5), we immediately obtain the following lemma, which will be necessary in Section 4.

Lemma 1. *It holds that* $\limsup_{x\to\infty} d(x)/x \leq 0$.

Under Assumption 1, we can obtain the following well known criterion for geometric ergodicity (see [8] and [15]).

Theorem 2. *Suppose that Assumption 1 holds. If there exists* $s > 0$ *such that* $E[\exp(sY)] < \infty$, *then* $\{X_k\}$ *is geometrically ergodic.*

4. Ergodicity of order φ. In this section, we obtain the main result concerning ergodicity of order φ by making use of Tweedie-Thorisson's method. First of all, we need to introduce several definitions due

to Thorisson [13].

The functions $\psi:[0, \infty] \longrightarrow [0, \infty]$ considered below are measurable, locally bounded and $\psi(\infty) = \infty$ (i.e., $\lim_{x\to\infty}\psi(x) = \infty$). Let $\tilde{\psi}$ be defined by $\tilde{\psi}(x) = \int_0^x \psi(y)dy$. Two functions ψ and θ are of the same order if

$$\limsup_{t\to\infty}\psi(t)/\theta(t) < \infty \quad \text{and} \quad \limsup_{t\to\infty}\theta(t)/\psi(t) < \infty.$$

This implies that $E[\psi(Y)] < \infty$ if and only if $E[\theta(Y)] < \infty$ for any nonnegative random variable Y.

Let Ψ_0 be the class of all concave non-decreasing ψ with $\psi(0) = 0$; Φ_0 the class of all convex ψ satisfying $\psi(2x) \le c\psi(x)$ for some $c < \infty$ and $\psi = \tilde{\theta}$ where $\theta(0) = 0$ and $\theta(x) \uparrow \infty$ as $x \to \infty$. Throughout the paper, let φ be a function of the same order as $x \longrightarrow x^n\varphi_0(x)$ where n is a nonnegative integer and $\varphi_0 \in \Psi_0$. In this case, we write $\varphi(x) \simeq x^n\varphi_0(x)$. If we define φ_j recursively by $\varphi_j = \tilde{\varphi}_{j-1}$, $j \ge 1$, then φ is also of the same order as φ_n (see Lemma 1(b) of [13]).

Under Assumptions 1 and 2, we shall show as follows that $\{X_k\}$ is ergodic of order φ if $E[\tilde{\varphi}(Y^+)] < \infty$ and some inequalities on $h^*(x)$ and the tail probability of $Y\cdot I(Y < 0)$ hold (see Theorem 3).

Lemma 2. *Suppose that Assumption 1 holds. If* $\psi \in \Lambda_0$, $E[Y^+\psi(Y^+)] < \infty$,

(4.1) $\quad \tilde{\psi}(x)P(Y \ge -x) \le \tilde{\psi}(h^*(x)), \qquad x \ge q'$,

(4.2) $\quad \psi(h^*(x)) \le c_0\psi(x), \qquad x \ge q'$,

and

(4.3) $\quad E[\psi(h^*(\tau_{[0,\, q']}))] \le c_1\psi(h^*(x)), \qquad x \ge q'$

hold for some constants, q' $(\ge q_0)$, $c_0 > 0$ *and* $c_1 > 0$, *then*

(4.4) $\quad \mathrm{E}[\tilde{\Psi}(h^*(\tau_{[0,\ q'']}))] \le c_2\tilde{\Psi}(h^*(x))$

holds for some constants, q'' $(\ge q')$ *and* $c_2 > 0$, *and all large enough* x.

Proposition 1. *Suppose that Assumptions 1 and 2 hold. Suppose also that there exist* q' $(\ge q_0)$ *and* $\tilde{r}$ (> 1) *such that*

(4.5) $\quad h^*(x) \le \tilde{r}x, \qquad x \ge q', \qquad \left(\text{i.e., } h(x) \ge x/\tilde{r}, \qquad x \ge h^*(q')\right).$

If $\mathrm{E}[\tilde{\varphi}(Y^+)] < \infty$, $\varphi(x) \simeq x^n\varphi_0(x)$ *and*

(4.6) $\quad \tilde{\varphi}_j(x)\mathrm{P}(Y \ge -x) \le \tilde{\varphi}_j(h^*(x)), \qquad x \ge q', \quad j = 0, 1, \ldots, n-1$

hold, then there exist positive constants q'' $(\ge q')$ *and* c *such that*

(4.7) $\quad \mathrm{E}_x[\varphi(h^*(\tau_{[0,\ q'']}))] \le c\varphi(h^*(x))$

for all large enough x.

Theorem 3. *Suppose that Assumptions 1 and 2 hold. If* $\mathrm{E}[\tilde{\varphi}(Y^+)] < \infty$, $\varphi \simeq \varphi_n$ *and for some* q' $(\ge q_0)$

(4.8) $\quad \tilde{\varphi}_j(x)\mathrm{P}(Y \ge -x) \le \tilde{\varphi}_j(h^*(x)), \qquad x \ge q', \quad j = 0, 1, \ldots, n$

hold, then there exists q'' $(\ge q')$ *such that*

(4.9) $\quad \mathrm{E}_x[\tilde{\varphi}(\tau_{[0,\ q'']})] < \infty$

for all large enough x. *Hence,* $\{X_k\}$ *is ergodic of order* φ.

We now turn to sufficient conditions for (4.8). By virtue of Theorem 3, we immediately have the following corollary, which is a simple extension of Theorem 5 of [13] (from the random walk (1.2) to the Markov chain (1.1)).

Corollary 2. *Suppose that Assumptions 1 and 2 hold. If* $\mathrm{E}[\tilde{\varphi}(Y^+)] < \infty$, $\varphi \simeq \varphi_n$ *and* (3.3) *hold, then* $\{X_k\}$ *is ergodic of order* φ.

From now on, we investigate sufficient conditions for (4.8) when (3.3) does not hold.

Recall that $h^*(x)$ is written as $h^*(x) = x - d(x)$ (i.e., (3.10)) . Since we are interested in the case where the inequality (3.3) fails to hold, we assume

(4.10) $\quad d(x) \geq 0, \quad x \geq q', \quad \left(\text{i.e., } h^*(x) \leq x, \quad x \geq q'\right).$

for some $q' > 0$. Then, it follows from Lemma 1 that

(4.11) $\quad \lim_{x\to\infty} d(x)/x = 0.$

We now obtain a criterion for (4.8) provided that φ_0 is a regularly varying function when the innequality (4.10) holds.

Corollary 3. *Suppose that Assumptions 1 and 2 hold. Let* $h^*(x)$ *has the form* (3.10) *and* $d(x)$ *satisfy* (4.10). *If* $\varphi_0 \in \Psi_0 \cap R_\sigma$ $(\sigma \geq 0)$, $\varphi \simeq \varphi_n$, $\mathrm{E}[\tilde{\varphi}(Y^+)] < \infty$ *and*

(4.12) $\quad \mathrm{P}(Y \geq -x) \leq \{1 + (\sigma + n + \varepsilon)d(x)/x\}^{-1}, \quad x \geq \tilde{q}$

for some constants $\tilde{q}$ $(\geq q)$ *and* $\varepsilon > 0$, *then* $\{X_k\}$ *is ergodic of order* φ. *Here,* R_σ *denotes the class of regularly varying functions with index* σ.

Remark 3. It is obvious to observe that the inequality (4.12) is implied by the inequality

(4.13) $\quad \mathrm{P}(Y < -x) \geq (\sigma + n + \varepsilon)d(x)/x, \quad x \geq \tilde{q}$

for some constants $\varepsilon > 0$ and $\tilde{q}$ $(\geq q)$.

Example 1'. Let $h^*(x)$ and $\mathrm{P}(Y < -x)$ have the same forms as in Example 1, and suppose $\varphi_0 \in R_\sigma$ $(\sigma \geq 0)$ and $\mathrm{E}[\tilde{\varphi}(Y^+)] < \infty$. Since $d(x) = x^{-\alpha}\xi(x)$, (4.13) is equivalent to

$$x^{-\beta}\eta(x) \geq (\sigma + n + \varepsilon)x^{-\alpha-1}\xi(x), \qquad x \geq \tilde{q}.$$

Hence, (4.13) holds if either of the following conditions holds:

(i) $\alpha > \beta - 1$

(ii) $\alpha = \beta - 1$ and $\limsup_{x\to\infty}\xi(x)/\eta(x) = 0$.

Example 2. Let $h^*(x)$ and $P(Y < -x)$ have the same forms as in Example 1 and suppose that $\alpha = 0$ and

(4.14) $\quad |E(Y)| + \varepsilon \leq \xi(x) \leq K, \qquad x \geq q$

for some positive constants ε, K and q. If $\beta > 1$, then $\{X_k\}$ is null (i.e., nonergodic). If $\beta = 1$ and $\eta(x) = (\log x)^{-\gamma}$ $(\gamma > 1)$, then $\{X_k\}$ is also null. These results can be proved by verifying all the conditions of Theorem 9.1(ii) of [14].

Remark 4. In view of Example 2, we obtain that (3.5) besides Assumption 1 is not a sufficient condition for ergodicity.

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